DAVE PRATT AND JANET AINLEY

THE CONSTRUCTION OF MEANINGS FOR GEOMETRIC CONSTRUCTION: TWO CONTRASTING CASES

ABSTRACT. Dynamic geometry software seems to offer new approaches to the teaching and learning of geometry. Interest has been particularly intense in Britain where geometry has almost disappeared from the curriculum. In this paper we draw on our experiences of using Logo with children and adult students as a way of thinking about the design of geometrical activities for young children who lack cultural support in this domain and who are at early stages of conceptual development. We explore how the nature of Cabri Geometry activities, in conjunction with previous connected experiences, may influence the construction of meaning for geometric construction.

PREAMBLE

In 19th Century England, schools were teaching Euclidean geometry to pupils preparing for entrance examinations to the universities of Oxford and Cambridge. Pupils were usually required to reproduce theorems by rote, even to the extent that answers which used letters different from those prescribed were marked incorrect (Howson, 1982). Indeed, in 1871, the first open meeting of the Association for the Improvement of Geometrical Teaching (later to become the Mathematical Association) was instigated by James Wilson (Rugby School) in protest at this state of affairs.[1]

Nevertheless, changes to the teaching of Euclidean geometry in England were relatively marginal; before the advent of *modern maths* in the 1960's, pupils continued to receive geometry as a series of theorems and proofs. For most pupils, this activity was formal, abstract and disconnected from any other familiar experience. There was no expectation that pupils would work actively in order to construct some meaning for these theorems. The pupils' job was clearly to learn the proofs as if pronouncing the catechism. The modern maths movement, in which transformation geometry replaced the strict Euclidean theorem-and-proof approach, gradually began to dominate the study of mathematics in secondary schools. However, even this approach, which had been hailed as more intuitive, was perceived as unsuccessful and largely rejected. As a result, children in Britain now encounter very little geometry at any age. The existence of an Attainment

International Journal of Computers for Mathematical Learning **1:** 293–322, 1997.

Target entitled *Shape Space and Measures* in the National Curriculum of England and Wales may seem to contradict this claim, but whilst such work is undoubtedly important, it is not used as a foundation for more advanced geometric concepts. The notion of geometric construction, the focus for this paper, is matched in its absence from the British curriculum by the notion of a theorem.

For us, the most powerful message that we draw from this brief historical account is that geometry has been presented for decades as purposeless, and therefore for most pupils has remained devoid of any meaning. Just as history could be presented as a series of dates with no encouragement for the learner to construct and attach personal significance to those dates, so geometry has been perceived by the vast majority of the population as remote, irrelevant and meaningless.

The central theme of this paper is to address the question of how recent developments in dynamic geometry software might offer new possibilities, allowing children to construct meaning for geometric construction. We find it helpful to relate this question to our own experiences over many years of observing children, and adult students, working with Logo, and to the 'Logo literature'. Later, we will present two contrasting experiences which we will interpret through the Logo lens in order to gain some insights into this broader question.

THE LOGO PARADIGM

Papert's early work with Logo (see for example Feurzeig et al. (1969); Papert et al. (1979); Papert (1972)) offered a new way of thinking about the learning of mathematics; an approach which proposed using Logo as a vehicle for learning about problem solving and problem posing. In particular, Papert proposed that children should play with and use mathematical concepts within a supportive computer-based environment, before embarking upon formal work with those concepts.

"When mathematizing familiar processes is a fluent, natural and enjoyable activity, then is the time to talk about mathematizing mathematical structures, as in a good pure course on modern algebra." (Papert (1972), p. 18.)

We find this a telling quote since it challenges us to ask what might be the familiar processes, suitable for mathematizing, out of which one day may emerge all sorts of mathematical notions. In this paper, we examine the nature of activities which may encourage the mathematizing of geometric construction.

In recent years, there have been several software developments which can be broadly categorised as dynamic geometry software. This type of software has stimulated much interest and excitement amongst mathematics educators: it is precisely tuned for the use of geometric construction tools. In fact, the resources and structures within different members of the family of dynamic geometry software do vary in not insignificant ways. We focus our discussion on *Cabri Geometry*, which was the software actually being used, though some of our conclusions may generalise across other implementations (e.g. *Geometer's Sketch Pad, Geometry Inventor*).

Papert has proposed that the turtle is a tool which the child can use to think about mathematics. In a parallel way, we wish to draw upon the considerable body of research on Logo to raise some questions about the use of dynamic geometry software. Children's work with Logo as represented in that research offers us a vision of children using primitive tools to build new tools and in so doing gaining mathematical insights. In Logo, there are transparent *windows* (see Noss and Hoyles (1996)) which open up new ways of looking at the world. For example, children drawing a house can be shown the use of variable as a means of drawing a whole street of houses of different sizes. By using this new idea, the child becomes familiar with the notion of variable and may eventually be able to use it independently. Although all the windows in Logo do not seem to be equally transparent (indeed some windows, such as those that look out upon the notion of list processing, seem in our experience to be quite opaque) children do seem to find Logo an unusually stimulating environment. It is now generally accepted that the teacher plays a crucial role in helping to demystify LogoMaths (or, to continue the metaphor, to de-mist(ify) the windows).

Do such windows exist in *Cabri Geometry* and, if so, how can we exploit them? How might children encounter geometric construction? Would they possess sufficient mental resources to construct meanings for the mathematical structures built into *Cabri Geometry* without substantial direct teaching which might then mitigate against the child's appropriation of the task?

There are many examples in the literature of children working creatively and imaginatively with Logo (for example see Papert (1982), Ainley and Goldstein (1988), Blythe (1990)), and this is contrasted with more formal and analytical approaches, prevalent in conventional teaching and learning of mathematics. Is a pluralistic stance, in which an informal bottom-up style of learning is given the same validity as a more formal top-down approach, possible in *Cabri Geometry*?

In Logo, children build meaningful products within an environment where the child is likely to stumble upon mathematical objects and structures. How might we encourage children to construct meaning in a *Cabri Geometry* environment? Before examining young children's sense making for geometric construction, we must first clarify what we mean by geometric objects and by geometric construction within *Cabri Geometry*.

WHAT IS GEOMETRIC CONSTRUCTION?

When we work with *Cabri Geometry*, we are likely to develop notions of geometric construction which are quite different from those conceived by children in 19th century England or indeed in current times. This is because the nature of the geometric knowledge is itself transformed, and because new pedagogical strategies become possible.

Geometric Objects in Cabri Geometry

Laborde (1995) has set out the important distinction between a drawing and a figure. A drawing incorporates many relations which are to be disregarded when considering the corresponding figure. For example, the drawing of a line contains thickness; the drawing of a tangent to a circle intersects the circle in a line segment. In contrast, the line as a figure is an ideal, which cannot be represented in reality as it has no thickness; the figure for a tangent to a circle meets the circle at a point, which has position but no dimension. Furthermore, a drawing is fixed as a single case, whereas the figure is often intended to represent an infinite set of cases. For example, contrast the drawing of a square with the concept of a square.

Children often find the distinction between drawing and figure problematical. Laborde (1995) suggests that the *Cabri Geometry* environment offers a new type of element, the *Cabri*-drawing based on a theoretical *Cabri*-figure. *Cabri Geometry* enables the user to draw objects not on a perceptual basis but on a geometric basis. For example, the tangent to the circle could be drawn as a line with the property that it is perpendicular to a radius of the circle. Because the *Cabri*-drawing of the tangent-line is based on the relationship of perpendicularity, this relationship will be maintained when the circle is transformed, say when dragging its centre to a new position or when dragging the radius point thus changing the size of the radius of the circle. By dragging certain elements of the *Cabri*-drawing, we might begin to see the *Cabri*-drawing, or construction, as a whole set of drawings, and therefore much closer to the corresponding theoretical *Cabri*-figure.

We will, however, suggest in this paper that this formulation is incomplete in the sense that we must consider also the nature of the activity within which such facilities in *Cabri Geometry* are exploited, as an important aspect of the child's abstraction from drawing to figure.

In creating *Cabri*-drawings, children will encounter and construct meanings for various types of *Cabri*-objects and *Cabri*-processes. We list here some which are especially pertinent to the rest of the paper. For the purpose of this formulation, we will intentionally blur psychological and mathematical manifestations of the elements, for example, using the term construction rather than *Cabri*-drawing as a way of incorporating both the drawing and the figure. It is worth noting that the originators of Euclidean geometry would probably not recognise the formulation which follows, which is clearly heavily influenced by the use of dynamic geometry software. We point to this as an example of situated cognition – the setting shaping the knowledge (Lave, 1988).

- *Basic elements*

  Certain elements of a system are provided (axiomatic) and can be built upon, using given functions, to construct new elements. However if a basic *element* is erased then the whole dependent construction becomes unsound. Examples of basic elements are points and lines. Children will encounter these elements as drawings, marks on the screen. An important issue is how children make sense of these elements; in particular do they take on attributes which we would associate with the theoretical *Cabri*-figure.
- *Functions*

  These are actions which can be carried out on a basic element (or indeed a second-order construction) to build a new construction. Thus, given a line-segment, we can ask for its mid-point. The function would be the constructing of the mid-point. Similarly, given two lines, we can ask for their intersection. In this case, intersecting would be the function. Functions appear as part of the menu system. We use the term *function* here in its mathematical sense of a mapping. We can envisage these actions as a process, but as we see later this process can be encapsulated into an object.
- *Constructions*

  When a function is applied to a basic element, we instantiate a construction. So, if we apply the mid-point function to a line segment, we instantiate a construction consisting of a line segment on which a new point is indicated. This process could be represented by the following schema, in which the function is represented by the symbol, M, in

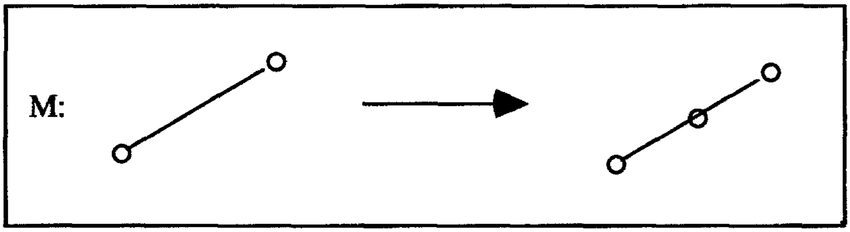

Figure 1. The mid-point function.

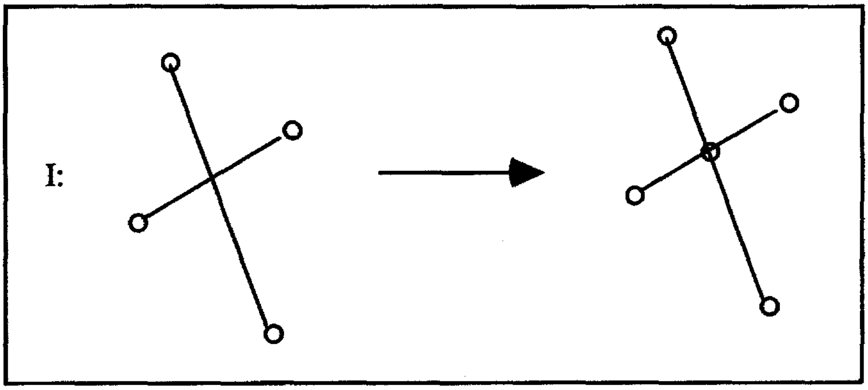

Figure 2. The intersecting function.

Figure 1. Similarly, the intersecting function is represented by the symbol, I, in Figure 2.

We can now use this new construction as the input to further functions to develop second-order constructions and so on. At some point, the sequence of functions that has thus been applied to the emerging construction may be turned into a macro (perhaps because it is recognised as having general worth i.e. a tool of more general applicability). In *Cabri Geometry*, the process of creating a macro involves identifying the inputs and outputs to the procedure which creates the construction. This sequences of operations can be named and the named function then appears in the menu. A macro is a sequence of functions encapsulated as a single function, with initial objects (input) as those which have been introduced during the sequence as basic elements, and final objects (output) as those which are to be displayed as the outcome of the whole process. The final objects would mostly be dependent on the initial objects, though they may include basic elements defined also as initial objects.

- *Functional Dependence*
  The nature of the relationship between basic elements and constructions is one of functional dependence. Thus when we drag the line segment around the screen, the mid-point moves with it. The mid-point, the output of the process of applying the function to the line segment, has a consistent relationship with its input. This dependence is emphasised when one tries to delete an element on which the other depends; the system points out that its dependants will also be deleted. Furthermore this functional relationship is apparent every time we wish to use a function. When we wish to construct a perpendicular bisector, the software requires that we declare the line segment onto which the bisector will be constructed. We see the usual structure of a functional dependence, one or more inputs, an operation, and an output.

GEOMETRY IN THE CURRICULUM

We would like now to return to our comparison of the Logo and *Cabri Geometry* environments. We suggest that the resources available to a child are deeply contingent upon cultural influences. One of the strengths of Logo is that the turtle graphics microworld offers an environment which taps straight into children's culture by offering them a world in which they can create drawings and movements. When children work with Logo, they have immediate resonance with the environment and the tools available. (At least in the early stages – it is less clear to us that the more sophisticated structures in Logo, such as list processing, are so immediately accessible). As explained above, Euclidean geometry traditionally has not only lain outside children's culture, but, in Britain at least, geometry even lies outside of the mathematics curriculum. The child's reaction to and interaction with *Cabri Geometry* will not be invariant across cultural backgrounds. For example, geometry has continued to be a central strand within the curriculum in France, where *Cabri Geometry* originated. Children brought up in the French culture are likely to respond differently from children in England and Wales, where the study of geometry has become narrow and marginalised.

However, the development of dynamic geometry software has re-awakened interest in geometry amongst British educators. This raises the question of how pupils and teachers can work with such software when they have little culture or geometric knowledge to support such activity. It is difficult to envisage entry points to the use of dynamic geometry software, other than giving children standard geometric problems to work on. However, these children may lack strategies for progressing with such problems, and their relationship to these problems is likely to be remote. These issues are keenly felt in the primary phase (5 to 11 years of age) where:

- children's conceptual structures are still in their early stages of development, and their understanding of geometry is limited to the experience of shape and space as opposed to any appreciation of more abstract geometric relationships;
- pedagogic practice in the primary phase is relatively child-centred and exploratory, demanding entry points which allow children to take ownership of the tasks they are given. Indeed, such tasks would need to be designed in such a way that they can be moulded by the children to their own purposes.

Much of the existing research with dynamic geometry software has focused on secondary or higher education students (see, for example: Laborde (1993, 1995) and Capponi & Sutherland (1992)) but there appears to be little or no literature at present, relating to primary schools. By presenting two contrasting cases from this younger age range, we aim to point up the central importance of designing for purpose when constructing dynamic geometry tasks. We will look at these two episodes through a theoretical framework proposed by Noss and Hoyles (1996).

WINDOWS AND WEBBING

Noss and Hoyles (1996) suggest that the computer can act as a *window* in two senses. In the first place, children can look through the window towards mathematical objects and structures. Thus we can reconstruct our questions about *Cabri Geometry* in these terms. We have described geometric objects and processes we perceived as embedded in *Cabri Geometry*. However, as the children peer through this window, we need to consider what sense they make of a construction, and indeed of a macro? How does this construction of meaning depend upon earlier experiences, rather than the immediate environment?

To help us to answer these questions, we draw on the other sense of window proposed by Noss and Hoyles. We, as researchers, can look through the computer as a window to observe, if not exactly what the children are thinking, then at least the manifestations of that thinking on the screen. The ways of manipulating a construction and the uses children make of functions, including macros, may give us some insights into the meanings they attach to those elements.

Noss and Hoyles also propose the term of *webbing* to describe a system of global and local support available both internally and externally to the child. The structure of local support available at any time is seen as the product of the learners' current understandings, forged and re-forged during activity, as well as the understandings built into it by others. They envisage this system as under the learner's control, signalling possible user paths rather than a unique goal. The notion of *webbing* offers us a way of thinking about the child's developing knowledge as dynamically constituted from interaction with both internal and external resources.

We can thus envisage the web as a large and complex network of resources, connected with the notion of geometric construction. Internally, the child may have access to shapes, such as a square or a circle, connected with certain properties of those shapes. These connections may result in a square being mentally constructed as a drawing rather than a figure. However, the web is organic and multidimensional, more like the global network of computing facilities than the complex but fixed connections in a fishing net. New resources are being constantly added to the child's web; others are removed or given less centrality.

However, there are also external resources available. The child could call upon a friend or the teacher, or a text book, or the structures within a piece of computer software. The child will have less control over the availability of these external resources. However, the child can decide whether to use such resources and to what purpose. Such decisions will depend on the internal resources of the child, and so we begin to envisage a complex dynamically interactive process with the child at the centre. This is the notion of webbing which will help us to make sense of the two contrasting episodes described and analysed below.

THE RESEARCH SETTING

The data in this study was collected as part of the ongoing research of the Primary Laptop Project, in which we are studying the effects on young children's mathematical learning when they have continuous and immediate access to portable computers. The computers are seen as part of a complex working environment, where many aspects integrate to support the children's learning. At the time when the data used in this paper was collected, three classes of children, aged between 8 and 12 years, had been using portable Macintosh systems for two out of the three terms of the year. The machines were generally shared between two children. Ownership of the machines by the children, and parental involvement, were encouraged by a number of strategies.

- The children were expected to look after their machine e.g. they had to make sure it was put away correctly, re-charge the battery, decide who took their machine home, etc.
- The children would often choose when to use the machine in school. Decisions not to use the machine were respected just as much as their choice to use it whenever they wanted. The exception to this rule was that the teachers and researchers would often design activities which required the use of the computer (as will be seen in the second episode described in this paper).
- As far as possible, the children were expected to decide how to maintain the desktop and their own folders for saving their work.
- The software on the machines, including the more gimmicky aspects (our description – not necessarily the children's), was there to be used whenever seemed appropriate. We avoided systems which over-protected the child in the name of protecting the software.
- The children were encouraged to 'show off' their work to their parents when they took their machines home. Indeed, the children seemed to gain much from this process, especially as they often ended up tutoring their parents.
- We often put the children into the role of tutors. For example, from time to time, the children in the project would need to hand over their machines to a new class. If the new class were not already familiar with using the hardware or the software, the experienced children would tutor them into this way of working. This peer tutoring became extended to specific activities where one class would show another class their projects which had emerged as a result of their work on the project. Indeed, this notion lies behind the second of the episodes described in this study.

We adopted a broadly *constructionist* (Harel & Papert, 1991) framework for our work within the project. The teachers and the researchers involved in the project team co-operated in order to plan activities within which are embedded mathematically powerful ideas. The children were encouraged to work on projects, developing an independence from the teacher but at the same time sharing their work and their ideas with each other. Such sharing, often guided by the teacher, helped to stimulate reflection on the important mathematical themes arising from the work.

In this stage of the project our research was essentially exploratory, rather than addressing clearly focused research questions. We were interested in exploring the range of mathematical activities that were possible for children in this environment, and in identifying areas for more focused research in the future. The researchers acted as a teacher/researcher pair. In other words, while one of us acted as a participant observer in the classroom, the other led the session and was clearly identified as the teacher. The normal classroom teacher was also in the classroom, acting as a second observer.

The observers kept field notes during normal class lessons, which typically included periods of relatively independent work by the children, and periods when the whole class came together to discuss ideas. These notes formed the basis for reflection and discussion by the project team between lessons. We recorded such notes over a period of several weeks as children worked on a specific coherent task. Since the children would, from time to time, move away from this particular task to carry out other work, the learning sequence was not continuous but the researchers were in a position to continue monitoring so that observation of the specific sequence could be continued.

This methodology was inevitably to some extent opportunistic in the obervations of the work of particular children: observers moved between groups during any particular lesson, partly in response to requests (for help or for approval and interest) from the children themselves. However, the periods in which the whole class came together allowed observers to maintain a sense of the progress of the whole class, and to identify potentially profitable areas for future detailed observation.

In the following section we present data collected from two different project classes, which offer contrasts both in the way in which the children were introduced to *Cabri Geometry*, and in the ways in which the children seemed to make sense of construction in their use of the software. The children involved were of different ages, and so some aspects of the two learning sequences are not directly comparible. Nevertheless, we suggest that the differences in the children's responses can be understood in relation to the contexts of the activities in which they used the software, rather than simply as a function of their ages.

In the weeks leading up to these episodes, both project classes had been using graphics software (a module within*ClarisWorks*) and *LogoWriter* both during lessons and in their private use of the laptops at home. We feel that these experiences may have had some impact on the children's perceptions of *Cabri Geometry*, and we discuss such connections later in the paper.

OBSERVATIONS

Episode 1: Spontaneous use of Cabri Geometry

Over a period of four months, a class of 8/9 year old children had become very fluent in using their computers. Throughout this time, *Cabri Geometry* had been available on the children's hard-disks but the project team had so far offered no activities which made explicit reference to it (not least because we had experienced some difficulty in deciding on appropriate introductory tasks).

It was normal practice in this classroom for the teacher to gather the children on the carpet to discuss the days' activities. In one such session it emerged that one child had in fact been 'playing' with *Cabri Geometry*. When the teacher asked the rest of the class, we found that most of the children had discovered and explored this software themselves.

Over a period of several lessons, the researcher systematically interviewed all the children in the class to investigate the nature of their explorations. These interviews also tried to probe into their perceptions of geometric construction. Below, we present edited parts from some of these interviews in order to give a flavour of the mental resources which the children brought to the task and how these shaped and were shaped by the structures within *Cabri Geometry* itself. We will later contrast this episode, in which the children 'discovered' *Cabri Geometry* for themselves with another in which the children were introduced to the software as part of a carefully designed activity.

Lynsey and Joy

When Lynsey and Joy were asked what they had already done with *Cabri Geometry*, they explained that they had "made circles and things". The following dialogue is taken directly from the field notes:

Researcher:	What did you do with them? (*i.e. the circles and things*)
Lynsey:	We just put them on the screen.
Researcher:	Did you make a pattern . . . or a picture . . . ?
Lynsey:	I made a man.
Joy:	I made a face.

Since they did not have these pictures with them, they loaded *Cabri Geometry* and began new pictures. Joy and Lynsey demonstrated knowledge of the CREATION menu. This menu allows the user to create primitive geometrical objects such as points, lines and line segments on the

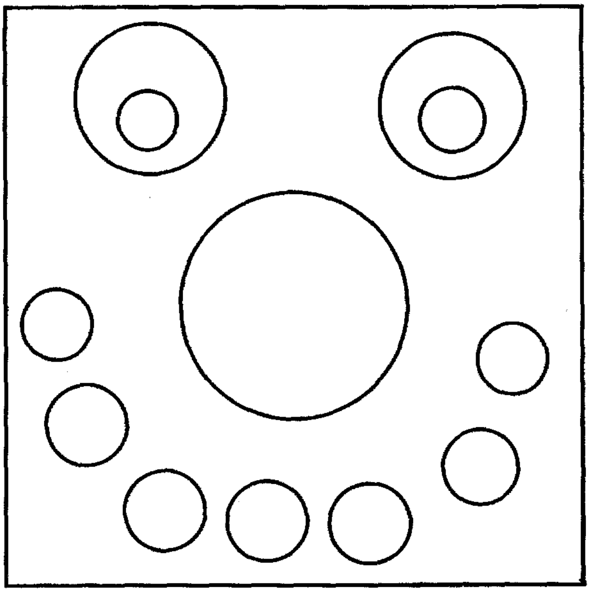

Figure 3. Verity's drawing of a face.

screen. For example, they knew how to create triangles, circles, lines and line segments. They knew how to move these around.

The CONSTRUCTION menu allows the user to combine objects through construction processes such as bisection or intersection. When asked about the CONSTRUCTION menu, they could not read words like *bisector*. They tended to choose a construction and then move the mouse around the screen. Sometimes they got no response and cancelled. At other times they constructed something randomly in which case they were asked to explore the arbitrary construction to see if they could figure out what had happened.

Later Lynsey and Joy called the researcher back to see "the alien" that they had created. This figure was drawn entirely from the CREATION menu. The researcher asked them if they had used CONSTRUCTION at all. They had not.

Phillipa and Verity

Verity had produced a picture of a face using circles (see Figure 3).

Phillipa had made a train, but in both cases, the two girls had only used the CREATION menu. The researcher asked the children what sort of program they thought *Cabri Geometry* was. It was clear that they saw it as a drawing package, such as they had used in *ClarisWorks*. Indeed,

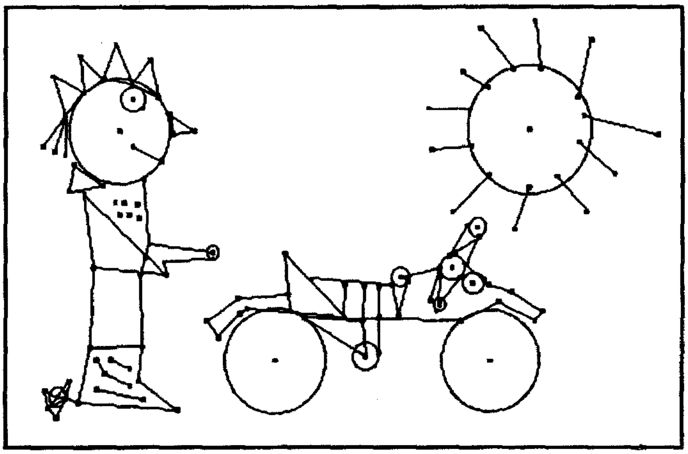

Figure 4.

Phillipa could not think of any advantages of using *Cabri Geometry* over the graphics program, a valid perspective from her point of view.

Ben, Andrew and Max

Andrew had produced an intricate picture of a motor bike and a punk rocker (see Figure 4). Despite its complexity, further inspection again showed that the picture was produced entirely through the CREATION menu.

Pictures by Ben and Max were similarly generated without reference to the CONSTRUCTION menu. The boys, perhaps looking to respond positively to the question, suggested that one advantage of *Cabri Geometry* over a graphics package was that it was easier to stretch shapes. The group enjoyed the idea of stretching the punk rocker's nose in their picture. At one point the nose became disconnected from the rest of his face. This did not disturb them; indeed they seemed to feel that it possibly added something to their picture!

In fact, the children, almost without exception, had used the CREATION menu with great imagination and persistence to generate complex and detailed drawings. It was also interesting that the children made no reference to the CONSTRUCTION menu; nor was there any evidence of them using constructions as part of their explorations. The children were creating drawings, made up of basic elements; there was no apparent need for these basic elements to be set into a geometric relationship with each other.

The one exception was the work of Bernard and Joe. Their interview proved to be particularly insightful.

Bernard and Joe

Bernard and Joe were a pair of very bright 8 year old boys who had become particularly fluent with the technology. Bernard's mathematical abilities, in particular, had become far more sharply focused in the eyes of his teacher during the period of the project. When asked about his exploration of *Cabri Geometry*, he first referred to items in the CREATION menu and he referred to his football pitch, drawn immaculately by eye using line segments circles and points. The following dialogue is from the field notes.

Researcher: Did you ever use the CONSTRUCTION menu?

Bernard (uniquely) did remember looking at this menu.

Bernard: I used perpendicular bisector.

Bernard struggled to pronounce the words.

Researcher: What did it do?

Bernard: It drew a long line.

Researcher: How did you do it?

Bernard and Joe were unsure, clearly struggling to remember. After a little more prompting they managed to construct a perpendicular bisector.

Researcher: Try picking up the line segment – move it around.

Bernard and Joe were impressed by the way that moving one line made the other move at the same time.

Bernard: It's like mechanical glue!

Researcher: Yes, that's right. If you had to tell someone else how to stick two pieces of wood together like that, what words would you use?

Bernard: Put some glue on the middle of one piece and then on the other and glue them together.

Researcher: Is it always in the middle?

Joe picked up the line segment and moved it around.

Joe: No ... er yes, yes it is.

Researcher: What direction would you tell them to stick it in?

Bernard: Across.

Bernard waved his hand to show what he meant.

Bernard had been impressed by the way that the mathematical relationship remained invariant when the original line segment had been dragged.

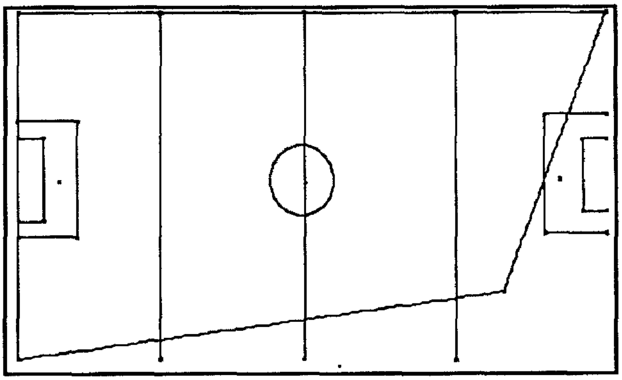

Figure 5.

However, we were interested in how he might make use of this idea. The researcher tried therefore to suggest a way in which Bernard might make use of the CONSTRUCTION menu in the hope that his actions on the computer would give us a window into his thinking about geometric construction.

The researcher showed Bernard how his football pitch was *messed up* (after Healy et al. (1994)) if we displaced a corner of the pitch (see Figure 5).

Bernard and Joe were excited by the idea that it might be possible to make their pitch stay in one piece even when parts of it were displaced. The researcher showed them how to carry out a few simple techniques, which would be needed for this task. Some time later, they were successfully applying the techniques that had been demonstrated. However, before long they returned to their original soccer pitch picture. The researcher wished to probe into the reasons for that decision:

Researcher:	Why have you gone back to that version?
Joe:	We've saved the other one.
Researcher:	But I could mess this one up. I could pick up that centre point and move it off the middle of the pitch.
Bernard:	Yes, but you aren't going to, are you, Dave?

Even though Bernard and Joe were initially excited by the idea of *mechanical glue*, the reality of creating their soccer pitch was more important to them than an investigation, which appeared to them to have no direct pay-off. They were engaged in creating a static image on the screen, and saw no purpose in putting effort into ensuring that it could be moved without messing up.

After these interviews, we decided to push the notion of messing up a further notch in the hope that this would enable them to engage with the notion of geometric construction. The teacher set a group a challenge: to draw a square in *Cabri Geometry.*

Predictably, every child used the CREATION menu, using line segments in most cases. The teacher asked them how they could be sure that their picture was in fact a square. In response to this question, many of the children got out their rulers to measure the dimensions on the screen. Verity realised that measuring in this way was rather crude, and asked: "Is there a way in *Cabri* that you can measure?"

The teacher showed her how to use the measure facility. In a short space of time, this knowledge had spread across the class. We have talked elsewhere (Ainley & Pratt, 1993 & 1995) about the portability of ideas in this sort of environment. The children do not distinguish between new *tricks* which enable them to achieve fairly pragmatic goals, and ideas which in our eyes open up completely new aspects of mathematics.

Through their actions, the children appeared to understand square as a drawing in which the four sides were equal in length. When the researcher introduced the notion of messing-up, the children could make very little sense of the activity. They could make a square just by drawing it on the screen such that the measured sides were equal in length. The idea that the square had to remain square when dragged appeared contrived and the children resisted the idea. There seemed no way into the notion of geometric construction.

The only purpose for the activity, from the children's perspective as well as the teacher's, was to learn about geometric construction. Since this learning objective was not set inside a wider context with some broader aim, it was difficult for the children to construct an understanding of how geometric construction might be helpful to them. At one level the *messing up* task could be taken on as a challenge or a puzzle, but, when this goal proved to be intractable, the children were left with few strategies other than seeking help from the teacher. Even when some children managed to construct a square, it was unclear to them just what they had learnt which was of any lasting value.

Our experience with Logo would suggest that it is by using tools purposefully towards the construction of a product that children discriminate the attributes of the tools and structures within the environment. The next episode describes a contrasting case in which just such an approach is used.

Episode 2 – The Drawing Kit Activity

Partly in response to this episode, the project team planned an activity which we hoped would place the children in a position of using geometric construction as a tool for developing a meaningful product. A group, consisting of the more mathematically able children from a class of 11/12 year olds was introduced to *Cabri Geometry* through this new activity. (The remainder of the class were also introduced to *Cabri Geometry* though a parallel activity, which, though interesting in itself, can not be included in the remit for this paper.) The teacher challenged this group to make a drawing kit for a class of younger children with whom they were paired for reading. It was explained that their 'reading partners' would be using the drawing kits to make their own pictures.

The teacher began by discussing what such a kit might contain. Some of the tools children might want for drawing are already available, but the range of ready made shapes is limited. The group brain-stormed the sorts of shapes their reading partners might need in such a kit;[2] the suggestions ranged from conventional geometric objects such as a square and a hexagon, to everyday objects such as a roof and a wheel. They also spent some time considering how these shapes had to behave. It was important that each shape could be moved around and positioned on the screen, and that its size could be altered to suit the requirements of the picture. They also needed to make it possible to produce as many of each shape as the child wanted, so their aim was not just to draw a single square or wheel, but to devise methods to produce these shapes. It was therefore important that they got a sense at this stage of the possibility of making macros, even though they were not able to follow the technical details at once.

The teacher demonstrated how to make an equilateral triangle with *Cabri Geometry* by creating a circle (by centre and radius point), constructing a *point on object* and then using the point as the centre of a second circle, whose radius point was the centre of the first circle. The two centres provided two vertices of the equilateral triangle. Constructing the intersection of the two circles gave the third vertex, as shown in Figure 6. The group were then encouraged to explore some shapes of their own, with the aim of eventually putting together a drawing kit for their reading partners.

Mark and Matthew

At this stage, the notion of constructing rather than drawing was unfamiliar. Matthew and Mark were trying to re-construct the teacher's method but instead of constructing a point on the circle, they merely placed a point so that it looked right. Our field notes commented that the "the visual impression was strong." At this point, the children's actions suggest that

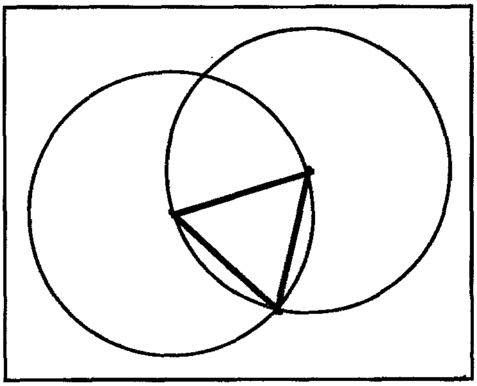

Figure 6.

they were not discriminating between the visual impression in the drawing and the mathematical relationship in the corresponding figure. However, the researcher took the opportunity to help the boys through this initial difficulty. This 'teaching' proved to be straight forward with the boys showing great delight when they were successful.

Mark and Matthew built a macro for a diamond (rhombus) using their macro for an equilateral triangle twice 'back to back' (see Figure 7). They seem to have used a macro within a macro intuitively without questioning that it would work. During this process, the boys were using language such as 'teaching the computer', 'procedure' and 'flip-side', language from their prior Logo experience, suggesting that they were making connections between building a macro and building a Logo procedure.

Another example occurred when Mark had constructed a circle and had placed (not constructed) points onto the circle and joined them to the centre so that they appeared to be at right angles, giving the appearance of a wheel with four spokes. When he tried to drag a point on the 'wheel', Mark noted that "several parts are not stamped on". Mark had previously used the primitive STAMP a great deal in his Logo work and it seems likely that he was here extending this vocabulary to the *Cabri Geometry* world. Mark recognised that this was not yet ready for inclusion in the kit. This was a vital moment in Mark's transition from drawing to figure. His actions indicated that he understood that the construction process could create drawings which were invariant when dragged and that this invariance was essential in the context of the drawing kit task. We see this partial abstraction from drawing to *Cabri*-drawing as an example of a situated abstraction (Hoyles and Noss, 1993). That is to say that the

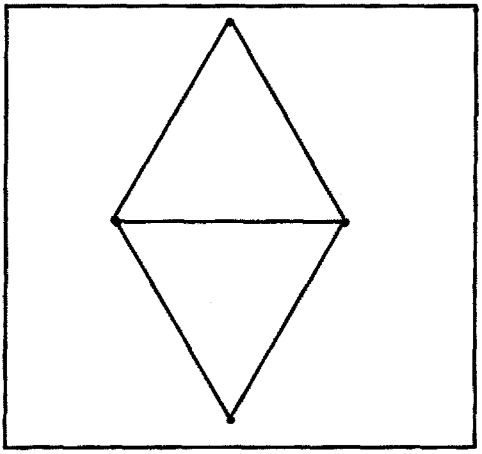

Figure 7. The rhombus – a macro containing other macros.

abstraction is fixed within the environment in which it is constructed and depends to some extent on the structures within that environment. The notion of figure would escape the situated nature of a *Cabri*-drawing but we do not claim that Mark was constructing that style of meaning for his construction. (We use the term style here rather than, say, level. Our way of thinking about these things is influenced by the constructionist school, who see the *formal* and *concrete* as styles rather than stages, since often it is entirely appropriate to behave and think concretely, especially when developing meaningful products on the computer. Over-emphasis on the formal can lead to inappropriate use of that style of thinking.)

It is significant that the job of teaching Mark how to construct the perpendicular line segments was straight forward, enabling him easily to complete his *Cabri*-drawing of a wheel, in such a way that it would be not be messed-up when dragged.

Later, Mark tried to make his wheel into a macro, but an error message indicated that there were insufficient initial objects. He was undismayed by this and simply started the macro again, but he was unable to resolve which other initial objects were needed. At this stage, Mark understood some important things. He recognised that he needed to make a macro, suggesting that he understood how a macro would help him move towards the completion of his task. He also understood that he needed not only to complete a *Cabri*-drawing but that the macro would need to know some things called initial objects. Mark had not yet sorted out which objects were the initial ones, and this suggests either that during the fairly long process of building his construction, he had lost sight of what the initial objects were, or that he had not yet constructed an unambiguous meaning for initial objects.

We see macros as an important structure within the *Cabri Geometry* environment, not only from the utilitarian standpoint that they would help the children to complete their task, but also because the macro represents the encapsulation of a sequence of functions as a new function.

After some further help, Mark was able to complete the macro for his wheel. The researcher noted Mark's persistence in completing the task and how he was able to make use of the teacher's help each time it was offered. From his on-screen macro-building actions and the way he talked about the process, we considered that Mark had made an abstraction of function, situated within the *Cabri Geometry* environment.

Luke and David

Luke and David had begun the activity by repeating the construction for an equilateral triangle, and were keen to turn this construction into a procedure for their reading partners to use. As with Mark, the connection between a procedure in Logo and a macro in *Cabri Geometry* seemed a strong one. They were shown how to make a macro and quickly showed their appreciation of this idea by using it to produce nested triangles, building three more triangles around each triangle to create a larger triangle (see Figure 8). The effect when one point was dragged around impressed the boys.

Luke and David decided to try to make a macro for a square and had been working on this task for some time when Mark came over to show them his wheel macro. (In fact this exchange was not entirely fortuitous, but had been engineered by the teacher.) Luke and David recognised that Mark's wheel macro contained the ideas that they needed for their square and they set about building the macro. However, they were puzzled when their macro failed to work.

In fact, Luke and David had built their macro using the four corners of the square as the initial objects, and so ended up with a 'floppy' quadrilateral. At this stage, Luke and David had reached a similar point to that of Mark and Matthew. After some support, they too were eventually happy that the macro should only depend upon two points. However, when the macro was used it left a point in the centre of the square. David went to delete the point, but Luke stopped him, realising that it would delete the square as well. David was confident that the macro was saved, and tried deleting to test out what happened. They resolved the problem by hiding the offending point.

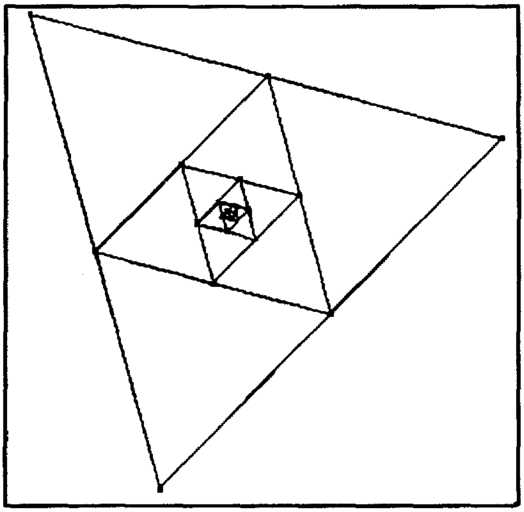

Figure 8.

These actions suggested that Luke and David had constructed a new meaning for object dependence. They understood that if you delete an independent object then dependent objects will also be deleted. They also understood that one of the attributes of an initial object is that other objects depend upon them. We see this situated abstraction as a new construction of meaning for functional dependence.

By this stage, the group as a whole had invented quite a range of constructions, some of which made conventional geometric objects and others which were more unusual but highly appropriate as tools for a drawing kit (see Figure 9). At this point there was a whole group discussion about the nature of initial objects and object dependence. A number of comments were made which indicate the children's developing construction of meaning for elements of the *Cabri Geometry* microworld.

- Becky suggested that the initial objects were: "centre and radius point", which happened to be true for many of their constructions.
- One girl asked what would happen if the centre of the circle were deleted. Other children described how other objects would be erased.
- Lauren proposed: "the initial objects are the ones that everything else depends on".
- The group talked through the construction of a regular hexagon, where each triangle was built on the previous one; they talked in terms of the computer 'knowing' about more and more points.

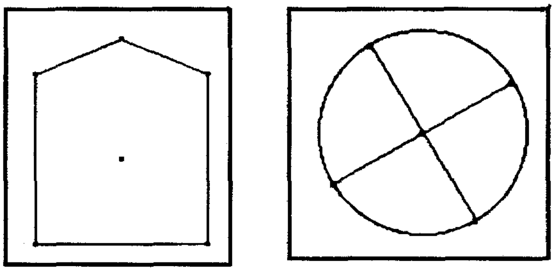

Figure 9.

- Luke, referring to the construction of an equilateral triangle which depended upon two initial objects, asked how the computer knew which way up to show the triangle. This led to a discussion about how the order of the initial objects is sometimes important.

At the end of the project the children were asked to write a report of their work. Luke and David's comments demonstrate their understanding that these were mathematically constructed objects rather than mere drawings. After listing and illustrating the shapes they had made they wrote: "They are all real shapes because you can move them without deforming the shape." Their use of the word *real* is interesting. One might have thought that a drawing was real and a figure was an abstraction. Here, we believe Luke and David use the term *real* as in *true*. They see the *Cabri*-drawing of a shape as true in that the constructed mathematical relationships prevents deformation. It is striking that this move towards abstraction was depicted in language which suggests that the abstract has become more concrete, more real.

DISCUSSION

In the introduction, we used our experiences with Logo to raise some questions about the use of *Cabri Geometry* in a context in which children were neither supported by the prevailing culture nor by possessing a sophisticated level of mathematical development. We asked, under such circumstances, what sense children might make of the notion of geometric construction, and whether it would be possible for them to adopt bottom-up approaches when they have limited recourse to top-down strategies.

The contrasting episodes above were described in an attempt to gain some insights on these questions. In one, children 'discovered' *Cabri Geometry* for themselves and went on to use it spontaneously; in the other, children were introduced to the software as part of a carefully designed activity. In the first episode, the children explored *Cabri Geometry* out of curiosity, whereas in the second they used it within a project set up by the teacher. By comparing these two episodes, we see how the purpose of the activity drives the webbing process through which the children construct meaning for the tools and structures within the *Cabri Geometry* environment.

Almost without exception, the children in the first episode had confidently and independently explored the dynamic geometry software. They had been sufficiently interested in the graphical nature of the software to want to find out what it could offer them. This exploration was free and unconstrained. We saw that although the children did gain some understanding of limited aspects of the software, the meanings they constructed were connected with drawing rather than geometric construction. The complexity and beauty of the drawings by Lynsey and Joy, and Ben, Andrew and Max is seductive but their relationship to drawing rather than construction is emphasised by Phillipa's observation that the package offered few advantages over a drawing package. They used *Cabri Geometry* as a drawing package and made connections and comparisons with the drawing and painting components of *ClarisWorks*, which were already familiar to them.

For the children engaged in drawing, the *webbing* process (Noss and Hoyles, 1996) did not lead to insights into the powerful mathematical ideas embedded within the *Cabri Geometry* microworld in the way that we have seen children engage with mathematical ideas when working in Logo. The CREATION menu made immediate sense to them since they could connect the menu items and the effects of using them with similar activities in the graphics component of *ClarisWorks*. They had a multitude of experiences in their web which enabled them to forge connections with the support offered in the CREATION menu. Although the CONSTRUCTION menu was there, part of the available web, nothing in the activity and nothing in their previous experience pointed them towards those particular structures. We wish to explore further why it was that these children made connections between *Cabri Geometry* and drawing and used the CREATION menu as their most immediate form of local support, in contrast to the children in the second episode who apparently were able to forge connections between *Cabri Geometry* and construction, using the CONSTRUCTION and CREATION menus as local support.

Observation of Ben, Andrew and Max give us a first clue. For them, the disconnected nose of their punk rocker was merely an enjoyable diversion. The unconstructed nose did not conflict with their aim of creating an interesting drawing. They had no need to move from a pure drawing to a *Cabri*-drawing. We found the remarks from Bernard particularly enlightening. Bernard, an intelligent boy, had some grasp on the notion of geometric construction, enough to offer us the vivid description of geometric construction as 'mechanical glue'. He recognised that construction offered the possibility of *sticking* objects together in a way which still allowed them to move. And yet, he saw no reason for working on this idea as his task, which was essentially to produce a static picture of a football pitch, did not demand such a facility.

These insights lead us to the conclusion that we must look carefully at the role that the activity itself is playing in the webbing process. In the first activity, the children were interested in drawing (static) pictures. The possibility of dynamic images never occurred to them, or if it did, they were unable to make connections between that possibility and the support offered within *Cabri Geometry*. The purpose of the activity, as construed by the children, shaped the webbing process in the sense that they targeted those support structures which were meaningful for them in relation to the perceived nature of the task.

We have observed similar episodes with children using Logo. A child drawing a picture in Logo will not necessarily make use of a powerful structure, or primitive, even when it is pointed out and explained to them by the teacher. This may sometimes be because the idea is too complex, but it is often the case that the intervention is badly timed in the sense that the proposed structure will not actually have a substantial pay-off for the child in her short term goal of drawing her picture. Thus, a repeat loop in order to draw the walls of a house hardly seems worth the candle, but in the context of animation the repeat loop takes on much greater significance.

Noss and Hoyles (1996) refer, rather enigmatically, to signposts which "assist in navigation" of the web. We might propose *purpose* as just one such signpost, introduced by the designer of the activity, usually the teacher. However, we feel dissatisfied with the notion of signposts, partly because they do not fit comfortably into the webbing metaphor. Furthermore, signposts are rather easily interpreted, whereas the purpose the teacher envisaged, may be construed quite differently by the learner. We prefer an image in which the designer of the activity, by careful consideration of how the child may interpret the purpose of the task, re-organises the external structures within the web, bringing new elements from the web into the local support domain. It is as if the designer picks up certain aspects of the web and pulls them into the local domain without breaking any of the connections built into the web. This transformation changes none of the connections in the web but has the effect of changing what is available locally to the learner. Inevitably, the connections the learner actually makes in use depend on many factors inaccessible to the designer, not least of which are the child's internal structures. Thus, the re-shaping, through purpose, of the web, aims to optimise in a stochastic sense the chance of the learner connecting with the mathematical concepts envisaged by the designer.

In the drawing kit activity, we re-organised those aspects of the web, the external structures, to which we, as well as the child, had access by stressing the constructional nature of the software. As a result, there seemed to be a tendency for these children to address some powerful geometrical ideas within quite a short period of working with the software. We believe that these children were able to concretise (after Wilensky (1993)) geometric construction because the drawing kit activity was designed in a way which encouraged the children to shape the *Cabri Geometry* setting in a distinctive way, and that the influence of the activity was more significant than differences in the age of the pupils. The 'drawing kit' children forged new connections through the structures available in the CONSTRUCTION menu.

We were struck by the way that Mark and Matthew referred to Logo concepts in trying to make sense of *Cabri Geometry*. They talked about stamping points onto a line and they saw macros as rather like Logo procedures. Similarly, Luke and David, when struggling with the notion of initial points in macros, seemed to make connections with inputs to Logo procedures. These connections between the children's Logo concepts and the tools within *Cabri Geometry* are examples of the children's *webbing* across internal and external resources, so that, for these children, *Cabri Geometry* was a quite different product from that understood by the children in the previous class.

In the Drawing Kit activity, the meanings that were constructed were mathematical because the nature of the task drew attention to the more mathematical structures within the web: the activity demanded that certain properties of the objects remain invariant when transformed in various ways. The children could appreciate the need for this invariance, since without it the drawing kit would not function properly, and they saw that construction offered them a way of satisfying this need. The activity was inherently motivating because these children wanted to make the product for their reading partners, but it was also well-designed in the sense that it promoted the possibility of the children making the mathematical connections with geometric construction.

For example, we saw many situations in which children identified and exploited the functional dependence between different geometric objects.

When Luke and David created the nested triangles, they witnessed this dependence when they admired the way in which dragging certain points transformed other parts of the diagram as well. There were many examples of children using macros and coming to terms with the precise role of the initial objects. Another situation where the increasing familiarity, or concretion, of functional dependence was clear occurred when David was prevented from deleting a point by Luke, who had realised that the rest of the diagram depended upon it. There was also the explicit comment by Lauren in the class discussion, when she declared that the initial objects are the ones on which everything else depends. We claim that the power of the drawing kit activity lay in the notion that the children are providing tools, in other words macros, for other children. The macro is an important structuring resource in this activity.

We have recognise similar features when observing children using Logo. A well-timed intervention by the teacher might suggest to a child that a drawing could become one of a family of such drawings. For example, the drawing of a house could be the first of a street of houses. This initial intervention can become the starting point for an introduction to the use of inputs to a procedure. The child will usually be delighted at the ease with which houses of different sizes can be easily drawn. The use of inputs has an immediate pay-off. However, through further work, the child may gradually learn new things about the use of inputs. For example, inputs can be called anything; inputs can be operated upon arithmetically; it is not necessary to replace every number by a new input, some numbers depend upon others. Each piece of knowledge represents the construction of new meaning, a situated abstraction, of the notion of variable.

The overriding issue here is the central influence of the nature of the activity in the webbing process. The web, as a massive interconnecting dynamic set of resources, both internal and external to the child, is inconceivable to the individual (though awareness of local aspects of the web may be possible). The term web is based, of course, on our recently constructed view of the network which connects computer-based resources around the globe. A striking feature of this web is that we explore certain local aspects of it, but the whole web is inconceivable, except in very general structural terms. Our explorations have to be guided by our sense of purpose, which may be playful (as in *surfing* the web) or they may be goal-oriented as when seeking out specific information. Even when our use is goal-oriented, we will find ourselves exploring connected areas; in other words our activity will not be entirely prescribed. In a similar way, the children in these episodes were guided by the structures in the activity. These structures, as well as those in the computer-based setting, were themselves part of the web. A well-designed activity (from the teacher's perspective) will optimise the chances of a child exploring and recognising the value of those structures within the web which will encourage the mathematizing process, without compromising the child's control and motivation, which are important if the children are to appropriate the task for themselves.

In this sense, the task of designing an activity is similar to that of designing a microworld, such as turtle graphics or *Cabri Geometry*. The microworld is likely to be much more general, capable of accommodating many activities and containing many powerful ideas, whereas the activity is likely to be more narrowly focused. Nevertheless each wishes to optimise the chance that the child will encounter powerful ideas within a creative and constructive environment.

The children's experiences of functional dependence were, of course, situated in the *Cabri Geometry* setting. We would not wish to suggest that these children would recognise functional dependence in another setting, for this is our construct not theirs. Ideas such as *deleting one point will delete its dependants* or *initial objects are the ones that everything else depends upon* may be seen as situated abstractions (Hoyles and Noss, 1992). In a constructionist approach, the learner, in building a meaningful product, learns how to use a mathematical idea and why that idea may be useful to them in a specific situation. At the same time they make other connections relating to that concept. The drawing kit children were learning how the notion of construction could help them to build robust diagrams in *Cabri Geometry*. Thus the notion of geometric construction was imbued with a sense of *utility* forged during the webbing process. We use the term 'utility' to mean the utility of a concept, which may be forged during activity on the task, as distinct from the *purpose*, which we reserve for the overall aim of the task as construed by the child. We would claim that one of the difficulties often encountered by teachers is that they do not separate purpose from utility, so that learning about the concept becomes the purpose of the task, giving the learner no opportunity to construct notions of its utility. It is our conjecture that it is this utility of a mathematical concept which is often missing in more conventional approaches to teaching and learning and its absence leads to a disconnected understanding of the concept in question.

We also note that most of the drawing kit children were themselves forging connections between *Cabri Geometry* and Logo and we wonder whether, given appropriate further activities which emphasise this connection, these children might construct abstractions which extend across Logo and *Cabri Geometry*. Certainly connections between macros and procedures seem to offer some hope in this respect. We might, for example, hope that a child would be able to connect the dependence between objects in *Cabri Geometry* to the dependence between procedure and screen drawing in Logo, a connection which we might see in terms of a relationship between algebra and geometry.

Finally we would like to return to our theme of using Logo as a tool to think about *Cabri Geometry*. We are aware of schools which teach Logo in a systematic prescriptive fashion, where the children learn Logo through closed activities. In these schools, we conjecture that the children's understanding of the mathematical structures embedded in Logo is likely to be limited and to exclude a concretised conception of the utility of those structures. In a similar way, we might expect such schools to give their children closed geometric problems to work with in *Cabri Geometry* and we would predict similar results for the children's conception of geometric construction. Such use of *Cabri Geometry* might not lead to significant advances over the treatment of geometry in 19th century England. However, we have seen that it is possible to design powerful activities, which optimise opportunities for the learner to construct meaning for the utility of the mathematical structures within *Cabri Geometry* by careful consideration of the children's view of the purpose of that activity.

NOTES

[1] We are grateful to Professor David Tall for drawing our attention to the literature describing the teaching of Euclid at this period of history.

[2] The children were using an early version of *Cabri Geometry* which offered a much more limited set of primitives than more recent versions. It is interesting to note that this specific activity may be perceived as less authentic by children using software which provides more features, such as shapes, as primitive.

REFERENCES

Ainley, J. and Goldstein, R. (1988). *Making Logo Work*. Blackwell.

Ainley, J. and Pratt, D. (1993) Portable mathematics and integrated technology. In B. Jaworski (Ed.), *Proceedings of the Technology in Mathematics Teaching Conference* (pp. 101–108). University of Birmingham.

Ainley, J. and Pratt, D. (1995). Planning for portability. In L. Burton and B. Jaworski (Eds.), *Technology in Mathematics Teaching – a Bridge Between Teaching and Learning* (pp. 435–448). Chartwell-Bratt.

Blythe, K. (1990). *Children Learning with Logo*. National Council for Educational Technology.

Capponi, B. and Sutherland, R. (1992). Trigonometry with Cabri in a French classroom. *MicroMath* 8(1): 32–33, Association of Teachers of Mathematics, UK.

Feurzeig W., Papert S., Bloom M., Grant R. and Solomon C. (1969) *Programming-Languages As A Conceptual Framework For Teaching Mathematics (Report No. 1889).* National Science Foundation, Office of Computing Activities, 1800 G Street, NW, Washington DC 20550

Gray, E. M. and Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth Conference of the International Group for the Psychology of Mathematics Education* 2 (pp. 72–79). Assisi, Italy.

Harel, I. and Papert, S. (Eds.) (1991). *Constructionism.* Norwood, New Jersey: Ablex.

Healy, L., Hoelzl, R., Hoyles, C. and Noss, R. (1994). Messing up. *Micromath* 10(1): 14–16, Association of Teachers of Mathematics, UK.

Howson, G. (1982). *A History of Mathematics Education.* Cambridge University Press, UK.

Hoyles C. and Noss R. (1993). Out of the Cul-De-Sac. *Proceedings of the Fifteenth Conference of the International Group for the Psychology of Mathematics Education North American Chapter* 1: 83–90, California.

Laborde, C. (1993). Do the pupils learn and what do they learn in a computer based environment? The case of Cabri-Géomètre. In B. Jaworski (Ed.), *Proceedings of the International Conference on Technology in Mathematics Teaching* 93 (pp. 39–52). Birmingham, UK.

Laborde, C. (1995). Designing tasks for learning geometry in a computer-based environment. In L. Burton, and B. Jaworski (Eds.), *Technology in Mathematics Teaching – A Bridge Between Teaching and Learning* (pp. 35–68). Chartwell-Bratt.

Lave, J. (1988). *Cognition in Practice.* CUP.

Noss, R. and Hoyles, C. (1992). Looking back and looking forward. In C. Hoyles and R. Noss (Eds.), *Learning Mathematics and Logo.* MIT.

Noss, R. and Hoyles, C. (in press). *Windows on Mathematical Meanings.* Kluwer.

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. *International Journal of Mathematics Education and Science and Technology* 3: 249–262.

Papert S., Watt D., diSessa, A. and Weir, S. (1979). Final report of the Brookline Logo Project Part II: Project summary and data analysis. *Massachusetts Institute of Technology Logo Memo No. 53.* MIT.

Papert, S. (1982) *Mindstorms: Children, Computers and Powerful Ideas.* Harvester Press.

Papert, S. (1984). Microworlds: Transforming education. In S. Evans and P. Clark (Eds.), *The Computer Culture.* White River Press.

Wilensky, U. J. (1993) Connected mathematics – Building concrete relationships with mathematical knowledge. *Doctoral Thesis.* Massachusetts Institute of Technology.

Mathematics Education Research Centre,
University of Warwick, CV4 7AL
UK