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COVERING A PLANE CONVEX BODY BY FOUR HOMOTHETICAL COPIES WITH THE SMALLEST POSITIVE RATIO

The famous conjecture of Hadwiger [5] that any convex body (i.e. a compact convex set with non-empty interior) of Euclidean n-space E^n can be covered by 2^n smaller positive homothetical copies remains unsolved for $n > 2$. For $n = 2$ the answer is affirmative as was proved in [8] by Levi. A natural question emerges about the smallest possible ratio of those four homothetical copies. The answer is given in the third part of this paper where it is proved that any convex body of E^2 can be covered by four homothetical copies with ratio $\frac{1}{2}\sqrt{2}$. An extreme example is the disk. It cannot be covered by four homothetical copies with a smaller positive ratio than $\frac{1}{2}\sqrt{2}$. An additional discussion of the smallest possible ratio is presented in the fourth part. The last part contains some corollaries concerning the question on covering of sets of a Minkowski plane with sets of smaller diameter. The first two parts are auxiliary.

1. Quasi-dual and dual parallelograms

We call parallelograms P and Q *quasi-dual* if the sides of P are parallel to the diagonals of Q and if the sides of Q are parallel to the diagonals of P.

This definition can be also expressed using vectors. Namely, parallelograms P and Q are quasi-dual if and only if for some of the possible denotations of the vectors determined by the pairs of parallel sides of P and Q by $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{q}_1, \mathfrak{q}_2$, respectively, the following pairs of vectors are parallel:

$$(*) \qquad \mathfrak{p}_1 + \mathfrak{p}_2 \| \mathfrak{q}_1, \quad \mathfrak{p}_1 - \mathfrak{p}_2 \| \mathfrak{q}_2, \quad \mathfrak{q}_1 + \mathfrak{q}_2 \| \mathfrak{p}_2, \quad \mathfrak{q}_1 - \mathfrak{q}_2 \| \mathfrak{p}_1.$$

According to the context, in the following two properties we assume that $\mathfrak{p}_1 \nparallel \mathfrak{p}_2$ and $\mathfrak{q}_1 \nparallel \mathfrak{q}_2$.

(i) *Any three conditions of* $(*)$ *imply the fourth one.*

(ii) *If* $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{q}_1, \mathfrak{q}_2$ *are pairs of vectors fulfilling* $(*)$, *then*

$$|\mathfrak{p}_1|/|\mathfrak{q}_1 - \mathfrak{q}_2| = |\mathfrak{p}_2|/|\mathfrak{q}_1 + \mathfrak{q}_2|,$$
$$|\mathfrak{q}_1|/|\mathfrak{p}_1 + \mathfrak{p}_2| = |\mathfrak{q}_2|/|\mathfrak{p}_1 - \mathfrak{p}_2|,$$

and $pq = \frac{1}{2}$, *where* $p = |\mathfrak{p}_1|/|\mathfrak{q}_1 - \mathfrak{q}_2|$ *and* $q = |\mathfrak{q}_1|/|\mathfrak{p}_1 + \mathfrak{p}_2|$.

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Let us briefly show the properties. For instance, assume that the first three conditions of (*) hold. Hence $\mathfrak{p}_1 + \mathfrak{p}_2 = 2r\mathfrak{q}_1$, $\mathfrak{p}_1 - \mathfrak{p}_2 = 2s\mathfrak{q}_2$, $\mathfrak{p}_2 = t(\mathfrak{q}_1 + \mathfrak{q}_2)$ for some numbers r, s, t different from 0. Therefore, $\mathfrak{p}_1 = r\mathfrak{q}_1 + s\mathfrak{q}_2$, $\mathfrak{p}_2 = r\mathfrak{q}_1 - s\mathfrak{q}_2$ and $r\mathfrak{q}_1 - s\mathfrak{q}_2 = t(\mathfrak{q}_1 + \mathfrak{q}_2)$. Thus $(r-t)\mathfrak{q}_1 = (s+t)\mathfrak{q}_2$. Hence $t = r - s$. This means that $\mathfrak{p}_1 = t(\mathfrak{q}_1 - \mathfrak{q}_2)$ and $\mathfrak{p}_2 = t(\mathfrak{q}_1 + \mathfrak{q}_2)$. So $\mathfrak{p}_1 || \mathfrak{q}_1 - \mathfrak{q}_2$ (which proves (i)), and $|\mathfrak{p}_1|/|\mathfrak{q}_1 - \mathfrak{q}_2| = |t| = |\mathfrak{p}_2|/|\mathfrak{q}_1 + \mathfrak{q}_2|$. Moreover, we get $\mathfrak{p}_1 + \mathfrak{p}_2 = 2t\mathfrak{q}_1$ and $\mathfrak{p}_1 - \mathfrak{p}_2 = -2t\mathfrak{q}_2$. Hence $|\mathfrak{q}_1|/|\mathfrak{p}_1 + \mathfrak{p}_2| = 1/|2t| = |\mathfrak{q}_2|/|\mathfrak{p}_1 - \mathfrak{p}_2|$. Consequently, (ii) holds.

If P and Q are quasi-dual parallelograms and if $p = q = \frac{1}{2}\sqrt{2}$ (see (ii)), then P and Q are called *dual*.

2. Some lemmas on parallelograms inscribed in a convex body

Let xy denote the closed segment joining points x and y, and let $|xy|$ mean the distance between x and y. By L_{xy} we denote the straight line through different points x and y. The boundary of a convex body C is denoted by bd C. Let (λ_1, λ_2) denote open interval, where $\lambda_1 < \lambda_2$ are real numbers. Half-open intervals are denoted by $[\lambda_1, \lambda_2)$ and $(\lambda_1, \lambda_2]$, and a closed interval by $[\lambda_1, \lambda_2]$.

Let an orientation of E^2 be fixed. By the angle between a line of a direction l and a line of a different direction m we understand the oriented angle $\alpha \in (0, \pi)$ between these lines and we denote the direction m by the symbol l_α.

LEMMA 1 (Zindler [9]). *Let $C \subset E^2$ be a convex body and let two different directions be given. Then one can inscribe a parallelogram in C with diagonals parallel to the directions.*

It is easy to verify (comp. also the considerations on pp. 48–50 of [9]) that there holds true

LEMMA 2. *If $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are parallelograms inscribed in a convex body $C \subset E^2$ and $a_1c_1 || a_2c_2$, $b_1d_1 || b_2d_2$, then $a_2b_2c_2d_2$ is a translate of $a_1b_1c_1d_1$ parallel to a side of $a_1b_1c_1d_1$.*

As a result of Lemmas 1 and 2, despite the possible non-uniqueness of inscribing a parallelogram with given directions of diagonals, the formulation of the following lemma is correct.

LEMMA 3. *Let $C \subset E^2$ be a convex body and l be a direction. For any $\alpha \in (0, \pi)$ we denote by $P(\alpha)$ a parallelogram inscribed in C with diagonals of directions l, l_α. The directions of sides of $P(\alpha)$ are continuous functions of α.*

Proof. If $P(\alpha)$ is not unique, we easily obtain from Lemma 2 that in a neighbourhood of α the directions of sides of our parallelogram change continuously (one of the directions is even constant in a neighbourhood of α).

Consider the case when $P(\alpha)$ is unique. In accordance with our orientation, let a, b, c, d be successive vertices of $P(\alpha)$ such that the diagonal bd is of the direction l and the diagonal ac is of the direction l_α. The centre of $P(\alpha)$ is denoted by s. Obviously, C lies in the union D of the strip between the straight lines L_{ab}, L_{cd} and the strip between the straight lines L_{ad}, L_{bc}.

Let $\varepsilon > 0$. Denote by a_ε the point of L_{ad} being at the distance ε from a on the opposite side of a to d. Let c_ε be the point of L_{bc} lying in the distance ε from c on the opposite side of c to b. Moreover, let w_ε denote the common point of the segments $a_\varepsilon c$, ab, and let z_ε denote the common point of the segments ac_ε, dc. By α_ε we mean $\sphericalangle aca_\varepsilon = \sphericalangle cac_\varepsilon$. Consider an angle α' such that $0 \leqslant \alpha' - \alpha \leqslant \alpha_\varepsilon$. Let $a'b'c'd'$ be a parallelogram inscribed in C such that the diagonal $b'd'$ is of the direction l (let $\mathbf{b'd'}$ be of the same sense as $\mathbf{bd}$) and that the diagonal $a'c'$ is of the direction $l_{\alpha'}$. Denote the centre of $a'b'c'd'$ by s'.

We show that the straight line $L_{a'c'}$ cuts both segments aa_ε, cc_ε. Suppose the contrary. For instance, let $L_{a'c'}$ cut L_{ad} at a point a^* such that $a_\varepsilon \in aa^*$ and $a^* \neq a_\varepsilon$ (in other cases, further reasoning is analogous). Thus we get from $\alpha \leqslant \alpha'$ that $L_{a'c'}$ cuts L_{bc} at a point c^* such that $c \in c_\varepsilon c^*$ and $c^* \neq c$. So the inclusions $P(\alpha) \subset C \subset D$ imply that s' lies in this angle with the vertex s and sides parallel to L_{ab}, L_{ad}, which contains b. Moreover, $s' \neq s$. From $b'd' \| bd$ and $P(\alpha) \subset C \subset D$ we see that s' lies on a side of this angle. Consider the case when s' belongs to the side parallel to L_{ab} (in the second case our further consideration is similar). From $P(\alpha) \subset C \subset D$ we get that b' and c' have positions such that $b \in ab'$, $b' \neq b$, $c \in dc'$, $c' \neq c$. Consequently, the segments ab' and dc' lie in bd C. Hence the translate of $P(\alpha)$ on the vector $\mathbf{cc'}$ (observe that $|cc'| \leqslant |bb'|$) is a parallelogram inscribed in C; a contradiction with the uniqueness of $P(\alpha)$.

Since $L_{a'c'}$ cuts the segments aa_ε and cc_ε, the point a' belongs to the triangle $aa_\varepsilon w_\varepsilon$ and c' belongs to the triangle $cc_\varepsilon z_\varepsilon$. This means that the motions of a and c are right-continuous.

Similarly, we show that the motions of a and c are left-continuous. Consequently, they are continuous. Hence, the centre s moves continuously. Containing s and being of the constant direction l, the straight line L_{bd} translates continuously. Hence, the convexity of C implies that b and d also move continuously. Since all the vertices of $P(\alpha)$ move continuously, the directions of the sides of $P(\alpha)$ change continuously.

LEMMA 4. *Let $C \subset E^2$ be a convex body. It is possible to inscribe in C a pair of quasi-dual parallelograms. Furthermore, a diagonal of one of the parallelograms (i.e. a pair of sides of the other) can be of any given direction.*

Proof. Let l be a given direction. Support C by two straight lines of the direction l. As the intersections we obtain two closed segments r_1r_2, v_1v_2. Obviously, it may occur that $r_1 = r_2$ or $v_1 = v_2$. Let, for instance, $\mathbf{r}_1\mathbf{r}_2 = \mu\mathbf{v}_1\mathbf{v}_2$ for some non-negative $\mu \leqslant 1$. Inscribe in C all segments of maximal length parallel to l. Their endpoints form two segments s_1u_1, s_2u_2 such that $\mathbf{s}_1\mathbf{u}_1 = \mathbf{s}_2\mathbf{u}_2$ (it may occur that $s_1 = u_1$ and $s_2 = u_2$). Moreover, assume that the order of the points r_1, s_1, u_1, v_1 on bd C coincides with the orientation of E^2. Let r, t_1, v, t_2 be the midpoints of segments r_1r_2, s_1u_1, v_1v_2, s_2u_2, respectively. The points determine disjointed arcs (without endpoints) $\widehat{rt_1}$, $\widehat{t_1v}$, $\widehat{vt_2}$, $\widehat{t_2r}$ of the boundary of C. Let δ_0 mean the angle between l and L_{rv}.

For any $x_1 \in \widehat{t_1v}$ we inscribe in C a parallelogram $x_1x_2y_2y_1$ such that $x_1x_2 \| l \| y_1y_2$ and that the conditions given below are fulfilled. Namely, if $x_1 \in t_1u_1$, then let y_1 be symmetric to x_1 w.r.t. t_1 (then the points x_2, y_2 are determined univocally). If $x_1 \in \widehat{u_1v_1}$, the parallelogram $x_1x_2y_2y_1$ is unique. If $x_1 \in v_1v$, we take x_2 symmetric to x_1 w.r.t. v. If we have $|x_1v| \geqslant |r_1r|$, then y_1, y_2 are unique. If $|x_1v| < |r_1r|$, we take y_1 on the segment r_1r in order to have $|x_1v| = |y_1r|$. The above construction ensures the uniqueness of the parallelogram $x_1x_2y_2y_1$ for any $x_1 \in \widehat{t_1v}$. From the construction of the parallelogram $x_1x_2y_2y_1$ we see that if x_1 moves continuously on the arc $\widehat{t_1v}$, the points x_2, y_2, y_1 also move continuously. The angle between l and the side x_1y_1 is denoted by γ.

We see from Lemma 1 that for any $\alpha \in (0, \pi)$ it is possible to inscribe in C a parallelogram $P(\alpha)$ with diagonals of the directions l, l_α. When α changes from 0 to π, then by Lemma 3 the angles between l and the sides of $P(\alpha)$ change continuously. One of them changes from 0 to some η_1 (in the interval I_1 of the form $(0, \eta_1)$ or $(0, \eta_1])$ and the second from some η_2 to π (this interval I_2 has the form (η_2, π) or the form $[\eta_2, \pi)$). The reader can show that the convexity of C implies $\eta_1 \leqslant \eta_2$.

CASE 1. *At least one of the intervals I_1, I_2 does not contain δ_0.* Let, for instance, $\delta_0 \notin I_1$. Denote by β the angle between l and a pair of sides of $P(\alpha)$ which change in the interval I_1. Simultaneously, for any $\beta \in I_1$, we denote the parallelogram $x_1x_2y_2y_1$ by $Q(\beta)$ provided the angle between l and the diagonal x_1y_2 is equal to β. Obviously, a unique $Q(\beta)$ exists for any $\beta \in I_1$. Since a continuous motion of x_1 on $\widehat{t_1v}$ implies continuous motions of y_2, y_1, we see that γ continuously depends on β. On the other hand, applying Lemma 3 to $P(\alpha)$ we observe that β continuously depends on α.

Consequently, γ is a continuous function of α on the interval $(0, \pi)$. Let γ_1 denote the maximum angle between l and the straight lines L_{rt_1}, L_{t_2v}. Let γ_2 be the minimum angle between l and L_{rt_2}, L_{t_1v}. Observe that $0 < \gamma_1 \leqslant \gamma_2 < \pi$ and that γ is always in the interval $[\gamma_1, \gamma_2]$. Since α proceeds from 0 to π and since γ continuously depends on α, there exists a value α_0 of α and a corresponding value γ_0 of γ such that $\alpha_0 = \gamma_0$. Let β_0 be the corresponding value of β. By virtue of (i) we see that $P(\alpha_0)$ and $Q(\beta_0)$ are the parallelograms we are looking for.

CASE 2. *Both the intervals I_1, I_2 contain δ_0.* This means that $I_1 = (0, \delta_0]$ and $I_2 = [\delta_0, \pi)$. The reader is left to observe that $s_1 \neq u_1$, $s_2 \neq u_2$ and $s_1u_1 || rv || s_2u_2$. Support C by the straight lines $L_{s_1u_1}$, $L_{s_2u_2}$ and denote the segments that are intersections of the lines with C by $s_1'u_1'$ and $s_2'u_2'$, respectively, such that $\mathbf{s}_1'\mathbf{u}_1'$, $\mathbf{s}_2'\mathbf{u}_2'$ are of the same sense as $\mathbf{s}_1\mathbf{u}_1$.

Subcase 2.1. Let $s_1' \neq r$ and $s_2' \neq r$. In this subcase we repeat the considerations of Case 1 but only for α in the interval $[\psi_1, \psi_2]$, where ψ_1, ψ_2 denote the angles between l and the segments ru_1', ru_2', respectively. This is now possible because β is always different from δ_0, and γ is always between ψ_1 and ψ_2.

Subcase 2.2. Let $s_1' = r$ or $s_2' = v$.

If $s_i' = r$, then $u_i' = v$ for $i \in \{1, 2\}$. Observe that for any direction different from l and l_{δ_0} it is possible to inscribe in C a parallelogram $Z_m = z_1z_2z_3z_4$ such that $z_1z_2 \subset rv$ and $z_2z_3 || l || z_4z_1$, and that one of the diagonals z_1z_3, z_2z_4 is of the direction m. On the other hand, Lemma 1 enables us to inscribe in C a parallelogram W whose diagonals are of the directions l and l_{δ_0}. Let m_0 be the direction of some pair of sides of W. Obviously, m_0 is different from l and l_{δ_0}. Property (i) implies that W and Z_{m_0} are the parallelograms we are looking for.

The proof of Lemma 4 is complete.

REMARK 1. In connection with Lemma 4, a conjecture appears that a pair of dual parallelograms can be inscribed in every plane convex body C. The proof would need some continuity arguments. The conjecture is true when C is centrally symmetric. This follows from a theorem of Grünbaum [4] that an affine regular octagon can be inscribed in every centrally symmetric plane convex body.

3. THE MAIN RESULT

THEOREM. *Every convex body $C \subset E^2$ can be covered by four positive homothetical copies whose ratio is not greater than $\frac{1}{2}\sqrt{2}$.*

Proof. By Lemma 4 we find successive points a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4

of the boundary of C such that the quadrilaterals $P = a_1a_2a_3a_4$ and $Q = b_1b_2b_3b_4$ are quasi-dual parallelograms (see Figure 1). By (ii) at least one of the numbers p and q is not greater than $\frac{1}{2}\sqrt{2}$. Let, for instance, $p \leqslant \frac{1}{2}\sqrt{2}$.

Denote by a the centre of P. Let c_1, c_2, c_3, c_4 be the common points of the pairs of lines containing the segments a_1b_4 and a_2b_2, a_2b_1 and a_3b_3, a_3b_2 and a_4b_4, a_4b_3 and a_1b_1, respectively (see Figure 1). Since a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 are successive points on the boundary of C, the set C is contained in the star-shaped set being the union of the quadrilaterals $aa_1c_1a_2, aa_2c_2a_3, aa_3c_3a_4, aa_4c_4a_1$. The homothety with centre c_i and ratio p is denoted by H_i, $i = 1, 2, 3, 4$. From $H_1(b_4) = a_1$, $H_1(b_2) = a_2$, $a_1a \| b_4b_3$ and $a_2a \| b_2b_3$ we conclude that $H_1(C)$ covers the triangle aa_1a_2. Let $x \in C$ be a point of the triangle $a_1a_2c_1$. Since $H_1(C)$ is convex and contains the points a_1, a_2, $H_1(x)$, it contains the whole triangle $a_1a_2H_1(x)$. Particularly,

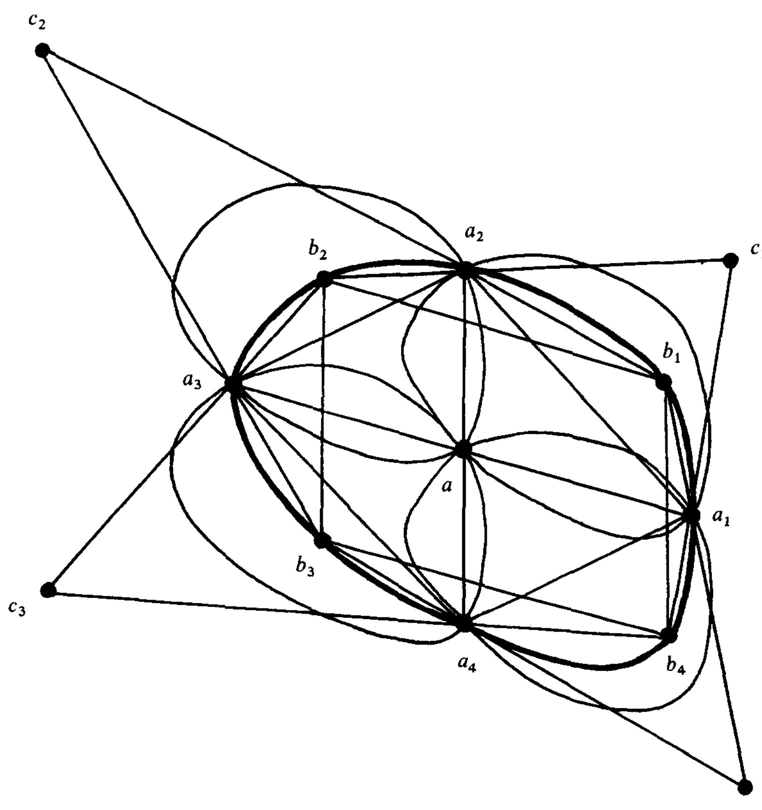

Fig. 1.

$x \in H_1(C)$. Thus $H_1(C)$ contains this part of C which lies in the quadrilateral $aa_1c_1a_2$. Similarly, the parts of C which lie in the quadrilaterals $aa_2c_2a_3$, $aa_3c_3a_4$, $aa_4c_4a_1$ are subsets of $H_2(C)$, $H_3(C)$, $H_4(C)$, respectively.

The proof is complete.

If P and Q are quasi-dual parallelograms inscribed in C and $p < 1$, then the covering with four homothetical copies constructed in the above proof is called *regular*.

4. Some additional properties of 4-coverings

A covering of a convex body C by m smaller positive homothetical copies is called an *m-covering* of C for short. For a further discussion of plane 4-coverings it is convenient to have a number characteristic $h_4(C)$ being a special case of the following definition.

For a convex body $C \subset E^n$ and a natural number m we define the *m-covering number* $h_m(C)$ as the smallest possible positive ratio of m homothetical copies of C whose union covers C. Actually, for the accuracy of this definition we should show that $h_m(C)$ exists. Obviously, it is sufficient to show that if the union of m homothetical copies $C_1^i, \ldots, C_m^i$ of C with a ratio $\lambda_i > 0$ covers C, $i = 1, 2, \ldots$, where $\lambda_1 > \lambda_2 > \cdots$ and $\lim \lambda_i = \lambda$, then the union of some m copies with a ratio λ also covers C. Obviously, we may assume that $C_k^i \cap C \neq \emptyset$ for $k = 1, \ldots, m$ and $i = 1, 2, \ldots$. Let s_k^i be the centre of the homothety mapping C onto C_k^i, where $k = 1, \ldots, m$ and $i = 1, 2, \ldots$. Since C is bounded, the set of all s_k^i is also bounded. By k successive selections we can find a sub-sequence i_j such that $s_k^{i_j}$ converges to some s_k for $k = 1, \ldots, m$, simultaneously. Hence, for any $\varepsilon > 0$ the union of the copies of C with the homothety centres $s_1, \ldots, s_m$ and with the ratio $\lambda + \varepsilon$ covers C. Since all the copies are convex bodies, in a standard way (considering any $x \in C$), we also obtain a covering of C for $\varepsilon = 0$. This confirms the existence of the number $h_m(C)$.

Our theorem may be expressed by the inequality

$$h_4(C) \leqslant \tfrac{1}{2}\sqrt{2}$$

for every convex body $C \subset E^2$. Natural questions emerge about the lower bound of $h_4(C)$, about the possible values of $h_4(C)$, about a description of those convex bodies $C \subset E^2$ for which $h_4(C) = \frac{1}{2}\sqrt{2}$, and about convex bodies C for which a regular 4-covering of the ratio $h_4(C)$ exists. The answers, partial answers and comments are presented in the following proposition and in Remarks 2 and 3.

PROPOSITION. *The following properties of* 4*-coverings of plane convex bodies hold:*

1. *For every convex body C we have $h_4(C) \geqslant \frac{1}{2}$. The equality holds if and only if C is a parallelogram.*
2. *For any ρ such that $\frac{1}{2} \leqslant \rho \leqslant \sqrt{2}/2$ there exists a (centrally symmetric) convex body C_ρ with $h_4(C_\rho) = \rho$.*
3. *For every convex body C whose boundary $r = r(\varphi)$ fulfils the polar equality $r(\varphi + \pi/4) = r(\varphi)$ we have $h_4(C) = \frac{1}{2}\sqrt{2}$.*
4. *For every centrally symmetric convex body C there exists a regular* 4*-covering of the ratio $h_4(C)$.*

Proof. 1. The inequality $h_4(C) \geqslant \frac{1}{2}$ results immediately from a comparison of the areas of C and its copies. Obviously, if C is a parallelogram, the equality holds.

Suppose that $h_4(C) = \frac{1}{2}$. Considering the areas we observe that the four copies are contained in C and that their interiors are disjointed. Thus every extreme point of C is a centre of some of our four homotheties. So C is a triangle or a quadrilateral. Since the copies have disjointed interiors, we easily obtain that C is a parallelogram.

2. Let $\frac{1}{2} \leqslant \rho \leqslant \frac{1}{2}\sqrt{2}$. Denote by C_ρ the convex hull of the following eight points $\langle 1,0\rangle$, $\langle \rho,\rho\rangle$, $\langle 0,1\rangle$, $\langle -\rho,\rho\rangle$, $\langle -1,0\rangle$, $\langle -\rho,-\rho\rangle$, $\langle 0,-1\rangle$, $\langle \rho,-\rho\rangle$ in a perpendicular coordinate system ($\langle x,y\rangle$ means the point of the coordinates x and y). Let $r = r_\rho(\varphi)$ be the polar equation of the boundary of C_ρ. An easy but tedious calculation shows that for every φ we have

$$|r_\rho(\varphi) r_\rho(\varphi + \pi/2)| \leqslant \rho |r_\rho(\varphi - \pi/4) r_\rho(\varphi + 3\pi/4)|$$

and that the equality holds for $\varphi = 0$. Consequently we get $h_4(C_\rho) = \rho$.

3. Let the boundary of C satisfy the assumed polar equality. Consider a 4-covering of C. Take into account an easily illustrated property that if H is a positive homothety and B is a convex body, then the set $H(B) \cap \operatorname{bd} B$ is connected. This enables us to observe that for some φ_0 the points $u = r(\varphi_0)$ and $v = r(\varphi_0 + \pi/2)$ belong to one of the four copies. Denote this copy by C^*. Let $u' = r(\varphi_0 - \pi/4)$ and $v' = r(\varphi_0 + 3\pi/4)$. The segment $u'v'$ is parallel to uv and contains the centre of C. Obviously, $|uv| = (\frac{1}{2}\sqrt{2}|u'v'|$. Since C is centrally symmetric, any segment inscribed in C and parallel to uv is not longer than the segment $u'v'$. Thus the ratio of the copy C^* is not smaller than $\frac{1}{2}\sqrt{2}$. Therefore, $h_4(C) \geqslant \frac{1}{2}\sqrt{2}$. By our theorem we obtain $h_4(\mathrm{C}) = \frac{1}{2}\sqrt{2}$.

4. Denote the centre of C by s. There exist homothetical copies C_1, C_2,

C_3, C_4 of C with a ratio $h_4(C)$ whose union covers C. The arcs $C_k \cap \operatorname{bd} C$, where $k = 1, 2, 3, 4$, are connected. Hence, symmetric points a, $c \in \operatorname{bd} C$ and a point $b \in \operatorname{bd} C$ exist such that one of the four copies contains a, b, and another contains b, c. Let d denote the point symmetric to b. The intersections of bd C with the straight lines through s parallel to ab and bc are denoted by z, x and w, y, respectively. Let $\mathbf{bc} = \lambda_1 \mathbf{wy}$ and $\mathbf{ab} = \lambda_2\, \mathbf{zx}$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. Let, for instance, $\lambda_1 \leqslant \lambda_2$. Since C is centrally symmetric, any segment inscribed in C and parallel to ab is not longer than the segment zx. Hence $\lambda_2 \leqslant h_4(C)$. On the other hand, arranging a construction similar to the one presented in the proof of our theorem for the points a, w, b, x, c, y, d, z we get a covering of C by four copies. Two of them have the coefficient λ_1 and the other two have λ_2. This gives $h_4(C) \leqslant \lambda_2$. Consequently, $\lambda_2 = h_4(C)$.

We show that $\lambda_1 = h_4(C)$. Suppose, on the contrary, that $\lambda_1 < h_4(C)$. We fix the positions of x and z and continuously change the positions of a, b, c, d on bd C such that $\mathbf{ab} = \mathbf{dc} \| \mathbf{zx}$ all the time and that λ_2 decreases. Observe that the positions of w and y change continuously. Thus λ_1 changes continuously. So for some positions of a, b, c, d we obtain $\lambda_1 = \lambda_2 < h_4(C)$. Repeating the construction given in the proof of our theorem we obtain a contradiction with the definition of $h_4(C)$.

We have shown that $\lambda_1 = \lambda_2 = h_4(C)$. Consequently, the parallelograms $P = abcd$ and $Q = wxyz$ are quasi-dual and $p = h_4(C)$, where p is defined in (ii). The proof of our theorem has shown the existence of a regular 4-covering of C with a ratio $h_4(C)$.

REMARK 2. The author conjectures that every plane convex body C with $h_4(C) = \frac{1}{2}\sqrt{2}$ is an affine image of a convex body whose boundary $r = r(\varphi)$ fulfils the polar equality $r(\varphi + \pi/4) = r(\varphi)$.

REMARK 3. If a convex body C is not centrally symmetric, a regular 4-covering with a ratio $h_4(C)$ may not exist. For instance, the coefficient of the copies in any regular 4-covering of a triangle T is never smaller than $\frac{2}{3}$. This results from the simple calculation that for every pair P, Q of quasi-dual parallelograms inscribed in T, the coefficients p and q (defined in (ii)) are not smaller than $\frac{2}{3}$. On the other hand, T can be covered by four copies of a smaller ratio $\frac{4}{7}$ provided the homothety centres are in the vertices and in the centre of T (from [1] it is even found that $h_4(T) = \frac{4}{7}$).

5. An application to a covering problem on Minkowski plane

Lenz [7] showed that every set of E^2 of diameter 1 can be covered by four balls of diameter $\frac{1}{2}\sqrt{2}$. One may conjecture that it is true also in every 2-

dimensional Minkowski space R^2. From our theorem we are able to obtain only the following three weaker properties:

COROLLARY 1. *Every set $A \subset R^2$ of diameter* 1 *can be covered by four sets of diameter* $\frac{1}{2}\sqrt{2}$.

COROLLARY 2. *Every set $A \subset R^2$ of diameter* 1 *can be covered by four balls of diameter* $\frac{2}{3}\sqrt{2}$.

COROLLARY 3. *Every centrally symmetric set $A \subset R^2$ of diameter* 1 *can be covered by four balls of diameter* $\frac{1}{2}\sqrt{2}$.

Corollary 1 results by an application of our theorem to the B-convex hull [6] of A (with the possible exception of the trivial case when A is contained in a segment). Applying a theorem of Bohnenblust [2] on the covering of every set of R^2 of diameter 1 with a ball of R^2 of diameter $\frac{4}{3}$ we obtain Corollary 2. Corollary 3 follows from our theorem and from the easily checked property that every centrally symmetric set of diameter 1 in R^2 is a subset of a ball of R^2 of diameter 1.

REMARK 4. In connection with Corollaries 1 and 2 we can formulate two stronger versions of the conjecture [3] that every set of diameter 1 in an n-dimensional Minkowski space R^n can be covered by 2^n sets of diameter smaller than 1. The first version is that every set of diameter 1 in R^n can be covered by 2^n sets of diameter $\frac{1}{2}\sqrt{2}$. The second is that every set of diameter 1 in R^n can be covered by 2^n balls of diameter smaller than 1.

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