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Social welfare functions and fairness

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Abstract. A definition of a social state is proposed that incorporates the notion of procedural fairness into Harsanyi's (1955) analytical framework. We show that, within the new framework, a Harsanyi-type social welfare function is immune to Diamond's (1967) criticism. Moreover, the resulting social welfare function embodies the notion of procedural fairness held by individual members of the society.

1. Introduction

Every society is repeatedly confronted with the need to choose among alternative courses of action (e.g., public policies, social institutions, constitutions and laws), about which its members have conflicting preferences. Social welfare functions are numerical representations of social preference relations over these courses of action, presumably embodying the ethical values of the society.

In a seminal work, Harsanyi (1955) identified courses of action with uncertain prospects, which he defines as follows:

> Since we admit the possibility of external economies and diseconomies of consumption, both social and individual prospects will, in general, specify the amounts of different commodities consumed and the stock of different goods held by all individuals at different future dates (up to the time horizon adopted) together with their respective probabilities. (Harsanyi 1955, p. 312)

Harsanyi postulated the existence of individual and social preference relations over the set of uncertain prospects and showed conditions under which the social welfare function takes the form of a weighted sum of individual utilities. Specifically, if both the individual and the social preference relations satisfy the axioms of

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expected utility theory and are represented by von Neumann–Morgenstern utility functions, and if everyone being indifferent between any two prospects implies that the prospects are indifferent according to the social preference relation, then the social utility function may be represented as a weighted sum of the individual utility functions. In this framework, the *individual preferences represent individual choice behavior* while the *social preferences represent moral judgment.*[1] The latter is defined as the impartial attitudes of an individual toward social policies or the preferences that he would have "if he gave equal weight to each individual's interests in choosing between alternative social situations" (Harsanyi 1977).

Harsanyi's social welfare function was criticized by Diamond (1967). The essence of Diamond's argument is that when the society is indifferent between two alternative allocations (sure prospects) about which individual members have conflicting preferences, it is strictly preferable that the choice between the two allocations be decided by a fair lottery. To grasp this point, consider a society consisting of two individuals, A and B, having equal claims to one unit of an indivisible good. The two feasible allocations are: (a) assign the unit to A and nothing to B and (b) assign the unit to B and nothing to A. Suppose that, since the two individuals have equal claims to the good the society is indifferent between these two allocations and each individual would prefer to receive the good. It seems intuitively reasonable that in this case it is strictly socially preferable that the allocation be selected by a procedure that gives each individual an equal chance of receiving the good (e.g., the individual that receives the good is selected by the flip of a fair coin). Moreover, if one takes a pragmatic view of fairness, then the fairness of the procedure may have significant consequences for the outcome. Thus, if the indivisible object in the above example is military service, the fairness of the draft procedure may affect the morale of the recruits and the performance of the military.[2] Yet, because of the linearity (in the probabilities) of the expected utility functional, this solution is inconsistent with Harsanyi's concept of a social welfare function.

Ideas on how to resolve the difficulty posed by Diamond's critique have been proposed by Deschamps and Gevers (1979), Sen (1970), Broome (1991, 1984), Hammond (1983), and, more recently, Epstein and Segal (1992). Since the present paper advances an alternative idea on how to resolve the same difficulty, expository considerations require that further discussion of the relevant contributions be postponed so that the difference between the present approach and previous treatments may be brought into light.

Briefly stated, the view advanced in the present paper is that Diamond's critique reveals a problem in Harsanyi's analytical framework rather than, as some writers have suggested, with the nature of the social preference relation. The problem, in my view, is in Harsanyi's (1955) definition of the choice space as the set of uncertain prospects. The fact that the procedure designed to attain fairness in the choice of allocation, (e.g., the flip of a fair coin, in the example of Diamond) has the form of an uncertain prospect created the impression that the allocation procedure is an uncertain prospect and therefore an element of the choice set. This, in turn, led to confusion between social (and individual) attitudes toward uncertainty and

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[1] For further elaboration see Harsanyi (1992). For an excellent exposition and discussion of Harsanyi's contribution see Weymark (1991).

[2] For further discussion of the pragmatic approach and its implications, see Karni and Schmeidler (1994).

fairness. In other words, the procedure by which the society strives to attain fairness is mistakenly modeled using the criteria according to which the society evaluates the relative merits of uncertain prospects.

To overcome this difficulty and to incorporate the notion of procedural fairness into Harsanyi's (1955) framework, I introduce the notion of social states, which has the interpretation of allocation procedures. I show that in the new framework, Harsanyi's theory yields a social welfare function that is a weighted sum of individual utility functions. The latter, however, are not necessarily linear in the probabilities of the allocation procedures and may, therefore, be immune to Diamond's criticism.

Moreover, unlike the theories of Harsanyi (1955, 1977) and of Epstein and Segal (1992), in which the concept of fairness (or morality) is captured entirely by a social welfare function, the approach advanced here is based on the idea that self-interest seeking individuals are nonetheless moral beings and that moral value judgments are manifested in their choice behavior. Thus, an individual's personal preferences, described by Harsanyi (1992) as "his preferences governing his everyday behavior," may represent both self-interest and moral values. According to this viewpoint, the social preference relation represents value judgments concerning the procedures for aggregating individual's preferences. Social welfare functions thus reflect both the moral judgements held by individual members of the society that govern their choice behavior and value judgments concerning the aggregation of individual preferences.

2. Social states

Before considering a formal exposition of the main idea it is important to understand the point of contention. To begin with, observe that the *essential aspect of the use of a randomizing device to determine the allocation in Diamond's example is social and individual concern about fairness.* The fact that the allocation procedure looks like Harsanyi's uncertain prospect is coincidental, and there is no reason to suppose that the axioms that govern rational choice in the face of uncertainty also govern choice among allocation procedures that embody different notions of fairness. To distinguish uncertain prospects from allocation procedures we introduce the notion of social states. Similarly to Savage's (1954) definition of a state of the world, we define a state of society, or a social state, as follows:

> A *social state* is a complete description of the situation of each individual in a society, leaving no relevant aspect unspecified.

Since the evaluation of the alternative allocations is not independent of the procedure by which these allocations are attained, allocations, in themselves, do not constitute comprehensive descriptions of society and are not social states. A coin flip to determine the allocations is a social state since, in addition to the allocation, it specifies the procedure by which the allocation was attained and thereby provides a description of the situation of each individual in society in every relevant aspect. In general, since procedural fairness is a relevant aspect of the situations of individuals in society, our approach requires that it is incorporated into the definition of social states. Indeed, for our purpose, social states are allocation procedures.

To incorporate the notion of social states into Harsanyi's (1955) framework we define lotteries over allocation procedures. These are randomizing devices that assign social states known probabilities. I shall refer to these new constructs by the name *social-state lotteries*. Applying Harsanyi's theory to social-state lotteries I conclude that the social preference relation is represented by a social welfare function that takes the form of a weighted sum of individual utilities defined over social states. Individual preference relations over the set of social-state lotteries are linear in the probabilities of these lotteries and, as in expected utility theory, provide an assessment of the relative merits of alternative risks. However, since the individual utility functions represent choice among procedures that embody different degrees of fairness, they are not necessarily linear in the probabilities of the allocation procedures. For such choices, the axioms of expected utility theory do not seem compelling.

3. Social welfare functions and individual utilities

Consider a society consisting of a finite number of individuals, $I = \{1, \ldots, n\}$. An *allocation* is a complete specification of the position of each individual in the society. Let X be a set of allocations. It is convenient to think of X as a subset of $\Re^n$ where the ith coordinate of $x \in X$ represents the total wealth of individual i. Let S denote the set of all social states. In the present context a social state is a random allocation procedure. Formally, a social state, s, is a probability measure on a σ-algebra on X.[3] Thus, for all $s \in S$, and a measurable subset Q of X, $s(Q)$ is the probability that an allocation in Q is realized by the allocation procedure s. We denote by δ_x the allocation procedure that assigns the unit probability mass to the allocation x.

Next we introduce a space of social-state lotteries. Assume that S is endowed with the topology of weak convergence and let L be the set of all Borel probability measures on S. We also assume that L is endowed with the topology of weak convergence. Note that with the usual mixture operation both S and L are mixture spaces.[4]

Following Harsanyi (1955), each individual i in I has a preference relation $\succeq_i$ on L, and there exists a social preference relation, $\succeq$, on L. In addition, (a) all these preference relations are weak orders and continuous in the sense that the upper and lower contour sets are closed in the topology of weak convergence, (e.g., in the case of the social preference relation, for each $\ell \in L$, the sets $\{\ell' \in L \mid \ell' \succeq \ell\}$ and $\{\ell' \in L \mid \ell \succeq \ell'\}$ are closed in the weak convergence topology) and (b) all individual preference relations and the societal preference relations satisfy the independence axiom of expected utility theory, and together they satisfy the (strong) Pareto principle, i.e., for all ℓ, $\ell' \in L$ $\ell \succeq_i \ell'$ for all $i \in I$, implies $\ell \succeq \ell'$ and if, in addition, $\ell \succ_i \ell'$ for some $i \in I$, then $\ell \succ \ell'$. Then the social preference relation may be represented as a linear combination of individual expected utilities. Formally, under the conditions specified above, there are functions $U_i: L \to \Re$ and $u_i: S \to \Re$ such that $U_i(\ell) = \int_S u_i(s)\, d\ell(s)$ represents the preference relation $\succeq_i$ for all $i \in I$.[5]

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[3] In general, the set of allocation procedures may include procedures other than lotteries (e.g., individuals may decide to compete for rewards or fight it out).
[4] For a definition of a mixture space, see Herstein and Milnor (1953).
[5] U_i is said to represent the preference relation $\succeq_i$ on L if $\ell \succeq_i \ell' \Leftrightarrow U_i(\ell) \geq U_i(\ell')$ for all ℓ, $\ell' \in L$. For a proof of existence of an expected utility representation under our assumption, see Grandmont (1972).

Moreover, there exist $\omega \in \Re^n_{++}$ such that for all ℓ and ℓ' in L,

$$\ell \succeq \ell' \Leftrightarrow \sum_{i \in I} \omega_i U_i(\ell) \geq \sum_{i \in I} \omega_i U_i(\ell').$$

In particular, the social evaluation of a social state, s, is given by:

$$\sum_{i \in I} \omega_i U_i(\delta_s) = \sum_{i \in I} \omega_i u_i(\mathrm{s}).$$

4. The veil of ignorance and the veil of amnesia

The definition of social states as allocation procedures rather than allocations is intended to formalize the idea that the evaluation of allocations is not divorced from the procedure by which they are attained. This requires that we distinguish between allocation procedures, which are lotteries on the set of allocations, and social-state lotteries, which are lotteries on the set of allocation procedures. The fundamental distinction between the two types of lotteries is that the preferences over the former exhibit intrinsic attitudes toward fairness, whereas the preference relation over the latter exhibit intrinsic attitudes toward risk. Consequently, when facing a social-state lottery involving a nondegenerate probability measure on the allocation procedures, it is reasonable to suppose that the reduction of compound lottery axiom does not apply.[6]

To grasp the point, consider again a society consisting of two individuals, A and B, and the two allocations $x_1 = (1, 0)$ and $x_2 = (0, 1)$, where, for $i = 1, 2$, the first and second coordinates of x_i represent the quantity of an indivisible good received by A and B, respectively. Consider next the allocation procedure δ_{x_i} that assigns the unit probability mass to the allocation x_i, $i = 1, 2$, and the allocation procedure $\mathrm{s} = [x_1, \frac{1}{2}; x_2, \frac{1}{2}]$ that assigns equal probabilities to x_1 and x_2. Turning next to the space of social-state lotteries, L, we must explain the distinction between the lotteries $\ell = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$ and $\ell' = \delta_s$. To begin with, note that elements of the set of social-state lotteries, L, may be regarded as two-stage compound lotteries in which a procedure is selected in the first stage and the allocation is selected in the second. However, since the axiom of reduction of compound lotteries does not apply, while ultimately ℓ and ℓ' assign the same probabilities to the alternative allocations, they are not necessarily equivalent in so far as the preference relations on L are concerned. More specifically, we claim that consideration of fairness may imply that ℓ' is strictly preferred to ℓ.

To grasp this claim consider the following thought experiment. Imagine that the members of a society convene in a presocietal stage to decide on allocation procedures to be used by the society of which they will be members. Suppose that, at this presocietal convention, the participants are aware that after they have decided on the allocation procedure to be employed, they enter the societal state. Moreover, *once in the societal state they will have no recollection of the lotteries that*

[6] For a definition of the axiom of reduction of compound lotteries see Samuelson (1952) or Luce and Raiffa (1957). Notice that the reduction of compound lotteries axiom may apply to compound social-state lotteries and to allocation procedures separately. For instance, let $\ell = \alpha\delta_S + (1-\alpha)\,\delta_{S'}$ and $\ell' = \alpha'\delta_S + (1-\alpha')\delta_{S''}$, then the compound social-state lottery $[p, \ell; (1-p), \ell']$ is equivalent to the one-stage social-state lottery $[(p\alpha + (1-p)\alpha')\,\delta_S + p(1-\alpha)\delta_{S'} + (1-p)\,(1-\alpha')\delta_{S''}]$.

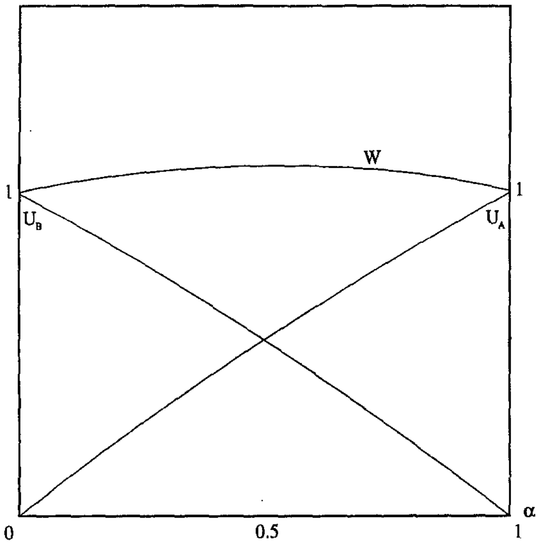

Fig. 1

were used to select the procedure and will know only which procedure is actually employed. Consequently, in the societal state, they will regard δ_{x_i} as an arbitrary procedure favoring one individual over the other while they will consider s to be a fair procedure. With this foresight, the individuals *in the presocietal stage* strictly prefer ℓ' over ℓ. Thus, $U_i(\ell') = \mu_i(\mathrm{s}) > \frac{1}{2}u_i(\delta_{x_1}) + \frac{1}{2}u_i(\delta_{x_2}) = U_i(\ell)$, $i = A, B$. By the Pareto principle this will also be the social preference. Hence, although the social welfare function, $W: L \to \Re$ is of the type suggested by Harsanyi (1955), namely, $W(\ell) = \sum_{i \in I} \omega_i U_i(\ell)$, for all $\ell \in L$, this function is immune to the criticism leveled by Diamond (1967). Note that, since they satisfy the axioms of expected utility theory, the individual utility functions, U_i, $i = 1, \ldots, n$, are linear in the probabilities of the social-state lotteries. However, as the above argument illustrates, they are not necessarily linear in the probabilities assigned by procedures to the allocations. Formally, $U_i(\ell) = \int_S u_i(\mathrm{s})\, d\ell(\mathrm{s})$. However, it is not necessarily the case that there is a function $v_i: X \to \Re$ such that $u_i(\mathrm{s}) = \int_X v_i(x)\, d\mathrm{s}(x)$.

Figure 1 depicts the example of Diamond. The space of social states is represented by the unit interval, where $\alpha \in [0,1]$ represents the procedure that assigns probability α to x_1 and probability $(1-\alpha)$ to x_2. Let $u_i: [0,1] \to [0,1]$, $i = A, B$ be the individual utility functions on the set of social states. Suppose that $u_A(\alpha) = u_B(1-\alpha)$ and let $u_A(\alpha) = \alpha^{1/2}$. The two utility functions represent conflicting preferences. Yet, the concavity of the functions capture the intrinsic sense of fairness that is present in individual's preferences. Clearly, $u_i(1/2) > \frac{1}{2}u_i(\delta_{x_1}) + \frac{1}{2}u_i(\delta_{x_2})$. Moreover, if the two individuals are given equal weights then the social welfare function, depicted by W, reflects the individuals' sense of fairness by attaining a unique maximum at $\alpha = 1/2$.

The use of a thought experiment involving a hypothetical presocietal stage of the decision-making process is not new.[7] To explain the nature of the social preference relation Harsanyi (1977) uses the presocietal stage to create a hypothetical environment in which individuals may exercise impartial evaluation of

[7] This motion is found in Vickrey (1945), Harsanyi (1953), and Rawls (1971).

alternative social institutions thereby expressing their "ethical judgment." To attain impartiality it is essential that individuals in the presocietal stage are not aware of their future (societal stage) identity (i.e., they are situated behind "a veil of ignorance"). The same veil of ignorance may be used in the present framework to interpret the ethical values expressed over social-state lotteries. In our case, however, the thought experiment has an additional purpose, namely, to allow us to compare alternative notions of procedural fairness. Essential for this is that the individuals are aware that, once in the societal stage, they will not remember which social-state lottery was used to select the procedure employed. Put differently, individuals are assumed to choose among social-state lotteries behind *a veil of amnesia*, i.e., knowing that once in the social state is determined, each person will have forgotten which social-state lottery was used to decide the social state. Consequently, the binary relation, $\succeq_i$, is interpreted as the preference relation governing individual i's choice behavior once he becomes fully informed of his exact position in the society but, at the same time, no longer remembering the lottery by which the social state was selected. By definition the individual preference relations incorporate the moral values that govern the individuals daily behavior. In the same vein, the impartial social preference relation, $\succeq$, is the hypothetical preference of an individual who must choose a social-state lottery being ignorant of his identity. This preference relation incorporates moral value judgements that would govern choice among institutions or procedures used to determine the allocations.

A real-life situation analogous in some respects to the hypothetical situation described above is provided by the following example.[8] Mom, who in this story is the embodiment of social ethics, has one ticket to Disney World, which she must give to one of her two children, Abigail or Benjamin. Mom is indifferent between giving the ticket to Benjamin and giving the ticket to Abigail. She can go into a separate room and, out of the children's sight, flip a coin to determine who gets the ticket and then announced the outcome to the two children. Alternatively, she can decide to flip the coin in the presence of both children. It is conceivable that the second procedure, which corresponds to $\ell = \delta_s$, is preferred by Mom to the first procedure, which, given the state of ignorance of the children, corresponds to the procedure $\ell = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$.

5. Concluding remarks

Epstein and Segal (1992) proposed a different approach to resolving the dilemma posed by Diamond's example. In particular, adopting Harsanyi's (1955) framework, they propose to replace the independence axiom imposed on the social preference relation by two (jointly weaker) conditions, namely, mixture symmetry and randomization preference.[9] They show that, under our Pareto condition, the social preference relation is representable by a quadratic social welfare function.

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[8] This example is due to Machina (1989).

[9] Using our notation these axioms may be stated as follows: *Mixture symmetry*: For all allocation procedure lotteries ℓ and ℓ' in L, $\ell \sim \ell'$ implies that $\alpha\ell + (1-\alpha)\ell' \sim (1-\alpha)\ell + \alpha\ell'$. *Randomization preference.* For all allocation procedure lotteries ℓ and ℓ' in L, if $\ell \sim \ell'$ and there is i and k such that $\ell \succ_i \ell'$ and $\ell' \succ_k \ell$, then $0.5\ell + 0.5\ell' \succ \ell$.

That is, there exists a function, $Q:\Re^I \to \Re$, such that for all uncertain prospects (in Harsanyi's terminology), ξ, ξ'

$$\xi \succeq \xi' \Leftrightarrow Q(U_1(\xi), \ldots, U_n(\xi)) \geq Q(U_1(\xi'), \ldots, U_n(\xi')),$$

where

$$Q(U_1(\xi), \ldots, U_n(\xi)) = \sum_{i \in I} b_i U_i(\xi) + \sum_{i \in I} \sum_{k \in I} a_{ik} U_i(\xi)\, U_k(\xi).$$

The preference relation represented by Q is quasi-concave. This social welfare function presumably captures the society's attitude toward fairness. Indeed, applied to the example of Diamond, the socially-preferred lottery will be the one that assigns equal probabilities to the two allocations.

The approach of Epstein and Segal differs from the approach of the present paper from a purely formal point of view (i.e., the functional forms of the social welfare functions in the two approaches are not necessarily the same). However, more importantly, by adopting Harsanyi's analytical framework, Epstein and Segal implicitly subscribe to the notion that the ethical considerations are detached from the personal preferences and are superimposed through the social preference relation. According to the viewpoint advanced in the present paper, the social welfare function is an aggregation procedure and, as such, is a carrier of those values that are relevant for the aggregation (e.g., the Pareto principle). Other moral values are embedded in the individual preference relations over allocation procedures. These values include self-interest as well as a subjective sense of fairness that governs the individual choice behavior among such procedures. The last observation points to an essential difference that, in principle, should permit an empirical test of some aspects of the theory. To grasp this point consider the following example.[10]

Consider a society that consists of three individuals, say A, B, and C. Suppose that the three individuals have equally valid claims to a unit of an indivisible good. Specifically, suppose that each individual requires a kidney transplant but there is only one kidney available. (The assumption that they have equally valid claims in this context means that the personal circumstances such as age, health, family responsibilities, etc., of the three individuals are identical.) Assume that the allocation procedures are lotteries, thus the set of allocation procedures is the two dimensional simplex $(p_A, p_B, p_C) \in \Re^3 \mid p_A + p_B + p_C = 1,\ \ p_i \geq 0,\ \ i = A, B, C\}$, where p_i is the probability that i receives the kidney. Suppose that individual A is asked to choose between the two allocation procedures s = (0.4, 0.6, 0.0) and s′ = (0.4, 0.3, 0.3). Since both allocation procedures assign to A the same probability of receiving the kidney, according to Epstein and Segal the individual should be indifferent between s and s′. According to the approach presented here it is conceivable that, since s′ treats B and C more fairly than s, A strictly prefers s′ over s. Note that A's preferences do not require him to be placed behind a veil of ignorance concerning his identity. Rather, by neutralizing the self-interest seeking aspect of A's preferences, this example helps bring to the fore the moral value judgment that plays a part in "governing his everyday behavior."

Deschamps and Gevers (1979) suggest a variation on Diamond's example involving a two-period procedure. The preference for the random allocation

[10] For a more elaborate discussion of this example and an axiomatic approach to modeling self-interest seeking moral behavior, see Karni and Safra (1995).

procedure is rationalized in terms of the concern for the individuals' welfare derived from their expectations in the first period. The essential reliance on the role of time may be of interest in its own right but, in my view, does not provide a satisfactory resolution of the problem posed by Diamond's critique.

Closer in spirit to our approach is a treatment proposed by Broome (1984, 1991). Specifically, Broome suggests overcoming the difficulty posed by Diamond by including the degree of fairness as an attribute of the allocation rather than of the process. One problem with this approach, pointed out by Broome himself, is that unfairness of a given allocation may not genuinely be a property of the outcome in a particular state but depends on what happens in other states. This problem aside, I take the view that fairness is, at least in part, in the process. It seems reasonable, therefore, to model social choice incorporating different aspects of the idea of fairness where they naturally belong. Thus, concepts of procedural fairness that are an intrinsic aspect of individual choice behavior should be incorporated into the individual's preference relations over allocation procedures, while concepts of fairness of the aggregation procedures should be expressed via the social preference relation.

Hammond (1983) regards consequentialism to be the source of the difficulty posed by Diamond's example. He proposes, but does not develop, the idea that consequentialism be weakened to allow the evaluation of the terminal outcomes of the choice process to be path dependent.[11] The approach of the present paper is different from that of Hammond because consequentialism is maintained for social-state lotteries; instead, the assumption of reduction of compound lotteries is rejected when a compound lottery consists of a social-state lottery and an allocation procedure.

As a final point, consider the question of whether the Epstein and Segal theory should be applied to our framework of social-state lotteries. It seems reasonable to require that, when facing a choice between two compound social-state lotteries, since the state are comprehensive depictions of the society, the society should prefer the lottery that assigns a higher probability to the preferred social state lottery. Yet, this is inconsistent with quadratic social welfare functions.[12] The reason that

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[11] The same idea is used by Machina (1989) to restore dynamic consistency in sequential choice under risk when the decision maker's preferences do not satisfy the independence axiom of expected utility theory.

[12] Suppose that the induced social preference relation on social states satisfies the Archimedean axiom of expected utility theory: i.e., for all $x, y, z \in X$, such that $\delta_x \succ \delta_y \succ \delta_z$ there are $\alpha, \beta \in (0,1)$ such that the social states $r(\alpha) = [\alpha, x; (1-\alpha), z]$ which assigns the probability α to the allocation x and the probability $(1-\alpha)$ to the allocation z and $r(\beta) = [\beta, x; (1-\beta), z]$ satisfy $\delta_{r(\alpha)} \succ \delta_y \succ \delta_{r(\beta)}$. Then, if the social preference relation $\succeq$ on L is representable by a (strictly) quadratic social welfare function, it violates the simple monotonicity condition that, for all social-state lotteries p and q in L, and $\alpha \in (0,1)$, $\delta_p \succ \delta_q$ implies $\delta_p \succ \alpha\delta_p + (1-\alpha)\,\delta_q \succ \delta_q$. To see this, let $I = \{1,2\}$, and consider the allocations $x_1 = (1,0)$, $x_2 = (0,1)$, $x_3 = (0,2)$ where the ith coordinate represents the quantity of a good allocated to individual i. Suppose that the society is indifferent between the allocation procedures δ_{x_1} and δ_{x_2}. By the strong Pareto condition, $\delta_{x_3} \succ \delta_{x_2}$. By transitivity $\delta_{x_3} \succ \delta_{x_1}$. By randomization preference, $0.5\delta_{x_1} + 0.5\delta_{x_2} \succ \delta_{x_2}$. By monotonicity $\delta_{x_3} \succ \delta_{x_1}$ implies $\delta_{x_3} \succ 0.5\delta_{x_3} + 0.5\delta_{x_2} \succ \delta_{x_1}$. By strong Pareto, $0.5\delta_{x_3} + 0.5\delta_{x_1} \succ 0.5\delta_{x_2} + 0.5\delta_{x_1}$. Hence, by transitivity, $\delta_{x_3} \succ 0.5\delta_{x_1} + 0.5\delta_{x_2}$. By the Archimedean axiom, there exists $\alpha > 0$ such that $0.5\delta_{x_1} + 0.5\delta_{x_2} \succ \delta_{\mathbf{z}} \succ \delta_{x_2}$, where $\mathbf{z} = [\alpha, x_3;\ (1-\alpha), x_2]$. Since $\delta_{\mathbf{z}} \succ \delta_{x_2}$, by strong Pareto and strict convexity, $0.5\delta_{x_1} + 0.5\delta_{\mathbf{z}} \succ 0.5\delta_{x_1} + 0.5\delta_{x_2} \succ \delta_{x_1}$. Hence, by the Archimedean axiom, there exists $\gamma \in (0,1)$ such that

a quadratic social welfare function does not make sense in the present framework is that the definition of a social state incorporates the notion of procedural fairness. Hence, no further advantage may be gained by additional randomization of the process by which the social state is selected.

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$\gamma(0.5\delta_{x_1} + 0.5\delta_z) + (1-\gamma)\delta_{x_1} \succ 0.5\delta_{x_1} + 0.5\delta_{x_2}$. But $\gamma(0.5\delta_{x_1} + 0.5\delta_z) + (1-\gamma)\delta_{x_1} = (0.5\gamma + (1-\gamma))\delta_{x_1} + (0.5\gamma)\delta_z$. Let $\beta = (1 - 0.5\gamma)$ to obtain $\beta\delta_{x_1} + (1-\beta)\delta_z \succ 0.5\delta_{x_1} + 0.5\delta_{x_2}$. By transitivity this implies $\beta\delta_{x_1} + (1-\beta)\delta_z \succ \delta_z$.