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Disagreement point monotonicity, transfer responsiveness, and the egalitarian bargaining solution*

Walter Bossert

Department of Economics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Abstract. This paper provides an axiomatization of the egalitarian bargaining solution. The central axiom used (together with some standard properties of bargaining solutions) in this characterization is a transfer responsiveness condition. First, it ensures that no transfer paradox can occur if bargaining power is transferred from one agent to another by decreasing one agent's and increasing the other agent's component of the disagreement point. Second, the extent of external effects of such a transfer is limited by requiring that agents not involved in the transfer neither gain more than the "winner" nor lose more than the "loser" of the transfer. *Journal of Economic Literature* Classification No.: C78.

1. Introduction

The bargaining problem as analyzed by Nash (1950) deals with the following situation: given a *feasible set* of utility vectors and a *disagreement point,* a group of n agents has to reach an agreement specifying an element of the feasible set as the outcome of a cooperative bargaining process. If the agents are unable to reach such a compromise, the outcome will be the disagreement point.

A *solution* to the bargaining problem is a function associating the resulting (unique) outcome with each possible pair of a feasible set and a disagreement point. Via axiomatic approaches, various solution concepts have been suggested (see, for instance, Nash 1950; Kalai and Smorodinsky 1975; Kalai 1977).

In the present paper, axioms for bargaining solutions concerning changes in the outcome caused by certain changes in the disagreement point are further analyzed. Thomson (1987) suggested the following *disagreement point monotonicity* conditions.

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First, suppose the bargaining position of one of the agents (agent i, say) is improved by increasing the i^{th} component of the disagreement point (while the feasible set and all other components of the disagreement point remain unchanged). It seems reasonable to assume that such an increase in the disagreement (or *threat*) point component of an agent increases his or her bargaining power, and, as a consequence, agent i should at least not lose from this change. The second of Thomson's (1987) disagreement point monotonicity axioms requires that agent i be *the only* agent who gains from such an improvement in her or his bargaining position. See Thomson (1987, pp 51–52) for a discussion of these properties.

Thomson (1987) has shown that the *Nash*, the *Kalai-Smorodinsky*, and the *egalitarian* solutions satisfy the first requirement, whereas the Nash and the Kalai-Smorodinsky solutions fail to satisfy the latter axiom, if there are at least three agents (if there are less than three agents, the second condition reduces to the first, given some standard properties of bargaining solutions).

These disagreement point monotonicity conditions are closely related to axioms requiring that no *transfer paradox* in the following sense can occur. Suppose bargaining power is "transferred" from an agent j to another agent i by increasing the i^{th} component of the disagreement point and, at the same time, decreasing the j^{th} component, that is, i's bargaining position improves and j's bargaining position worsens. Assuming again that the other agents' disagreement point components and the feasible set are unchanged, it seems very natural to require that i gains (or at least does not lose) and j loses (or at least does not gain) from such a transfer.

In this paper, the following strengthening of the above described *transfer responsiveness* condition is used. In order to limit the extent of *external effects* of a transfer of bargaining power via the disagreement point, it is required that the agents *not* involved in this transfer neither lose more than agent j nor gain more than agent i. This *strong transfer responsiveness* axiom thus ensures that third parties are not affected too seriously by such a transfer. Given that the primary goal of a transfer of bargaining power is to make the "winner" better off at the expense of the "loser", it seems very plausible to rule out that *other* agents gain even more than the winner or lose more than the loser. This can be considered a *fairness* condition regarding the outcome of transfers of bargaining power via the disagreement point.

The strong transfer responsiveness condition – together with some standard axioms for bargaining solutions – can be used to provide an alternative characterization of the egalitarian bargaining solution, the solution that selects the maximal point of the feasible set such that the utility gains of all agents over the disagreement outcome are equal. This characterization is the main result of the paper.

Clearly, the above described strong transfer responsiveness axiom requires interpersonal comparisons of utility gains and losses – this is a well-known feature of egalitarian and other solution concepts. For instance, Kalai (1977), Myerson (1981), Nielsen (1983), Moulin (1985), Chun and Thomson (1990) discuss solution concepts that require the possibility of comparing utilities (in particular, in most cases, utility gains and losses) interpersonally. It is clear that if we accept these solution concepts (the egalitarian solution and some of its generalizations are among the most thoroughly discussed concepts in these earlier contributions), we *have* to accept the possibility of interpersonal comparisons of utility, irrespective of whether this assumption is made explicitly (as, for instance, in Nielsen 1983) or is implicit in other assumptions.

A strong argument in favour of incorporating the possibility of interpersonal utility comparisons is their practical relevance. Interpersonal comparisons of well-being can be used and *are* (explicitly or implicitly) used in practice as an important and powerful tool in reaching a compromise or a decision for a group of agents. For instance, if we think of a bargaining solution as providing outcomes proposed by an impartial arbitrator, it seems very natural to include the possibility of comparing gains and losses of utility interpersonally in order to arrive at the conclusion that one outcome is preferable to another one. In fact, an arbitrator will often use the argument that the loss of some agents caused by choosing outcome A instead of B is less serious than the damage caused to another group of agents, if B is chosen instead of A. This is precisely the type of interpersonal comparison that is needed to state the above strong transfer responsiveness condition and that is commonly used to make the egalitarian bargaining solution a meaningful concept. See, for instance, Sen (1974, 1977), d'Aspremont and Gevers (1977), Roberts (1980), Blackorby, Donaldson, and Weymark (1984), Bossert (1991) for more detailed discussions of interpersonal comparisons in collective decision making.

In Sect. 2, bargaining problems and some properties commonly required from bargaining solutions are introduced. Disagreement point monotonicity and transfer responsiveness conditions are discussed in Sect. 3. The most important result of this section shows that the strong transfer responsiveness condition has, combined with a mild continuity requirement, surprisingly strong implications, namely that if an agent's disagreement point component increases, all *other* agents must lose *the same amount* as a consequence. This result is very useful for the proof of the characterization theorem, which constitutes the main part of Sect. 4. After stating and proving the characterization result, it is shown via some examples that the axioms used are independent. The paper concludes with Sect. 5, which contains a brief discussion of the relationships between this and earlier axiomatizations of egalitarian-type solutions.

2. Bargaining problems and bargaining solutions

Let $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$, resp.) denote the set of all (all nonnegative, all positive, resp.) real numbers. $\mathbb{R}^n$ ($\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$, resp.) is the n-fold Cartesian product of $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$, resp.). The notation for vector inequalities is $\geq$, $>$, $\gg$. For an arbitrary nonempty set C, a sequence of elements in C is denoted by $\{c^r\}_{r\in I}$, where I is the set of positive integers. Convergence of a sequence of subsets of $\mathbb{R}^n$ is defined in terms of the Hausdorff topology.

An *n-person bargaining problem* is a pair (S, d), where $S \subseteq \mathbb{R}^n$ is the set of feasible utility vectors, and $d \in \mathbb{R}^n$ is the disagreement point. Throughout the paper, it is assumed that there are at least three agents, that is, $n \geq 3$. A set $S \subseteq \mathbb{R}^n$ is *comprehensive* if and only if

$$\forall x \in S, \forall y \in \mathbb{R}^n, x > y \Rightarrow y \in S \ .$$

$S \subseteq \mathbb{R}^n$ *is strictly comprehensive* if and only if

$$\forall x \in S, \forall y \in \mathbb{R}^n, x > y \Rightarrow y \in S \text{ and } \exists\, z \in S \text{ such that } z \gg y \ .$$

Σ^n is the set of bargaining problems (S, d) such that S is convex, closed, and comprehensive, there exists $x \in S$ such that $x \gg d$, and

$$\exists\, p \in \mathbb{R}^n_{++}, m \in \mathbb{R} \quad \text{such that}\, px \leq m \,\forall\, x \in S \;. \tag{1}$$

Σ^n is the standard class of bargaining problems considered in the literature (sometimes, comprehensiveness is not required). (1) is used instead of the usual boundedness condition – see also Chun and Thomson (1990, p 952). It is worth pointing out that convexity is not a crucial assumption in the present paper – the characterization result goes through irrespective of whether or not convexity of S is required. The subclass of Σ^n of strictly comprehensive problems is denoted by $\tilde{\Sigma}^n$.

The *ideal point*, $b(S, d) \in \mathbb{R}^n$, of $(S, d) \in \Sigma^n$ is defined by

$$b_i(S, d) = \max\{x_i \mid x \in S \text{ and } x \geq d\} \,\forall\, i \in \{1, \ldots, n\} \;.$$

A *bargaining solution* (on Σ^n) is a function $F: \Sigma^n \mapsto \mathbb{R}^n$ such that $F(S, d) \in S$ for all $(S, d) \in \Sigma^n$.

The following is a list of standard properties of bargaining solutions. All these properties are well-established and hence do not require any discussion here.

Continuity: $\forall\, (S, d) \in \Sigma^n, \forall \{(S^r, d^r)\}_{r \in I}$ such that $(S^r, d^r) \in \Sigma^n \forall r \in I$,

$$\lim_{r \to \infty} S^r = S \text{ and } \lim_{r \to \infty} d^r = d \Rightarrow \lim_{r \to \infty} F(S^r, d^r) = F(S, d) \;.$$

d-**continuity:** $\forall\, (S, d) \in \Sigma^n, \forall \{d^r\}_{r \in I}$ such that $(S, d^r) \in \Sigma^n \forall\, r \in I$,

$$\lim_{r \to \infty} d^r = d \Rightarrow \lim_{r \to \infty} F(S, d^r) = F(S, d) \;.$$

Individual rationality: $\forall\, (S, d) \in \Sigma^n, F(S, d) \geq d$.

Weak Pareto optimality: $\forall\, (S, d) \in \Sigma^n,\ \forall\, x \in \mathbb{R}^n,\ x \gg F(S, d) \Rightarrow x \notin S$.

The solution that is of most interest in this paper is the *egalitarian* solution E (see Kalai 1977 and Thomson and Lensberg 1989, p 21). For each $(S, d) \in \Sigma^n$, $E(S, d)$ is given by the point $x \geq d$ on the boundary of S with $x_i - d_i = x_j - d_j$ for all $i, j \in \{1, \ldots, n\}$.

E is a special case of the *solutions of egalitarian character* E^λ (a subclass of Roth's 1979 solutions of proportional character), defined by

$$E^\lambda(S, d) = \lambda E(S, d) + (1 - \lambda) d \;\; \forall\, (S, d) \in \Sigma^n \;,$$

where $\lambda \in [0, 1]$. Clearly, $\lambda = 1$ leads to the egalitarian solution.

Another generalization of E is the class of *weighted egalitarian* solutions W^β – see Chun and Thomson (1990) (introduced under the name *proportional* solutions by Kalai 1977). Let $\Delta^n = \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} x_i = 1 \right\}$. For $\beta \in \Delta^n$, $W^\beta(S, d)$ is defined as the (unique) point of intersection of the set of weakly Pareto optimal points in S and the straight line passing through the points d and $(d + \beta)$. Choosing $\beta_i = \beta_j$ for all $i, j \in \{1, \ldots, n\}$ leads to the egalitarian solution.

All of the solutions introduced above satisfy **continuity** (and hence the weaker condition *d*-**continuity**) and **individual rationality**, all but E^λ with $\lambda \in [0, 1)$ satisfy **weak Pareto optimality.**

3. Disagreement point changes

If an agent's bargaining position improves by increasing the respective component of the disagreement point (other things remaining unchanged), it seems intuitive to require that this agent should not be worse off than before. Thomson (1987) showed that this condition is satisfied by the Nash, the Kalai-Smorodinsky, and the egalitarian solutions. A stronger version (see Thomson 1987) requires that none of the other agents gains from such a change in the disagreement point. The formal definition of this axiom is

Strong d-monotonicity: $\forall (S,d), (S',d') \in \Sigma^n, \ \forall i \in \{1,\ldots,n\}$,

$$S=S', d_i' > d_i, \text{ and } d_k' = d_k \forall k \in \{1,\ldots,n\} \backslash \{i\}$$

$$\Rightarrow F_i(S',d') \geq F_i(S,d) \text{ and } F_k(S',d') \leq F_k(S,d) \forall k \in \{1,\ldots,n\} \backslash \{i\} \ .$$

E satisfies **strong d-monotonicity** for all $n \geq 2$, whereas the Nash and the Kalai-Smorodinsky solutions do not satisfy this axiom, if there are at least three agents. Furthermore, Thomson (1987) noted that strong disagreement point monotonicity ensures that no transfer paradox can occur, that is, it is ruled out that agent i loses or agent j gains if d_i increases and d_j decreases (ceteris paribus). This property – the exclusion of a transfer paradox with respect to disagreement point changes – is defined as

Transfer responsiveness: $\forall (S,d), (S',d') \in \Sigma^n, \ \forall i,j \in \{1,\ldots,n\}$,

$$S=S', d_i' > d_i, d_j' < d_j, \text{ and } d_k' = d_k \forall k \in \{1,\ldots,n\} \backslash \{i,j\}$$

$$\Rightarrow F_i(S',d') \geq F_i(S,d) \text{ and } F_j(S',d') \leq F_j(S,d) \ .$$

For solutions satisfying **d-continuity**, **transfer responsiveness** is *equivalent* to **strong d-monotonicity** (the proof that **d-continuity** and **transfer responsiveness** imply **strong d-monotonicity** is analogous to the proof of Lemma 2 below). See Thomson (1987) for further details concerning these two axioms.

The egalitarian solution E is not the only solution satisfying the above transfer responsiveness condition for $n \geq 3$; other examples are the weighted egalitarian solutions. However, by using a particular strengthening of this axiom, a characterization of the egalitarian solution can be provided. A reasonable strengthening seems to be to limit the extent of external effects of bargaining power transfers via the disagreement point on the agents *not* involved in the transfer.

Consider a transfer of bargaining power from agent j to agent i by decreasing d_j and at the same time increasing d_i, keeping all other components of the disagreement point unchanged. In order to limit the external effects of this transfer, one can require that nobody loses more than agent j and that nobody gains more than agent i. As mentioned in the introduction, formulating such a condition requires interpersonal comparisons of utility gains and losses. I refer to this axiom as

Strong transfer responsiveness: $\forall (S,d), (S',d') \in \Sigma^n, \ \forall i,j \in \{1,\ldots,n\}$,

$$S=S', d_i' > d_i, d_j' < d_j, \text{ and } d_k' = d_k \forall k \in \{1,\ldots,n\} \backslash \{i,j\}$$

$$\Rightarrow F_i(S',d') \geq F_i(S,d), F_j(S',d') \leq F_j(S,d), \text{ and}$$

$$F_i(S',d')-F_i(S,d) \geq F_k(S',d')-F_k(S,d)$$
$$\geq F_j(S',d')-F_j(S,d) \forall k \in \{1,\ldots,n\} \backslash \{i,j\} .$$

As is shown in Sect. 4, **strong transfer responsiveness** can be used to provide a characterization of the egalitarian solution. This section is concluded with a result that will be very useful in the proof of this characterization theorem. Combined with *d*-**continuity**, **strong transfer responsiveness** has surprisingly strong consequences. *d*-**continuity** and **strong transfer responsiveness** together imply that if d_i is increased while all $d_k, k \in \{1,\ldots,n\} \backslash \{i\}$, remain unchanged, the losses of all agents $k \in \{1,\ldots,n\} \backslash \{i\}$ are *the same.* This property is introduced as

Equal-loss *d*-**monotonicity:** $\forall (S,d), (S',d') \in \Sigma^n, \ \forall i \in \{1,\ldots,n\}$,

$$S=S', d_i' > d_i, \text{ and } d_k' = d_k \forall k \in \{1,\ldots,n\} \backslash \{i\}$$
$$\Rightarrow F_i(S',d') \geq F_i(S,d), F_k(S',d') \leq F_k(S,d) \forall k \in \{1,\ldots,n\} \backslash \{i\}, \text{ and}$$
$$F_k(S',d')-F_k(S,d)=F_j(S',d')-F_j(S,d) \forall j,k \in \{1,\ldots,n\} \backslash \{i\} .$$

Equal-loss *d*-**monotonicity** requires that if an agent's bargaining position improves, *all* other agents have to share *equally* the burden that is thereby imposed on them.

In the presence of *d*-**continuity**, **equal-loss** *d*-**monotonicity** is equivalent to **strong transfer responsiveness**. This is demonstrated via two lemmas.

Lemma 1. *Let* $n \geq 3$, *and let* F *be a solution on* Σ^n. *If* F *satisfies* **equal-loss** *d*-**monotonicity**, *then* F *satisfies* **strong transfer responsiveness**.

Proof. Let (S,d), $(S',d') \in \Sigma^n$ be such that $S'=S$ and $d'=(d_1,\ldots,d_i+\delta_i,\ldots,d_j-\delta_j,\ldots,d_n)$ for some $i,j \in \{1,\ldots,n\}$, $\delta_i, \delta_j \in \mathbb{R}_{++}$. Furthermore, let $\bar{d}=(d_1,\ldots,d_j-\delta_j,\ldots,d_n)$. Then $(S,\bar{d}) \in \Sigma^n$, and **equal-loss** *d*-**monotonicity** implies

$$F_i(S',d') \geq F_i(S,\bar{d}) \geq F_i(S,d) \text{ and } F_j(S',d') \leq F_j(S,\bar{d}) \leq F_j(S,d) ,$$

and furthermore,

$$F_i(S,\bar{d})-F_i(S,d)=F_k(S,\bar{d})-F_k(S,d) \geq F_j(S,\bar{d})-F_j(S,d) \text{ and}$$
$$F_i(S',d')-F_i(S,\bar{d}) \geq F_k(S',d')-F_k(S,\bar{d})=F_j(S',d')-F_j(S,\bar{d})$$

for all $k \in \{1,\ldots,n\} \backslash \{i,j\}$. Adding these (in)equalities, it follows

$$F_i(S',d')-F_i(S,d) \geq F_k(S',d')-F_k(S,d) \geq F_j(S',d')-F_j(S,d)$$

for all $k \in \{1,\ldots,n\} \backslash \{i,j\}$. □

Lemma 2. *Let* $n \geq 3$, *and let* F *be a solution on* Σ^n. *If* F *satisfies* *d*-**continuity** *and* **strong transfer responsiveness**, *then* F *satisfies* **equal-loss** *d*-**monotonicity**.

Proof. Suppose by way of contradiction that F satisfies *d*-**continuity** and **strong transfer responsiveness**, and there exist (S,d), $(S',d') \in \Sigma^n$, $i \in \{1,\ldots,n\}$ such that $S'=S$, $d_i' > d_i$, $d_k' = d_k$ for all $k \in \{1,\ldots,n\} \backslash \{i\}$, and

(i) $F_i(S',d') < F_i(S,d)$

or

(ii) $F_j(S',d') > F_j(S,d)$ for some $j \in \{1,\dots,n\} \setminus \{i\}$

or

(iii) $F_j(S',d') - F_j(S,d) > F_k(S',d') - F_k(S,d)$ for some $j,k \in \{1,\dots,n\} \setminus \{i\}$.

In case (i), by d-**continuity**, there exists $\varepsilon_j \in \mathbb{R}_{++}$ such that

$$F_i(S,d) - F_i(S',d') > F_i(S,\bar{d}) - F_i(S',d')$$

where $\bar{d} = (d_1,\dots,d_i',\dots,d_j - \varepsilon_j,\dots,d_n)$, that is, $F_i(S,d) > F_i(S,\bar{d})$, contradicting **strong transfer responsiveness.**

In case (ii), d-**continuity** implies that there exists $\varepsilon_j \in \mathbb{R}_{++}$ such that, with $\bar{d} = (d_1,\dots,d_i',\dots,d_j - \varepsilon_j,\dots,d_n)$,

$$F_j(S',d') - F_j(S,d) > F_j(S',d') - F_j(S,\bar{d}) \ ,$$

and hence, $F_j(S,d) < F_j(S,\bar{d})$, again contradicting **strong transfer responsiveness.**

In case (iii), let

$$\varepsilon = F_j(S',d') - F_j(S,d) - F_k(S',d') + F_k(S,d) > 0 \qquad (2)$$

and $\bar{d} = (d_1,\dots,d_i',\dots,d_j - \varepsilon_j,\dots,d_n)$ with $\varepsilon_j \in \mathbb{R}_{++}$. By d-**continuity** and the fact that the cases (i) and (ii) cannot occur, ε_j can be chosen such that

$$0 \geq F_j(S,\bar{d}) - F_j(S',d') > -\frac{\varepsilon}{2} \quad \text{and}$$

$$0 \geq F_k(S',d') - F_k(S,\bar{d}) > -\frac{\varepsilon}{2} \ .$$

This, together with (2), implies

$$F_j(S,\bar{d}) - F_j(S,d) - F_k(S,\bar{d}) + F_k(S,d) > \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \ ,$$

that is, $F_j(S,\bar{d}) - F_j(S,d) > F_k(S,\bar{d}) - F_k(S,d)$, which is a contradiction to **strong transfer responsiveness.** □

Lemmas 1 and 2 immediately imply

Theorem 1. *Let $n \geq 3$, and let F be a solution on Σ^n satisfying* d-**continuity**. *Then F satisfies* **equal-loss** d-**monotonicity** *if and only if F satisfies* **strong transfer responsiveness.**

4. A characterization of the egalitarian solution

In this section, the main result of the paper is stated and proven. The two crucial steps of the proof are given by the following two lemmas. Lemma 3 demonstrates that the egalitarian solution satisfies **equal-loss** d-**monotonicity** – and thus **strong transfer responsiveness** –, whereas Lemma 4 shows that any solution satisfying **equal-loss** d-**monotonicity** and some standard properties must coincide with the egalitarian solution on the class of strictly comprehensive problems. The remainder of the proof then follows from an immediate continuity argument.

Lemma 3. *Let $n \geq 3$. The egalitarian solution E on Σ^n satisfies* **equal-loss** ***d*****-monotonicity**.

Proof. Suppose (S,d), $(S',d') \in \Sigma^n$, $S=S', d_i' > d_i$ for some $i \in \{1,\ldots,n\}$, and $d_k' = d_k$ for all $k \in \{1,\ldots,n\} \setminus \{i\}$. From Theorem 6 in Thomson (1987), $E_i(S',d') \geq E_i(S,d)$ and $E_k(S',d') \leq E_k(S,d)$ for all $k \in \{1,\ldots,n\} \setminus \{i\}$. By definition of E,

$$E_j(S',d') - d_j' = E_k(S',d') - d_k' \tag{3}$$

and

$$E_j(S,d) - d_j = E_k(S,d) - d_k \tag{4}$$

for all $j,k \in \{1,\ldots,n\} \setminus \{i\}$, respectively. Noting that $d_j' = d_j$ and $d_k' = d_k$ and subtracting (4) from (3), it follows

$$E_j(S',d') - E_j(S,d) = E_k(S',d') - E_k(S,d)$$

for all $j,k \in \{1,\ldots,n\} \setminus \{i\}$. □

Lemma 4. *Let $n \geq 3$, and let F be a solution on Σ^n. If F satisfies* **individual rationality**, **weak Pareto optimality**, *and* **equal-loss** ***d*****-monotonicity**, *then F coincides with E on $\tilde{\Sigma}^n$.*

Proof. Suppose that F satisfies **individual rationality**, **weak Pareto optimality**, and **equal-loss** ***d*****-monotonicity**, and, by way of contradiction, there exists $(S,d) \in \tilde{\Sigma}^n$ such that $F(S,d) \neq E(S,d)$. Choose $j \in \{1,\ldots,n\}$ such that

$$F_j(S,d) - d_j \leq F_k(S,d) - d_k \forall k \in \{1,\ldots,n\} .$$

Since $F(S,d) \neq E(S,d)$ and F satisfies **weak Pareto optimality**, there must exist an $i \in \{1,\ldots,n\}$ such that $F_i(S,d) - d_i > F_j(S,d) - d_j$. Let

$$\varepsilon = F_j(S,d) - d_j . \tag{5}$$

Since $n \geq 3$, $\{1,\ldots,n\} \setminus \{i,j\} \neq \emptyset$. From the strict comprehensiveness of S, it follows that there exist $h \in \{1,\ldots,n\} \setminus \{i,j\}$ and d_h' such that $(S,d') \in \tilde{\Sigma}^n$, $d_h < d_h' < b_h(S,d)$, and $x_i < F_i(S,d) - \varepsilon$ for all $x \in S$ such that $x \geq d'$, where $d' = (d_1,\ldots,d_h',\ldots,d_n)$. Consequently, from **individual rationality**,

$$F_i(S,d) - F_i(S,d') > \varepsilon ,$$

which implies, by **equal-loss** ***d*****-monotonicity**,

$$F_j(S,d) - F_j(S,d') > \varepsilon . \tag{6}$$

By (5), (6), and $d_j' = d_j$, it follows $F_j(S,d') < d_j'$. But this is a contradiction to **individual rationality**. □

The argument used in the proof of Lemma 4 is illustrated for $n=3$ in Fig. 1. The area enclosed by the points b_i, b_j, b_h is the set of weakly Pareto optimal points $x \in S$ such that $x \geq d$. $F_i(S,d) - d_i$ is greater than $F_j(S,d) - d_j = \varepsilon$. If d_h is increased to d_h', the new disagreement point is d'. The set of weakly Pareto

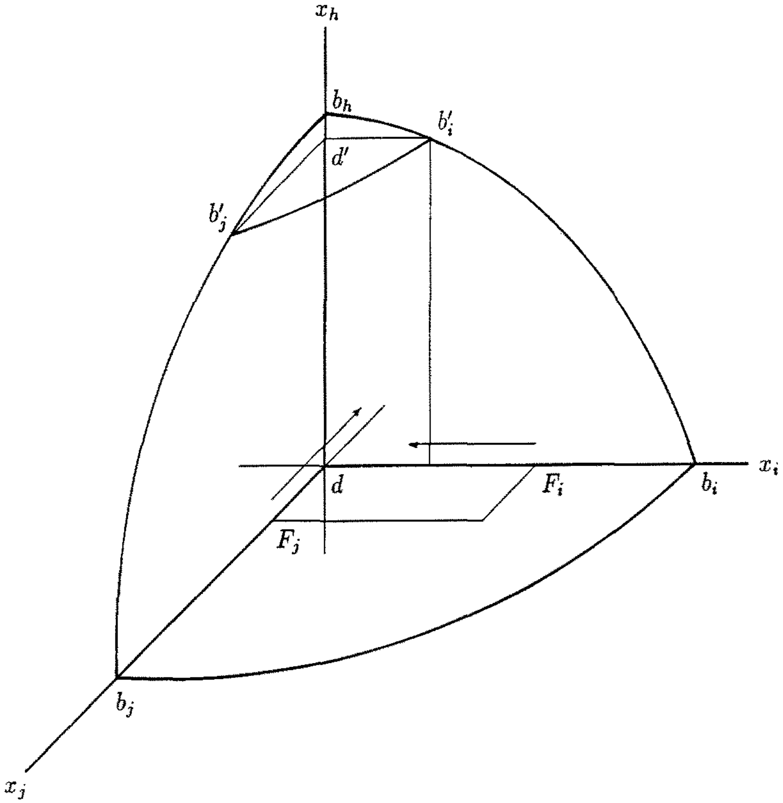

Fig. 1. Illustration of Lemma 4

optimal points $x \in S$ such that $x \geq d'$ is given by the area enclosed by b'_i, b'_j, b_h. Given **individual rationality**, the new outcome $F(S, d')$ must be such that agent i loses more than ε. Since **equal-loss** ***d*-monotonicity** is satisfied, j must lose more than ε as well, which leads to $F_j(S, d') < d'_j$, contradicting **individual rationality**.

These results now can be used to prove the characterization theorem.

Theorem 2. *Let $n \geq 3$, and let F be a solution on Σ^n. Then F satisfies* **continuity**, **individual rationality**, **weak Pareto optimality**, *and* **strong transfer responsiveness** *if and only if $F = E$.*

Proof. It is clear that E satisfies **continuity**, **individual rationality**, **weak Pareto optimality**, and, by Theorem 1 and Lemma 3, **strong transfer responsiveness**. On the other hand, by Theorem 1 and Lemma 4, each solution F satisfying ***d*-continuity**, **individual rationality**, **weak Pareto optimality**, and **strong transfer responsiveness** must coincide with E on $\tilde{\Sigma}^n$. To complete the proof, simply note that each element (S, d) of $\Sigma^n \backslash \tilde{\Sigma}^n$ is the limit of a sequence $\{(S^r, d^r)\}_{r \in I}$ with $(S^r, d^r) \in \tilde{\Sigma}^n$ for all $r \in I$, and apply **continuity**. □

That **strong transfer responsiveness** is crucial in Theorem 2 is clear. To see that all other properties are needed as well, observe the following examples which establish the independence of the axioms used in this characterization.

If **weak Pareto optimality** is dropped, further solutions become available; for instance, the solutions E^{λ} with $\lambda \in [0,1)$ satisfy all of the axioms required in the statement of Theorem 2 except for **weak Pareto optimality**.

Similarly, if **continuity** is dropped, a solution $F \neq E$ satisfying the remaining axioms can be defined by

$$F(S,d)=\begin{cases} b(S,d) & \text{if} \quad b(S,d)\in S \\ E(S,d) & \text{if} \quad b(S,d)\notin S \end{cases} \quad \forall (S,d)\in \Sigma^n .$$

A solution which satisfies all of the required properties but **individual rationality** is obtained by defining $F(S,d)$ to be the weakly Pareto optimal point x in S such that $x_i = x_j$ for all $i,j \in \{1,\ldots,n\}$.

Note that if only the set of strictly comprehensive problems $\tilde{\Sigma}^n$ is considered the domain of F, a characterization result analogous to Theorem 2 can be obtained without using **continuity**, but merely *d*-**continuity** – this is an immediate consequence of the fact that only the last step in the proof of Theorem 2 requires **continuity**.

Defining a solution on $\tilde{\Sigma}^n$ as a mapping $\tilde{F}: \tilde{\Sigma}^n \mapsto \mathbb{R}^n$ such that $\tilde{F}(S,d) \in S$ for all $(S,d) \in \tilde{\Sigma}^n$, it follows (denoting the restriction of E to $\tilde{\Sigma}^n$ by $\tilde{E}$)

Corollary 1. *Let $n \geq 3$, and let $\tilde{F}$ be a solution on $\tilde{\Sigma}^n$. Then $\tilde{F}$ satisfies d-***continuity**, **individual rationality**, **weak Pareto optimality**, *and* **strong transfer responsiveness** *if and only if $\tilde{F} = \tilde{E}$.*

Since the axioms used in Corollary 1 (and in all other results except for Theorem 2) do not involve any changes in S, alternative formulations could (with some obvious changes in notation) be obtained by stating all axioms and results for each *fixed S.* This is an interesting feature of the results of this paper. As Chun and Thomson (1990, p 959) pointed out, it seems that changes in the disagreement point might be easier to implement in experimental testing than changes in the shape of the feasible set. Keeping the feasible set fixed should therefore particularly facilitate such experiments.

5. Comparison with earlier results

Alternative characterizations of egalitarian and related solutions to bargaining and social choice problems can be found, for instance, in Myerson (1981), Nielsen (1983), Moulin (1985), Peters (1986), Thomson and Lensberg (1989), and Chun and Thomson (1990). This section is devoted to a brief discussion of the main features of these axiomatizations to illustrate in which respects Theorem 2 differs from those earlier results.

Myerson's (1981) joint characterization of egalitarian and utilitarian social choice functions uses a concavity condition as the crucial axiom, together with weak Pareto optimality and *independence of irrelevant alternatives* (see also Nash 1950).

Nielsen (1983) uses the possibility of making interpersonal comparisons of gains from the disagreement point together with Pareto optimality, a strong version of individual rationality, and independence of irrelevant alternatives to characterize a solution that coincides with the egalitarian solution on $\tilde{\Sigma}^n$.

Peters (1986) uses a *partial superadditivity* condition with respect to the feasible set to characterize the weighted egalitarian solutions.

Thomson and Lensberg's (1989, p 22) characterization of E is a particular version of Kalai's (1977) axiomatization of the weighted egalitarian (or *proportional*) solutions, obtained by adding a symmetry condition to Kalai's original set of axioms. The driving force in this characterization is a monotonicity condition concerning changes in S. It requires that if S is expanded to some arbitrary feasible superset of S, none of the agents lose.

The four axiomatizations described above are quite different from the characterization established in Theorem 2. Moulin's (1985) axiomatization of *equal sharing above a convex decision* social choice functions could be considered somewhat more similar to the approach in the present paper in that Moulin also uses a "no transfer paradox" axiom (which is similar to Thomson's 1987 transfer responsiveness condition, but does not share the distinguishing (stronger) features of **strong transfer responsiveness**). However, the class of problems analyzed in Moulin (1985) is quite different from the one discussed here, and furthermore, the transfers considered there do not take place via changes in a disagreement point.

Chun and Thomson's (1990) characterization of the class of weighted egalitarian solutions is mainly based on a disagreement point *concavity* condition that applies to situations where the disagreement point is not known. Since changes in the disagreement point for fixed feasible sets are important in this characterization as well, it seems appropriate to have a closer look at the relationships between the strong transfer responsiveness condition and the above mentioned disagreement point concavity axiom, which is defined as (see Chun and Thomson 1990, p 953)

***d*-concavity:** $\forall (S,d),\ (S',d') \in \Sigma^n, \forall \alpha \in [0,1]$,

$$S=S' \Rightarrow F(S,\alpha d+(1-\alpha)d') \geq \alpha F(S,d)+(1-\alpha)F(S'd') \ .$$

Despite the similarity of the axiom systems used, the two characterizations differ in various substantial respects. One feature that distinguishes Chun and Thomson's (1990) axiomatization from the present one is the former's reliance on the convexity assumption on S. If this assumption is dropped, the weighted egalitarian solutions do not necessarily satisfy ***d*-concavity**. To illustrate that, let $n=2$, S be the comprehensive hull of $\{(1,2),\ (1,1),\ (2,1)\}$, $d=(1/2,\ -1/2)$, and $d'=(-1/2, 1/2)$. The egalitarian solution yields $E(S,d)=(2,1)$, $E(S,d')=(1,2)$, and $E(S,d/2+d'/2)=(1,1)$, violating ***d*-concavity**. Clearly, similar examples can be found for higher dimensions.

It is a trivial observation that the entire set of axioms used in Theorem 2 implies ***d*-concavity** for solutions defined on Σ^n, if $n \geq 3$. Even with the convexity assumption, however, **strong transfer responsiveness** and ***d*-concavity** by themselves are logically independent. Clearly, any weighted egalitarian solution W^β with $\beta_i \neq \beta_j$ for some $i,j \in \{1,\ldots,n\}$ satisfies ***d*-concavity**, but violates **strong transfer responsiveness**. For any convex, closed, comprehensive $S \subseteq \mathbb{R}^n$ satisfying (1), choose an arbitrary point $\hat{x}$ in the interior of S, and define $S^*(S)=S \cap \{x \in \mathbb{R}^n | \exists\, i \in \{1,\ldots,n\}$ such that $x_i \leq \hat{x}_i\}$. Now define a solution E^* in the following way. For $(S,d) \in \Sigma^n$, let $E^*(S,d)$ be the point x on the boundary of $S^*(S)$ such that $x_i - d_i = x_j - d_j$ for all $i,j \in \{1,\ldots,n\}$. E^* satisfies **equal-loss *d*-monotonicity**, but violates ***d*-concavity**. Notice that the above example also satisfies ***d*-continuity**, which further underlines the independence of the two results (and establishes the logical independence of **equal-loss *d*-monotonicity** and ***d*-concavity**).

Summarizing these observations, Theorem 2 provides an alternative characterization of the egalitarian bargaining solution that differs significantly from earlier axiomatizations.

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