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Queue allocation of indivisible goods

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Abstract. A model with a finite number of indivisible goods (houses) and the same number of individuals is considered. The allocation of houses among the individuals according to a queue order is analysed. First an allocation mechanism is constructed where it is a dominant strategy for the individuals to truthfully report their preferences. Second it is demonstrated that in order to obtain the desired allocation, the individuals must not in general report their complete ranking of the houses, but only their maximal elements in recursively defined choice sets.

1. Introduction

In this paper the allocation of indivisible goods by a queue-method is analysed. We may think of a planner distributing houses (or buildings sites, or jobs, or day-care places etc.) among a finite number of individuals. There is no divisible good (money) that can be used to (fully) compensate differences in value of the indivisible goods or alternatively, the planner has to set prices on the goods but he has no exact information about the correct equilibrium prices.[1] As a consequence, it becomes important for efficiency that *indifferences* in the preferences of one individual are properly taken into account in the allocation procedure.

This non-market allocation situation is common for many goods provided by local governments. The planner is then faced not only with a pure allocation but also with an informational problem. To achieve an efficient and maybe also a

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[1] For an analysis of equilibrium in a model with indivisible goods and money, see e.g. Gale (1984), Quinzii (1984) or Svensson (1984). Implementation problems are studied in Svensson (1991).

fair allocation he has to collect a number of individual characteristics, in particular individual preferences.

Two problems are obvious. First, information may be private and the individuals may have incentives not to tell the truth. Second, if the number of individuals is large, it may be costly and even unnecessary for the individuals to report their complete preferences over the entire set of houses.

These two problems are analyzed in the paper. First we look for an allocation mechanism, the outcomes of which are efficient and weakly fair (to be defined later) allocations, and where truth-telling is a *dominant strategy*. Second, we show that the outcomes of the mechanism, the queue-allocations, may be achieved by a recursively defined procedure, where at each step and given the data from previous steps a choice set for one individual is defined. The individual then has to report only his maximal elements in the choice set. Then the procedure continues. So instead of each individual reporting his complete ranking of all houses, each individual is given a choice set, in general smaller than the set of all houses, and is required to report his maximal elements.

The model with indivisible goods used in this paper has previously been studied in e.g. Shapley and Scarf (1974). They showed the existence of a competitive equilibrium. Roth (1982) constructed a procedure (a mechanism) for achieving a competitive equilibrium, where it is a dominant strategy for the individuals to reveal their true preferences.

The existence of a strategy proof mechanism in Roth's study is remarkable – in the general case implementation in dominant strategies is impossible. But the result is due to the special character of the model, in particular the absence of a divisible good. That is also the reason why implementation in dominant strategies is possible in our case.

2. The model

Let $N=\{1,2,\ldots,n\}$ be a finite 'society' of *n individuals* and let $A=\{a_1,a_2,\ldots,a_n\}$ be the same number of *indivisible goods*. The elements of A are called houses. An *allocation* is an *injective function* $f:N\rightarrow A$. Each individual $i\in N$ has a *preference order* R_i over A. Preferences are assumed to be a complete, reflexive and transitive binary relation. Strict preference is denoted P_i and I_i denotes indifference. As usual an allocation f is *Pareto optimal* if there is not other allocation g such that $g(i)\,R_i f(i)$ for all $i\in N$ and $g(i)\,P_i f(i)$ for some i.

Equity is usually defined as a symmetry requirement, but in models with indivisible goods only, equity may easily be an empty concept. In the present study equity will be substituted by a ranking expressed by a permutation π of the individuals N. Hence, individual i is before individual j in the social ranking if $\pi(i)<\pi(j)$. The society's ranking of the individuals reflects some kind of justice; it may be a ranking according to needs, or a ranking according to queue-time etc., but in our analysis it is an *exogenous ranking*. Hence we neither consider the normative problem what the ranking ought to be, nor how to collect individual characteristics which the ranking could be based on.

In order to make only 'internal' interpersonal utility comparisons, equity is often formalised as a *fairness (no-envy) criterion*, i.e. f is a fair allocation if $f(i)\,R_i f(j)$ for all $i,j\in N$. Here[2] the symmetry condition is replaced by the

[2] The existence of fairness in a model with indivisible goods and money is analysed in e.g. Maskin (1987) or Svensson (1983).

social ranking π and we call an allocation f *weakly fair* if for all $i \in N$, $f(i) R_i f(j)$ for all j such that $\pi(j) > \pi(i)$. The concept of weak fairness operationalizes the society's ranking of the individuals. Also note that a fair allocation is weakly fair for any choice of π.

One can easily prove that a fair allocation is also Pareto-optimal. Furthermore, if there are no indifferences in the individual preferences R_i, a weakly fair allocation is also Pareto optimal. On the other hand, if e.g. individual 1 is indifferent between a_1 and a_2, while individual 2 prefers a_1 to a_2, the allocation $f(1) = a_1$ and $f(2) = a_2$ is weakly fair (if $\pi(1) = 1$ and $\pi(2) = 2$) but not Pareto optimal. So in general a weakly fair allocation is not necessarily Pareto optimal.

We assume that the social preference is to achieve a weakly fair and Pareto optimal allocation. This has to be implemented by a *mechanism* $F(R)$, where $R = (R_1, R_2, \ldots, R_n)$ is an n-tuple of individual preferences, a preference profile, and where the outcomes $F(R)$ are weakly fair and Pareto optimal allocations. Our first objective is to construct a mechanism such that it is a dominant strategy for the individuals to report their true preferences.

3. The mechanism

With no loss of generality we now assume that $\pi(i) = i$ if nothing else is explicitly said. Denote by R_{-i} the profile $(R_1, R_2, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ of all individual preferences except the preferences R_i of individual i. Also let $R = (R_i, R_{-i})$. Truthtelling is then a dominant strategy for the mechanism F if for all i: $f(i) R_i f'(i)$ for all conceivable preference profiles R', where $f \in F(R_i, R'_{-i})$ and $f' \in F(R')$. Moreover, for a subset $B \subset A$, let $m_i(B) = \{a \in B; a R_i b$ for all $b \in B\}$, i.e. i's maximal elements in B.

To define the mechanism F (when $\pi(i) = i$), consider a given preference profile R and define recursively a decreasing sequence $\{S_i\}_{i=1}^{n}$ of (maximal) *choice sets* according to:

- $S_1 = A$ and
- $S_{j+1} = \{a \in A$; there is an allocation f such that $f(i) \in m_i(S_i)$ for $i \leq j$ and $f(j+1) = a\}$.

Now let $F(R) = \{$allocations $f; f(i) \in m_i(S_i)$ for all $i\}$ and call the elements in $F(R)$ *queue allocations.* The mechanism F may be multivalued, but a direct consequence of the definition of F is that the individual utility levels are *uniquely determined*: if f and g are two queue allocations, then every individual i is indifferent between $f(i)$ and $g(i)$.

Theorem 1. *Truthtelling is a dominant strategy for F and corresponding outcomes are Pareto optimal and weakly fair allocations.*

Proof. When individual i is to announce his preferences R'_i, not necessarily equal to his true preferences R_i, he is faced with a choice in a set of alternatives given by S_i. Hence, the choice set for i is independent of his 'choice' R'_i of preferences. There is nothing to gain by reporting anything else but the true preferences R_i.

It is also immediately clear that the only way to increase the utility of individual i is to increase his choice set S_i, which is not possible without decreasing the utility for some individual $k, k < i$. Hence an outcome is Pareto optimal.

Finally, an outcome is weakly fair since the choice sets S_i are obviously decreasing, not necessarily strictly, with increasing i. Q.E.D.

We may now note that the mechanism F does not fully implement the social preferences. The following *example* shows that for some preference profile there is a weakly fair and Pareto optimal allocation that is not the outcome of F.

Suppose $n=3$ and that individual preferences are given by:

$$a_1 I_1 a_3 P_1 a_2\,, \quad a_1 P_2 a_2 P_2 a_3 \quad \text{and} \quad a_3 P_3 a_1 P_3 a_2$$

One easily finds that $S_1 = S_2 = A$ and $S_3 = \{a_2\}$ and hence that $F(R) = f$, where $f(1) = a_3$, $f(2) = a_1$ and $f(3) = a_2$, while the allocation g, where $g(1) = a_1$, $g(2) = a_2$ and $g(3) = a_3$, is weakly fair and Pareto optimal.

On the other hand, we can show that every Pareto optimal allocation is a queue allocation for *some queue order* π. See Theorem 2 below. That means that to implement an arbitrary weakly fair and Pareto optimal allocation by the mechanism F one may have to change the original queue order (from $\pi(i) = i$ to some other order).

Note that if the original social ranking of the individuals also reflects a ranking of interpersonally comparable utilities then the allocations implemented by F are exactly those satisfying the Rawlsian leximin criterion; first the utility of individual one has to be maximal, then the utility of individual 2 has to be maximal, etc.

Theorem 2. *Let f be a Pareto optimal allocation. Then there is a ranking π (not necessarily $\pi(i) = i$) and a decreasing sequence $\{S_i\}_{i=1}^n$ of choice sets such that $S_i \subset A$ and $f(i) \in m_i(S_{\pi(i)})$.*

Proof. Since f is Pareto optimal there is an individual i such that $f(i) R_i f(j)$ for all j: Otherwise there would be a finite sequence $i_1, i_2, \ldots, i_k$ such that $f(i_{j+1}) P_{i_j} f(i_j)$ for $j < k$ and $f(i_1) P_{i_k} f(i_k)$. But the allocation $g(i_j) = f(i_{j+1})$ for $j < k$, $g(i_k) = f(i_1)$ and $g(i) = f(i)$ if $i \neq i_j$ is a Pareto improvement – a contradiction.

Now let $G_1 = \{i; f(i) R_i f(j) \forall j\}$ and let $\# G_1 = n_1$. Also let $S_1 = S_2 = \ldots = S_{n_1} = A$. Choose a bijection $\pi_1 : G_1 \to \{1, 2, \ldots, n_1\}$. Then the theorem is true for π_1 and $\{S_i\}_{i=1}^{n_1}$ if N is replaced by G_1 and A is replaced by $f(G_1)$.

Then consider the set $N - G_1$ of individuals and the set $A - f(G_1)$ of houses and repeat the procedure above. We get $G_2 \subset N$ and $S_{n_1+1} = S_{n_2+2} = \ldots = S_{n_2} = A - f(G_1)$ and π_2. Continue this procedure until the entire N has been allocated on groups G_{n_j}. The proof of the theorem is then complete if π is defined as $\pi(i) = \pi_j(i)$ when $i \in G_j$. Q.E.D.

4. Sequential calculations of the choice sets

The existence of a well-behaved mechanism F is established, but the mechanism is unnecessarily demanding as to the information requirement concerning individual preferences. Every individual must in general not report his preferences over the entire set A of indivisible goods. It is, of course, sufficient if he reports his maximal elements in the choice set S_i.

Accordingly we want a less information demanding mechanism or procedure for achieving allocations $f \in F(R)$. The procedure may be characterised by the sequence $\{S_i\}_{i=1}^n$ of choice sets, where the various sets are defined in n steps; $i=1,2,\dots,n$. At the first step individual 1 is given the choice set $S_1=A$ and reports his maximal elements $m_1(S_1)$. If the maximal element is unique, $f(1)$ is defined, otherwise the definition of $f(1)$ remains. Given this information, S_2 is defined and in the next step individual 2 reports his maximal elements in S_2 and so on. This leads to a complete definition of the various choice sets S_i.

We want a simple rule for calculating the choice sets, or alternatively at each step i we want to calculate the difference $S_i - S_{i+1}$ given the reported maximal elements in $S_j, j \leq i$. The difference $S_i - S_{i+1}$, that will be denoted A_i, is in many cases the empty set. For instance, if an allocation is fair then we may set $S_i = S_{i+1} = A$ for all i, i.e. $A_i = \emptyset$ for all i. In other cases, however, the sets A_i are necessarily nonempty and a nonempty A_i means that the elements in A_i have to be allocated at step i in the allocation process. Of course, the elements in A_i have to be allocated to a group of individuals, denoted N_i, where $\# A_i = \# N_i$.

In order to formulate the rule determining the sets A_i and N_i at each step i some lemmas will be needed. Lemma 1 below is a reformulation of a lemma that is proved in Svensson 1984 (Lemma 1 there). The following notation and assumptions will be employed in Lemma 1 and Lemma 2. There we assume that the number $\# N$ of individuals is not larger than the number $\# A$ of houses. (In the rest of the paper we have equality.) Moreover, for each $i \in N$ there is a nonempty set $m_i \subset A$, which may be interpreted as i's maximal elements in A. For $G \subset N$ the set $\cup_{i \in G} m_i$ of the union of the maximal sets of the group is denoted m_G.

Lemma 1. *There is an allocation f such that $f(i) \in m_i$ if and only if $\# G \leq \# m_G$ for all $G \subset N$.*

An interpretation of Lemma 1 is that m_i represents individual i's maximal elements in A. An allocation f such that $f(i) \in m_i$ is then an envy-free (or fair) allocation and the condition $\# G \leq \# m_G$ is a characterisation of envy-free allocations.

Lemma 2. *If $\# G \leq \# m_G$ for all $G \subset N$ then there is a unique solution (possibly the empty set) to the problem: $\max \# G$ s.t. $G \subset N$ and $\# G = \# m_G$.*

Proof. Suppose that G and H are two solutions and that $G \neq H$. But then $\#(G \cup H) < \# m_{G \cup H}$. By Lemma 1 there is an allocation f such that $f(i) \in m_i$ for all $i \in G \cup H$. Since $\#(G \cup H) < \# m_{G \cup H}$ there is $i \in G \cup H$ and $a \in m_{G \cup H}$ such that $a \in m_i$ and for all j, $f(j) \neq a$. If $i \in G$ then $\# G < \# m_G$ and if $i \in H$ then $\# H < \# m_H$, contradicting G and H being two solutions of the maximisation problem. Hence $G = H$. Q.E.D.

We will now recursively define a sequence $\{A_j\}_{j=1}^n$ of subsets of A and a sequence $\{N_j\}_{j=1}^n$ of subsets of N. In analogy with earlier notation, for $G \subset N$, $G \neq \emptyset$, let $m_G = \cup_{i \in G} m_i(S_i)$ (and $m_i = m_i(S_i)$). Suppose that $A_1, A_2, \dots, A_{j-1}$ and $N_1, N_2, \dots, N_{j-1}$ are defined. Also let $A_j^u = \cup_{i \leq j} A_i$, $A_0^u = \emptyset$ and $I_j = \{i \in N; i \leq j$ and $i \notin N_k$ if $k < j\}$. Then N_j is defined as the solution to:

$\max \# G$, s.t.

$G \subset I_j$ and $\# G = \#(m_G - A^u_{j-1})$.

Given the definition of N_j, the set A_j is defined as:

$$A_j = m_{N_j} - A^u_{j-1} \ .$$

Hence the various sets N_j and A_j are found in the following way. In the first step one has to check if the number $\# m_1$ of maximal elements for individual one is equal to one. If that is the case we set $A_1 = m_1$ and $N_1 = \{1\}$. Then the planner repeat the procedure with the sets $A - A_1$ and $N - N_1$ instead of A and N.

On the other hand, the number $\# m_1$ of maximal elements may be greater than one. In that case suppose that $\# G < \# m_G$ for all $G \neq \emptyset$, $G \subset N$ and $i < j$ if $i \in G$, while there is some nonempty $G \subset N$ with $i \leq j$ if $i \in G$ such that $\# G = \# m_G$. Then find the largest G with this property, which is unique by Theorem 3 below, and denote it N_j. Also set $A_1 = A_2 = \ldots = A_{j-1} = \emptyset$, $N_1 = N_2 = \ldots = N_{j-1} = \emptyset$ and finally $A_j = m_{N_j}$.

Now we can repeat the procedure starting with the sets $A - A_j$ and $N - N_j$ instead of the sets A and N. After a finite number of repetitions we have a complete definition of all sets A_j and N_j (some of them empty) such that $A = \cup_j A_j$ and $N = \cup_j N_j$. In Theorem 4 below we demonstrate that the set A_j is exactly the difference $S_j - S_{j+1}$ between choice sets of individuals j and $j+1$. Hence we have a rule for calculating the various choice sets with minimal information about individual preferences.

Theorem 3. *The various sets N_j and A_j are uniquely defined.*

Proof. This follows immediately from Lemma 2.

Theorem 4. *For all $j = 1, 2, \ldots, n$:*

1. $S_{j+1} = S_j - A_j$ (for $j < n$),
2. $f(N_j) = A_j$ for every allocation f such that $f(i) \in m_i$ for $i \in N_j$.

Proof. The second point above follows directly from the definition of the sets A_j and N_j: Let f be an allocation such that $f(i) \in m_i$ for $i \in N_j$. Since $A_j = m_{N_j} - A^u_{j-1}$, $f(N_j) \subset A_j$. But $\# N_j = \# m_{N_j} - A^u_{j-1} = \# A_j$. But then $f(N_j) = A_j$.

Now to point 1. First note that $\# G \leq \# m_G$ when $G \subset N$, because by the definition of the choice sets S_i there is an allocation f such that $f(i) \in m_i$ for all $i \in N$. For the rest of the proof consider a specific $j, j < n$, and assume first that $N_j \neq I_j$.

By point 2 follows that $S_{j+1} \subset S_j - A_j$. Since the number $\# N_j$ is maximal we have $\# G < \#(m_G - A^u_{j-1})$ when $G \subset I_j - N_j$ and $G \neq \emptyset$. But then for any $a \in A$, $\# G \leq \# \cup_{i \in G}(m_i - \{a\})$ when $G \subset N$. In particular we may choose $a \in S_j - A_j$. But then there is an allocation $f^a: I_j - N_j \to A$ such that $f^a(i) \in m_i$, $f^a(i) \neq a$. This follows from Lemma 1. Moreover, by point 2, $a \notin A_k$ for $k \leq j$ if $a \in S_j - A_j$. But then there is an allocation $g: \cup_{k \leq j} N_k \to A$ such that $g(i) \in m_i, g(i) \neq a$.

By combining the two allocations f^a and g one easily sees that there is an allocation $f: I_j \rightarrow A$ such that $f(i) \in m_i$, $f(i) \neq a$ for $i \leq j$. But this entails that $S_j \subset S_{j+1}$and hence $S_{j+1} = S_j$, so point 1 is true in this case.

Finally in the case $N_k = I_k$ point 1 is trivial. Q.E.D.

5. An example

To illustrate the use of Theorem 4 when allocating indivisible goods we consider a simple example. Assume that a planner has to allocate five houses $A = \{h_1, \ldots, h_5\}$ among five individuals $N = \{1, \ldots, 5\}$. The individuals are ranked according to their index (queue number), i.e. individual 1 has to choose first, then individual 2 has to make a choice etc. The planner knows this order but he does not know the individual preferences for the houses. He can ask for the complete individual preference ordering and given this information allocate the houses Pareto optimally and in accordance with the queue-order. Theorem 1 shows how to give the individuals incentives to reveal their true preferences for the houses while Theorem 4 demonstrates how to avoid collecting information not needed for calculating the an outcome of the mechanism. Suppose that individual preferences are defined as:

i	h_1	h_2	h_3	h_4	h_5
1	1	1	1	1	2
2	3	1	1	2	3
3	3	1	1	2	2
4	2	1	1	3	2
5	1	2	3	3	4

Here individual 1 e.g. is indifferent among h_1, h_2, h_3 and h_4 which all are strictly better than h_5. Individual 2 regard h_2 and h_3 as equally good but better than the rest of the houses. h_4 comes on the second place in his ranking while h_1 and h_5 are the worst ones. In the same way the matrix gives the other individuals' ranking of the houses.

Now let us see how Theorem 4 can be used to find an optimal allocation. The planner shall first give to individual 1 – the first one in the queue – a choice set S_1 and the individual responds with his maximal elements in S_1. Given this information, the planner calculates individual 2's choice set S_2 and possibly also determines which house to be allotted to individual 1. Then the procedure continues. In our case an optimal allocation f is determined as follows:

1. Individual 1 receives $S_1 = \{h_1, \ldots, h_5\}$ and he responds with the set $m_1 = \{h_1, \ldots, h_4\}$ of maximal elements.

2. $N_1 = \emptyset$ and $A_1 = \emptyset$. Hence the planner let $S_2 = S_1 - A_1 = \{h_1, \ldots, h_5\}$, i.e. no restrictions on 2's choice set. The respond from individual 2 is $m_2 = \{h_2, h_3\}$.

3. Since $\#\{1,2\} = 2 < 4 = \#m_{\{1,2\}}$ and $\#\{2\} = 1 < 2 = \#m_2$, $N_2 = \emptyset$ and $A_2 = \emptyset$. Hence $S_3 = S_2 - \emptyset = \{h_1, \ldots, h_5\}$ and yet no further restrictions on the choice sets. The respond from individual 3 is $m_3 = \{h_1, h_3\}$.

4. But now $N_3 \neq \emptyset$ because $\#\{2,3\} = 2 = \#m_{\{2,3\}}$. Moreover, $\#\{1,2,3\} = 3 < 4 = \#m_{\{1,2,3\}}$ so $N_3 = \{2,3\}$ and hence $A_3 = \{h_2, h_3\}$. But an optimal allocation f must have $f(N_3) = A_3$ and $f(2) = h_2$, $f(3) = h_3$ will do.

5. Now the planner has to restrict individual 4's choice set. By Theorem 4, $S_4 = S_3 - A_3 = \{h_1, h_4, h_5\}$. The respond from individual 4 to S_4 is $m_4 = \{h_1, h_5\}$. But $\#\{1,4\} = 2 < 3 = \#(m_{\{1,4\}} - A_3^u)$, so $N_4 = \emptyset$ and $A_4 = \emptyset$. Hence $S_5 = S_4 - \emptyset = \{h_1, h_4, h_5\}$. The respond from individual 5 is $m_5 = \{h_1\}$.
6. Now we have $\#\{1,4,5\} = 3 = \#(m_{\{1,4,5\}} - A_4^u)$ and $N_5 = \emptyset$. Obviously $f(5) = h_1$ necessarily, while we must have $f(1) = h_4$ and $f(4) = h_5$.

The result of the recursive allocation procedure may be summarised as follows. If $B = \{h_1, h_4, h_5\}$ then

i	S_i	$f(i)$	ranking of $f(i)$
1	A	h_4	1
2	A	h_2	1
3	A	h_3	1
4	B	h_5	2
5	B	h_1	1

Hence the first three individuals in the queue have received the complete set A as their choice set. For the remaining two individuals the choice sets have been reduced by two houses. As a whole the choice sets of the various individuals are comparatively large and this demonstrates the importance of taking indifferences in individual preferences into account accurately.

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