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Freedom of choice

A comparison of different rankings of opportunity sets

Marlies Klemisch-Ahlert*

Fachbereich Wirtschaftswissenschaften, Universität Osnabrück, Postfach 4469, 49034 Osnabrück, Germany

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Abstract. Inspired by the discussion of different functions of freedom of choice (instrumental versus intrinsic value) by Sen and others and an axiomatic characterization of an intrinsic aspect by Pattanaik and Xu, we compare unique axiomatic characterizations of three classes of rankings of opportunity sets in terms of freedom of choice: First, we investigate the simple cardinality-based ranking proposed by Pattanaik and Xu and a generalization of this. Secondly, we propose a new criterion that is based on the comparison of the ranges of sets of options. Thirdly, we solve possibly occuring conflicts between these two criteria.

1. Introduction

There are two kinds of motivation for us to deal with the topic "freedom of choice". The first one is the importance to include considerations of freedom of choice into economic theories and the discussion of the different functions of freedom in this context. Sen [9] distinguishes between the instrumental function of freedom and its intrinsic value. "But the existence of instrumental relevance (i.e., the value of things as means to other ends) does not entail a denial of the intrinsic importance as well (i.e., their value as ends on their own right)" (Sen [9] p. 270).

The second motivation stems from the approach of Pattanaik and Xu [6] who formalize one of the various aspects of the intrinsic value of freedom of choice, namely the *opportunity* for choice. Pattanaik and Xu consider sets of available options, a person or a society has to choose from. It is their aim to characterize

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a ranking of these sets in terms of freedom of choice. This ranking is not utility-based, because in their opinion the intrinsic value of freedom should be independent of the utility evaluations of the available options.

This paper is also concentrated on the intrinsic value of freedom of choice. We will present a development and comparison of different rankings of opportunity sets. Since freedom of choice is a very complex phenomenon, every attempt to formalize it can only pick out one or at least only a few of the large number of aspects of freedom. First we consider the aspect "number of elements of an opportunity set" proposed by Pattanaik and Xu and a generalization of this. Secondly we investigate the "range of the set of options". We observe that conflicts between the rankings belonging to these two criteria may occur. Therefore, in a third step we try to solve this conflict. We present three different ways out of the problem.

Let X be a finite set of options. Pattanaik and Xu define the axioms of "Indifference between No-Choice Situations", "Strict Monotonicity" and "Independence" for a ranking of all nonempty subsets of X. This ordering is the simple cardinality-based ordering which is defined by the comparison of the number of elements in the sets. Sets with a greater number of elements are ranked higher than smaller sets.

We investigate the reasons why the ordering Pattanaik and Xu obtain is that simple. The axiom of Indifference between No-Choice Situations implies that no information about the kind of the options under consideration can be involved in the ranking. Are there other orderings that are cardinality-based in some sense fulfilling the second and third axiom formulated by Pattanaik and Xu? Is it possible to include some cardinally measured information about the alternatives into the ranking? The positive answer is presented in Sect. 2. There we show that the "core" of all these orderings is the reflexive and transitive relation defined by the inclusion of sets. This ranking itself has the properties of Strict Monotonicity and Independence.

Criticism on the axiom of Independence leads us to some ideas to weaken this requirement. The simple cardinality-based ordering does not capture intuitions like the *range* of the set of options. "Indeed, our formal structure itself does not contain any information about closeness or similarity of different alternatives" (Pattanaik and Xu [6]). In Sect. 3 we define a ranking relation that depends on the shapes of the sets. For this purpose we have to assume that the opportunities can be represented in an n-dimensional real space.

In this context it is necessary to discuss on which spaces freedom of choice should be modelled. Mainly there are four candidates for the space of options. First we mention the traditionally used set of commodity bundles, modelled as the nonnegative orthant of $\mathbb{R}^n, n \in \mathbb{N}$. A set of opportunities in this case is a subset of $\mathbb{R}^n$ consisting of representations of commodity bundles that are feasible e.g. for a consumer. The set is formed through economic restrictions (e.g. prices and budgets) or through non-budgetary restraints like rationing of goods or ethical restrictions. This is the space Lindbeck [5] uses to analyse the consequences of welfare state policies on the freedom of choice of individuals.

The second candidate is the space of characteristics introduced by Lancaster [4], which with some reservations can also be modelled as the non-negative orthant of $\mathbb{R}^n$. In the space of characteristics each component of an option denotes the quantity (if it is measurable) of a certain material characteristic which is available if this option is realized.

Thirdly, there is the Rawlsian approach of primary goods (Rawls [7]). It seems very difficult to quantity primary goods like "rights" or "self-respect". Only if this problem can be solved, the space of n-tuples of primary goods would also be representable in the n-dimensional Euclidean space.

Sen argues that these three candidates for a space of opportunities are not really concerned with the intrinsic value of freedom of choice. "But primary goods as well as commodities and incomes are *means* to ends. If the positive conception of freedom is to reflect our ability to achieve valuable functionings and well-being then there is clearly a case for for viewing this freedom in terms of alternative bundles of functionings that a person may be able to achieve" (Sen [9] p. 278). Sen suggests to choose the capability set of a person as the object on that freedom of choice should be defined. The capability set can be formally represented by a set of functioning n-tuples (cf. Sen [8]). If we try to map this set into the space $\mathbb{R}^n$, the question how to measure functionings numerically becomes essential. In a comment on Sen's capability approach Gaertner [2] argues, "Elementary functionings such as life expectancy, infant mortality or adult literacy rate are relatively easy to measure, even on a cardinal scale. But how about more complex functionings such as achieving self-respect, to take part in social and (or) political life, to be happy in one's job?" This problem seems difficult to solve. Possibly some of the rather complex functionings can be decomposed, at least partly, in several components that are measurable. If the problem of measurement of functionings could be solved in some way, then we would be able to represent capability sets in the space $\mathbb{R}^n$.

In spite of all difficulties of measurement, in Sects. 3 and 4 we concentrate on opportunity sets that are subsets of $\mathbb{R}^n$. At some points of the discussion in the following sections we give examples of candidates for opportunity sets fitting the interpretation of our suppostions and results.

In Sect. 3 we give up the axiom of Independence and replace it by a weaker type of independence axiom. We use a stronger type of monotonicity axiom instead that takes the convex hulls of the opportunity sets into account. With these new axioms we obtain a unique characterization of a relation that can be described as the generalized version of the inclusion of the convex hulls of opportunity sets.

Since in many examples of choice situations not only the number of elements of an opportunity set is important, but also the shape and the range of the set, in Sect. 4 we investigate several combinations of these two criteria. A "small" conflict-eliminating relation is obtained, if the two criteria are combined via conjunction. We show that there are two general ways of solving possibly occuring conflicts between these criteria, independent of a certain given situation. These ways lead to lexicographic combinations of the criteria. The two lexicographic relations we obtain and the conjunction-relation can be distinguished by respective dominance or non-dominance properties. In the main results of Sect. 4 we uniquely characterize the three relations by sets of axioms. We conclude with some remarks on an open problem.

2. Cardinal rankings of opportunity sets

In [6], Pattanaik and Xu propose three axioms that uniquely characterize the simple cardinality-based ordering $\succsim_{\#}$. Let us first define the axioms and the ordering.

Let X be the universal finite set of alternatives and Z the set of all nonempty subsets of X. $\succsim$ denotes a binary relation defined over Z, i.e. a subset of $Z \times Z$. If A, B are elements of Z, instead of $(A, B) \in \succsim$ we will write $A \succsim B$. We assume $\succsim$ to be reflexive and transitive. $A \succsim B$ means that the degree of freedom of choice from A is at least as great as the degree of freedom of B. $\succ$ ("greater degree of freedom") and $\sim$ ("the same degree of freedom") are defined as usual.

Pattanaik and Xu [6] introduce some properties of $\succsim$:

Property 2.1. (Indifference between no-choice situations). For all $x, y \in X, \{x\} \sim \{y\}$.

Property 2.2. (Strict monotonicity). For all distinct $x, y \in X$ $\{x, y\} \succ \{x\}$.

Property 2.3. (Independence). For all $A, B \in Z$ and for all $x \in X \setminus (A \cup B)$, $[A \succsim B \Leftrightarrow A \cup \{x\} \succsim B \cup \{x\}]$.

We now define the simple cardinality-based ordering $\succsim_{\#}$ over Z.

Definition 2.1. For all $A, B \in Z$, $A \succsim_{\#} B \Leftrightarrow \# A \geq \# B$.

(For all $A \in Z$, $\# A$ denotes the cardinality of A, i.e. the number of elements in A.)

Pattanaik and Xu prove the following theorem.

Theorem 2.1. *There is a unique reflexive and transitive relation $\succsim$ over Z that satisfied the Properties 2.1, 2.2 and 2.3. This relation is the simple cardinality-based ordering $\succsim_{\#}$.*

Ordering the degree of freedom of choice by comparing the cardinality of sets means that each alternative $x \in X$ has a certain weight which is the same for all alternatives in X, let us say a weight of 1.

If we interpret the given set X of opportunities to be a set of bundles of capabilites then there may exist bundles x and y in X such that x and y consist of very different combinations of capabilities. For example, x may guarantee the nutritional well-being of a person together with no restrictions on the basic liberties, like freedom of thought and freedom of movement and others, whereas alternative y may imply starvation and may be characterized through severe restrictions on the basic liberties. We feel that the opportunities x and y should not have the same weight in the valuation of freedom of choice when they occur in given sets of available opportunities. In this case, even the singletons $\{x\}$ and $\{y\}$ should not be judged to be indifferent in terms of freedom of choice.

The aim of this section is to point out the meaning of the property of Indifference between No-Choice Situations to the characterization of a ranking in terms of freedom of choice. In the first step of our investigation we will make the assumption that every alternative $x \in X$ has a certain fixed weight $\alpha(x)$. The question arises, how to define the weights $\alpha(x)$ for opportunities $x \in X$. We do not think of an explicit valuation of all opportunities x, but we can use α in order to reduce the impact of certain alternatives on the ranking in terms of freedom of choice, e.g. alternatives that are inacceptable from the point of view of human rights or of other criteria that might be fundamental for the choice situation to be modelled. Assuming this idea, Property 2.1. (Indifference between No-Choice Situations) implies the equality of the weights of all alternatives in X.

Let us now define an ordering over Z that fulfills the Properties 2.2. and 2.3. but not always Property 2.1.

Let $\alpha : X \to]0, \infty[$ be a mapping that to every $x \in X$ assigns a weight $\alpha(x) > 0$.

Definition 2.2. For every given $\alpha : X \to]0, \infty[$ we define an ordering over Z. For all $A, B \in Z$

$$A \succsim_{\sum\alpha} B \Leftrightarrow \sum_{x \in A} \alpha(x) \geq \sum_{x \in B} \alpha(x) \ .$$

Remark 2.1. $\succsim_{\sum\alpha}$ is reflexive, transitive and complete and has Properties 2.2. and 2.3. $\succsim_{\sum\alpha}$ is equivalent to $\succsim_{\#}$ if and only if α is constant over X, i.e. $\alpha(x) = \alpha(y)$ for all $x, y \in X$.

The application of a weight function α to X leads to a generalization of the simple cardinality-based ordering. Via α it is possible to involve some information about the alternatives in X in the definition of freedom of choice. If we give up the property of Indifference between No-Choice Situations for a moment, we can ask whether the Properties 2.2. and 2.3. together with a "cardinal" definition of degree of freedom for sets with one element may lead to a unique ordering that is defined by some "cardinal" comparison.

Definition 2.3. An ordering $\succsim$ on Z is called generated by a mapping $\alpha : X \to \mathbb{R}$ if and only if there exists a mapping $d_\alpha : Z \to \mathbb{R}$ with

(i) for all $x \in X \quad d_\alpha(\{x\}) = \alpha(x)$,
(ii) for every $A = \{x_1, \ldots, x_n\} \in Z \quad d_\alpha(A)$ only depends on the *vector* of weights $(\alpha(x_1), \ldots, \alpha(x_n))$
(i.e. $d_\alpha(A)$ depends on the set of values $\{\alpha(x_1), \ldots, \alpha(x_n)\}$ and on the frequency of every value in the vector $(\alpha(x_1), \ldots, \alpha(x_n))$),
(iii) for all $A, B \in Z \quad A \succsim B \Leftrightarrow d_\alpha(A) \geq d_\alpha(B)$.

Remark 2.2. It is easy to construct an ordering $\succsim$ on Z that is generated by a certain α but does not fulfill the Properties 2.2. and 2.3.

For a given weight function α we replace Property 2.1. by the following property that holds for every ordering on Z that is generated by α.

Property 2.4. For all $x, y \in X \quad \{x\} \succsim \{y\} \Leftrightarrow \alpha(x) \geqq \alpha(y)$.

Any weight function α induces an ordering on X. Property 2.4 then requires the ranking $\succsim$ to be an extension rule for this ordering (cf. Kannai and Peleg [3], Bossert [1]). This means that a characterization of a ranking in terms of freedom of choice fulfilling Property 2.4. is also a characterization of an extension of a preference over X to the power set of X.

It is easy to see, that given α, there are other orderings over Z besides $\succsim_{\sum\alpha}$ that are generated by α and fulfill the Properties 2.2., 2.3. and 2.4.

Therefore, defining the weights of the alternatives in X and assuming the properties of Strict Monotonicity and Independence does not lead to a unique ordering on Z. If the property of Indifference between No-Choice Situations is combined with the other two properties it becomes responsible for the fact that Pattanaik and Xu in their theorem obtain a unique ordering.

Definition 2.4. We define a binary relation $\succsim_{\subseteq}$ over Z.

For all $A, B \in Z$, $A \subseteq B \Rightarrow B \succsim_{\subseteq} A$ and A and B are not comparable in all other cases.

If sets A and B can be compared in terms of inclusion, then $\succsim_{\subseteq}$ is defined by the direction of the inclusion. In cases where neither A includes B nor B includes, A, A and B cannot be compared. $\succsim_{\subseteq}$ is reflexive and transitive but not complete.

Lemma 2.1. The binary relation $\succsim_{\subseteq}$ is the intersection of all binary relations $\succsim_{\sum\alpha}$ with $\alpha: X \to]0, \infty[$.

Proof. If sets $A, B \in Z$ can be compared by $\succsim_{\subseteq}$, then $A \succsim_{\subseteq} B$ implies $A \succsim_{\sum\alpha} B$ for every $\alpha: X \to]0, \infty[$.

If $A, B \in Z$ cannot be compared by $\succsim_{\subseteq}$ then $\neg(A \subseteq B)$ and $\neg(B \subseteq A)$ hold. Then there exists an element $x \in A$ with $x \notin B$ and an element $y \in B$ with $y \notin A$. If A and B have more than one element, there exist mappings α and $\alpha': X \to]0, \infty[$ such that

$$\sum_{z \in A \setminus \{x\}} \alpha(z) = \sum_{z \in B \setminus \{y\}} \alpha(z)$$

and the same for α',
and $\alpha(x) > \alpha(y)$ and $\alpha'(x) < \alpha'(y)$.

$$\text{Then } \sum_{z \in A} \alpha(z) > \sum_{z \in B} \alpha(z) \quad \text{and} \quad \sum_{z \in A} \alpha'(z) < \sum_{z \in B} \alpha'(z) .$$

This implies $A \succ_{\sum\alpha} B$ but $A \prec_{\sum\alpha'} B$.

If w.l.o.g. A has only one element, i.e. $A = \{x\}$, then there exists an α with $\alpha(x) > \sum_{z \in B} \alpha(z)$ because $x \notin B$, and an $\alpha'$with $\alpha'(x) < \alpha'(y)$ for $y \in B$, $y \neq x$. In this case too $A \succsim_{\sum\alpha} B$ and $A \succ_{\sum\alpha'} B$ hold.

Therefore the pairs (A, B) and (B, A) are not elements of the intersection of $\succsim_{\sum\alpha}$ and $\succsim_{\sum\alpha'}$. □

If we consider the set of all reflexive, transitive and complete orderings over Z that fulfill the Properties 2.2. and 2.3. then the intersection of these is the incomplete binary relation $\succsim_{\subseteq}$. This follows from the facts that $\succsim_{\subseteq}$ is the smallest reflexive and transitive binary relation that fulfills the Properties 2.2. and 2.3. and that $\succsim_{\subseteq}$ cannot be enlarged even if we consider only the intersection of all orderings $\succsim_{\sum\alpha}$ with $\alpha: X \to]0, \infty[$, as we have shown in the proof of Lemma 2.1.

Definition 2.5. Let $\succsim$ be a reflexive and transitive ranking on Z. An extension of $\succsim$ is a ranking on Z that is reflexive, transitive and complete and contains $\succsim$.

Thus the following theorem holds.

Theorem 2.2. *Considering the set of all reflexive, transitive, and complete orderings over Z, the Properties 2.2. and 2.3. uniquely characterize the class of all extensions of* $\succsim_{\subseteq}$.

Pattanaik and Xu criticize the independence property IND (Property 2.3.): "Thus the major failure of IND lies in not taking into account the extents to which the different alternatives are 'close' or similar to each other." The search for possible models that enable us to capture some ideas of closeness of opportunities or similarity of sets of opportunities leads us to the consideration of opportunities in n-dimensional real spaces.

3. Ranking opportunity sets in *n*-dimensional spaces

Let a commodity space, a characteristic space, a space of primary goods or a space of capabilities be represented by $\mathbb{R}^n$ and let the set of alternatives X be the whole set $\mathbb{R}^n$, for a first analysis. For most applications, the space $\mathbb{R}^n_+$ of the nonnegative n-dimensional real space is appropriate. However, if we e.g. consider the choice of a person between alternatives that are characterized by a social redistribution of goods, then a negative component means a loss of a certain good for this person.

We are now going to characterize a binary relation $\succsim$ on the set Z of all non-empty finite subsets of $\mathbb{R}^n$, that is reflexive and transitive. We try to weaken the independence property 2.3., but in order to compensate this we have to require a monotonicity property more comprising than 2.2. We assume that the location of a set does not have an influence on the degree of freedom as long as the shape of the set remains unchanged, and we formulate this idea in two indifference properties.

We require:

Property 3.1. (Indifference between shifted situations). For all $A \in Z$ and $t \in \mathbb{R}^n$

$A \sim B$ holds where B is defined by $B = \{a + t \mid a \in A\}$.

Property 3.1. means that the degree of freedom of a set is not changed if we add a fixed vector $t \in \mathbb{R}^n$ to each alternative. The shape of the set remains the same and the range for the possible decisions is unchanged. Only the location differs. In this section we do not introduce weight functions. Our requirement of Indifference between Shifted Situations implies the indifference between singletons, because a singleton $\{y\}$ can be obtained by another singleton $\{x\}$ by a shifting with $t = y - x$.

We also require:

Property 3.2. (Indifference between reflected situations). For all $A \in Z$ and for all $i \in \{1, \dots, n\}$ the following holds:

if $B = \{(b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_n) \mid (b_1, \dots, b_{i-1}, -b_i, b_{i+1}, \dots, b_n) \in A\}$

then $A \sim B$.

B is the result of a reflection of A with respect to component i. The degree of freedom is not influenced by this kind of change of the set of feasible alternatives.

If the dimensions $1, \dots, n$ of a commodity space or a space of characteristics are of equal value with respect to freedom of choice, further shape preserving mappings may be allowed, such as:

- permutations of the dimensions
- revolving situations round a center.

We think that different dimensions of the space of alternatives may be of different importance for the freedom of a choice. Especially the scale of each dimension plays a role. Therefore, we do not require properties of indifference with respect to further shape preserving mappings.

For $A \subset \mathbb{R}^n$ we define $\operatorname{conv}(A)$ to be the convex hull of A, i.e. the smallest convex set containing A. With the notion of the convex hull of a set A we try to capture the idea of "closeness" of an alternative x to the set A or the "similarity" of x to the alternatives in A. If and only if $x \in \operatorname{conv}(A)$ holds then x is similar to A. We want this to be equivalent to the fact that the degree of freedom of choice is constant if A is changed to $A \cup \{x\}$.

Property 3.3. (Convex hull monotonicity). For all $A \in Z$

$$[x \notin \operatorname{conv}(A) \Rightarrow A \cup \{x\} \succ A] \quad \text{and}$$

$$[x \in \operatorname{conv}(A) \Rightarrow A \cup \{x\} \sim A] \; .$$

If an element x joins A such that $x \notin \operatorname{conv}(A)$, then x is an alternative that was not in the range of A before, i.e. x is not "similar" to the elements of A. Then $A \cup \{x\}$ has a greater degree of freedom than A. The second part of Property 3.3. means that only those alternatives improve the degree of freedom that are not in the convex hull of a given set A.

Property 3.3. implies Property 2.2. (Strict Monotonicity). Property 3.3. also implies $A \cup \{x\} \succsim \text{A}$ for all $A \in Z$ and $x \in X$. It is easy to prove by induction that Property 3.3. and transitivity imply the following property.

Property 3.4. (Weak monotonicity). For all $A, B \in Z \quad A \cup B \succsim A$.

Weak Monotonicity means that an enlargement of the set of opportunities does not reduce the freedom of choice. A motivation for this property is that a person that has to choose an alternative from $A \cup B$ is allowed to consider only opportunities in A. Therefore this person has at least the freedom to choose from A.

We give three motivations for the consideration of convex hulls of sets of alternatives in this context.

Assume there is one person that has to choose m-times one element from a set of feasible alternatives A. We can model this situation as a decision in an enlarged space of commodity bundles or characteristic vectors $A_m := \left\{ \sum_{i=1}^{m} a_i \,|\, a_i \in A \right\}$. In this space, the feasible sums of commodity n-tuples or characteristic n-tuples are represented. We normalize the situation A_m to the set $\frac{1}{m} A_m := \left\{ \frac{1}{m} \cdot a \,|\, a \in A_m \right\}$. If we compare $\frac{1}{m} A_m$ to A we find that certain convex linear combinations of alternatives in A have been created. $\bigcup_{m \in \mathbb{N}} \frac{1}{m} A_m$ is a dense subset of the convex hull of A. In this case, freedom of choice is approached by the consideration of repeated choice situations. Here, the convex hull of A is an appropriate variable to formulate a monotonicity condition.

The same idea is applicable to choices of m individuals. Each individual chooses an element from a given set of alternatives A. Here, we also deduce the freedom of the choice for the whole group of individuals from the set $\mathbf{A}_m$ resp. $\frac{1}{m} A_m$.

A third motivation for convex linear combinations is the possibility for persons to choose a lottery between a finite number of alternatives. Every alternative in $\operatorname{conv}(A)$ can be interpreted as an expected realization in $\mathbb{R}^n$, if the person that decides uses a lottery between some elements in A. If we consider expected outcomes of lotteries as "within the range" of feasible bundles of goods or constellations of characteristics, primary goods or functionings, a pure alternative with the same quantities is nothing "new" to choose. We should emphasize that we do not argue in the sense of preferences or utilities, which may differ between an expected bundle of commodities and an equally defined deterministic bundle.

There are several problems with these intuitions:

If we consider m decisions of one person in our model, we are only interested in the feasible sums of n-tuples after the m decisions. Analogously, in the case of a decision of m individuals only the aggregated bundles of commodities or n-tuples of characteristics, primary goods or functionings are taken into account. The seperated decision situation, the choice from set A, loses some importance. The alternatives in A are now alternatives among others that are not purely achievable but are generated by the alternatives in A.

In the third case, we enlarge the set of possible activities or hypothetical activities of an individual in so far that we allow the choice of lotteries between pure alternatives. In addition, the only property of a lottery that matters is its expected outcome in the respective space of options. It often happens that the same expectation is generated by different lotteries or pure alternatives. This multiplicity does not influence the degree of freedom in this model.

Nevertheless, Property 3.3. captures an idea of "freedom as a range of possibilities to choose from".

Examples for freedom of choice with an intuition that is perhaps more supported by the range of the feasible set than by the numbers of feasible elements are for instance the following.

For freedom of movement the boundary of a region is an important factor. But we admit that also the number of points that can be chosen has some influence on this kind of freedom. For the right of free speech the extreme opinions that are allowed to be expressed (if the locations of opinions are representable in an n-dimensional characteristic space) are the main factors that determine this kind of freedom. The fact that certain normal opinions or mixed opinions between extreme ones are additionaly admissable does not enlarge the freedom of choice granted by this right.

Instead of Property 2.3. (Independence) we require Property 3.5.:

Property 3.5. (Restricted independence). For all $A, B \in Z$ with $\operatorname{conv}(B) \subseteq \operatorname{conv}(A)$ and for all $x \in X \setminus (A \cup B)$

$$A \succsim B \Rightarrow A \cup \{x\} \succsim B \cup \{x\} \quad \text{and}$$

$$A \sim B \Rightarrow A \cup \{x\} \sim B \cup \{x\} \quad \text{hold} .$$

Remark 3.1. It is easy to see that 2.3. (Independence) is stronger than 3.5. (Restricted Independence).

In Property 3.5. we only compare sets of alternatives A and B with $\operatorname{conv}(B) \subseteq \operatorname{conv}(A)$. Therefore, the location of the sets of alternatives under consideration is a very special one. Property 3.5. means that $\succsim$ and indifference are preserved if we add a "new" element to the sets A and B.

The sets are assumed to be comparable in terms of inclusion of convex hulls. With this assumption, some of the criticisms on Property 2.3. (such as that 2.3. does not capture the closeness of the similarity of options) are no longer applicable.

Definition 3.1. We define a binary relation on Z as follows:

For $A, B \in Z \quad A \succsim_{\text{conv}} B$ iff

there exists a finite sequence of shiftings and reflections

$f_1, \ldots, f_m$ such that for $C = f_1 \circ \ldots \circ f_m(B) \quad \text{conv}(C) \subseteq \text{conv}(A)$.

We define $\succsim_{\text{conv}}$ and $\sim_{\text{conv}}$ as usual.

Lemma 3.1. *$\succsim_{\text{conv}}$ is reflexive, transitive and fulfills the Properties 3.1, 3.2, 3.3, 3.4. and 3.5. and does not have Property 2.3. (Independence).*

The proof of Lemma 3.1. is easy.

Theorem 3.1. *Let D be a subset of $Z \times Z$ such that*

$$(A, B) \in D \Leftrightarrow A \succsim_{\text{conv}} B \vee B \succsim_{\text{conv}} A \ .$$

$\succsim_{\text{conv}}$ is the only reflexive and transitive binary relation that is defined for all pairs in D (i.e. if $(A, B) \in D$ then $A \succsim B$ or $B \succsim A$ holds) and that fulfills the Properties 3.1, 3.2, and 3.3.

The proof of Theorem 3.1. can be found in the Appendix. There we also show the independence of the three properties.

Notice that Restricted Independence is not used in the characterization of $\succsim_{\text{conv}}$. It is implied by the other properties.

We have proved that the properties of

- Indifference between Shifted Situations,
- Indifference between Reflected Situations, and
- Convex Hull Monotonicity

together with reflexivity and transitivity uniquely characterize the relation defined in Definition 3.1. on the set D. This relation compares decision situations A and B in the following manner: If there exists a certain "allowed" transformation T such that $\text{conv}(T(B)) \subseteq \text{conv}(A)$ then $A \succsim_{\text{conv}} B$. Thus for the comparison of the degree of freedom the shapes of the convex hulls of the sets are relevant, not their location.

The relation $\succsim_{\text{conv}}$ is not complete. Not every pair of situations in $Z \times Z$ can be compared by $\succsim_{\text{conv}}$. If one allows further shape preserving transformations within the indifference properties and also Definition 3.1, then more comparisons are possible. But as we argued above, we tried to choose indifference properties (3.1, 3.2) which are plausible in general.

The only property that depends on the idea of taking average or expected outcomes in the alternative space into consideration, when the degree of freedom is to be characterized, is "Convex Hull Monotonicity". The other properties in Theorem 3.1. do not need this intuition.

Since we think that in many examples of choice situations not only the number of elements is important, but also the convex hull of the set, we are now searching for some fundamental properties of rankings of freedom of choice combining these two characteristics.

4. Lexicographic combinations of rankings

This section may serve as an exercise showing how to deal with conflicts between different aspects of freedom of choice. We propose some combinations of the rankings $\succsim_{\#}$ and $\succsim_{\text{conv}}$. We consider X to be a finite subset of $\mathbb{R}^n$, because $\succsim_{\#}$ is defined only for finite X.

Definition 4.1. For all $A, B \in Z \quad A \succsim_{\# \wedge \text{conv}} B \Leftrightarrow A \succsim_{\#} B$ and $A \succsim_{\text{conv}} B$.

If we search for an enlargement of the domain of $\succsim_{\# \wedge \text{conv}}$, we have to decide the priority of $\succsim_{\#}$ and $\succsim_{\text{conv}}$.

Definition 4.2. For all $A, B \in Z$

$$A \succsim_{\text{lex}(\#,\text{conv})} B \Leftrightarrow (A \succ_{\#} B) \quad \text{or} \quad (A \sim_{\#} B \quad \text{and} \quad A \succsim_{\text{conv}} B) \; .$$

Definition 4.3. For all $A, B \in Z$

$$A \succsim_{\text{lex}(\text{conv},\#)} B \Leftrightarrow (A \succ_{\text{conv}} B) \quad \text{or} \quad (A \sim_{\text{conv}} B \quad \text{and} \quad A \succsim_{\#} B) \; .$$

We have defined two lexicographic relations with respect to the criteria $\succsim_{\#}$ and $\succsim_{\text{conv}}$.

Lemma 4.1. *The relations* $\succsim_{\# \wedge \text{conv}}$, $\succsim_{\text{lex}(\#,\text{conv})}$ *and* $\succsim_{\text{lex}(\text{conv},\#)}$ *fulfill the following indifference properties:*

Property 2.1 (Indifference between No-Choice Situations),
Property 3.1 (Indifference between Shifted Situations), and
Property 3.2 (Indifference between Reflected Situations).

Lemma 4.2. $\succsim_{\# \wedge \text{conv}}$, $\succsim_{\text{lex}(\#,\text{conv})}$ *and* $\succsim_{\text{lex}(\text{conv},\#)}$ *fulfill the following monotonicity properties:*

Property 2.2 (Strict Monotonicity) and
Property 3.4 (Weak Monotonicity).

The lemma above can be proven easily.

Note that the condition of Convex Hull Monotonicity is violated by all three relations.

Since the criticism of the property of Independence of the simple cardinality based ordering was a motivation for us to search for less strong independence requirements, we now investigate the independence properties of the relations considered in this section. Property 2.3 (Independence) is not fulfilled by any of the relations

$$\succsim_{\# \wedge \text{conv}} , \quad \succsim_{\text{lex}(\#,\text{conv})} , \quad \succsim_{\text{lex}(\text{conv},\#)} .$$

Property 3.5 (Restricted Independence) is fulfilled by all of the three relations. In order to prove this statement we have to use the fact that for $\succsim_{\#}$ and $\succsim_{\text{conv}}$ the property of Restricted Independence holds.

In order to characterize the three relations defined in this section we need the definition of a further type of property.

Property 4.1. (Weak exchange monotonicity). For all $A \in Z$, $k \in \mathbb{N}$, $x_1, \ldots, x_k \in A$, $y_1, \ldots, y_k \in X \setminus A$ such that $x_1, \ldots, x_k \in \text{conv}((A \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\})$ $(A \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}) \succsim A$ holds.

The property of Weak Exchange Monotonicity means that the freedom of choice remains at least as high as in A if we replace a number of k alternatives of A by k "new" alternatives that do not reduce the convex hull of the set of alternatives. This property is fulfilled by $\succsim_{\#}$, $\succsim_{\text{conv}}$, $\succsim_{\# \wedge \text{conv}}$, $\succsim_{\text{lex}(\#,\text{conv})}$ and $\succsim_{\text{lex}(\text{conv}, \#)}$.

Property 4.2. (Strict exchange monotonicity). We assume Property 4.1. and in addition the following property:

If in the cases described in Property 4.1. there exists an $i \in \{1,\dots,k\}$ with $y_i \notin \text{conv}(A)$, then

$$(A \setminus \{x_1,\dots,x_k\}) \cup \{y_1,\dots,y_k\} \succ A \ .$$

This property is fulfilled by $\succsim_{\text{conv}}$, $\succsim_{\# \wedge \text{conv}}$, $\succsim_{\text{lex}(\#,\text{conv})}$ and $\succsim_{\text{lex}(\text{conv}, \#)}$ but not by $\succsim_{\#}$. Evidently, Strict Exchange Monotonicity implies Weak Exchange Monotonicity.

We now search for properties that enable us to clarify the differences between the three relations. Let us consider changes of a given set of opportunities $A \in Z$. Imagine to add certain new opportunities $x_1,\dots,x_k$, $k \in \mathbb{N}$, to A thar are "similar" to the elements of A (i.e. $x_1,\dots,x_k \in \text{conv}(A)$), such that we obtain the enlarged set $A \cup \{x_1,\dots,x_k\}$. In contrast to this procedure let us now replace some opportunities $y_1,\dots,y_l$, $l \in \mathbb{N}$, by some new alternatives $z_1,\dots,z_m \notin A$, $m \in \mathbb{N}$, $m \leqq l$, such that the convex hull of the new set $(A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$ is enlarged in comparison to A. This means that we "draw some opportunities of A a little bit outwards". This procedure is allowed to reduce the number of elements, a fact that will only strengthen the conflict of the criteria. Between $A \cup \{x_1,\dots,x_k\}$ and $(A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$ there is a conflict in the comparison with respect to the numbers of elements and the convex hulls of the sets. Now we have to decide, wether we are able to compare the resulting sets $A \cup \{x_1,\dots,x_k\}$ and $(A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$ in terms of freedom of choice or not, and if so, how to rank them. If we decide that we are not able to compare these situations, then we obtain a property of $\succsim_{\# \wedge \text{conv}}$. If we decide to compare these situations whenever they occure in the way $A \cup \{x_1,\dots,x_k\} \succ (A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$ we define a property of $\succsim_{\text{lex}(\#,\text{conv})}$, because the criterion "number of elements" is stronger than the criterion "inclusion of the convex hulls". If we decide the relation to be defined always the other way round we end up with a property of $\succsim_{\text{lex}(\text{conv}, \#)}$.

We now define the corresponding properties.

Property 4.3. (Non-dominance). Let $A \in Z, k, l, m \in \mathbb{N}$, $m \leqq l$,

$$x_1,\dots,x_k \in X \setminus A,\ y_1,\dots,y_l \in A \quad \text{and} \quad z_1,\dots,z_m \in X \ ,$$

be given such that $x_1,\dots,x_k \in \text{conv}(A) \setminus A$ and $\exists\, i \in \{1,\dots,m\}$ with $z_i \notin \text{conv}(A)$ and $y_1,\dots,y_l \in \text{conv}((A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\})$. Then

$$\neg\, A \cup \{x_1,\dots,x_k\} \succsim (A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$$

and

$$\neg\, A \cup \{x_1,\dots,x_k\} \precsim (A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\}$$

If for all $i \in \{1,\dots,m\}$ $z_i \in \text{conv}(A)$ then

$$A \cup \{x_1,\dots,x_k\} \succ (A \setminus \{y_1,\dots,y_l\}) \cup \{z_1,\dots,z_m\} \ .$$

In the property of Non-Dominance in the case of the described conflict between the criterion $\#$ and the criterion "Inclusion of the Convex Hulls", no ranking decision is made. Only if the criterion "Inclusion of the Convex Hulls" is indifferent (i.e. $\forall i \in \{1,\ldots,m\}\ z_i \in \operatorname{conv}(A)$) then an enlargement of the number of elements leads to a strict relation.

The property of Non-Dominance is fulfilled by $\succsim_{\# \wedge \operatorname{conv}}$ and is not fulfilled by $\succsim_{\operatorname{lex}(\#,\operatorname{conv})}$, $\succsim_{\operatorname{lex}(\operatorname{conv},\#)}$, $\succsim_{\#}$ and $\succsim_{\operatorname{conv}}$.

Property 4.4. (Dominance of $\#$). Let $A, x_1,\ldots,x_k, y_1,\ldots,y_l$ and $z_1,\ldots,z_m$ be given like in Property 4.3. without an assumption about the positions of $z_1,\ldots,z_m$ in relation to $\operatorname{conv}(A)$. Then $A \cup \{x_1,\ldots,x_k\} \succ (A \setminus \{y_1,\ldots,y_l\}) \cup \{z_1,\ldots,z_m\}$.

In this property the described conflict between the two criteria is always solved by the dominance of the criterion $\#$.

The property of Dominance of $\#$ is fulfilled by $\succsim_{\operatorname{lex}(\#,\operatorname{conv})}$ and of course by $\succsim_{\#}$, but not by $\succsim_{\# \wedge \operatorname{conv}}$ and $\succsim_{\operatorname{lex}(\operatorname{conv},\#)}$.

Property 4.5. (Dominance of the inclusion of convex hulls). Let $A, x_1,\ldots,x_k$, $y_1,\ldots,y_l$ and $z_1,\ldots,z_m$ be given like in Property 4.3. such that there exists an $i \in \{1,\ldots,m\}$ with $z_i \notin \operatorname{conv}(A)$. Then

$$A \cup \{x_1,\ldots,x_k\} \prec (A \setminus \{y_1,\ldots,y_l\}) \cup \{z_1,\ldots,z_m\} .$$

If for all $i \in \{1,\ldots,m\}\ z_i \in \operatorname{conv}(A)$ then

$$A \cup \{x_1,\ldots,x_k\} \succ (A \setminus \{y_1,\ldots,y_l\}) \cup \{z_1,\ldots,z_m\} .$$

If there is a conflict between the criteria, this conflict is solved by the dominance of the criterion "Inclusion of the Convex Hulls". Only if this criterion leads to indifference, an enlargement of the number of elements leads to the reversed strict relation.

This is a property of the relation $\succsim_{\operatorname{lex}(\operatorname{conv},\#)}$, but not of the relations $\succsim_{\operatorname{conv}}$, $\succsim_{\# \wedge \operatorname{conv}}$, $\succsim_{\operatorname{lex}(\#,\operatorname{conv})}$ and $\succsim_{\#}$.

If we assume Non-Dominance for a ranking of opportunity sets, this property restricts the set of pairs that can be compared more than the comparison by $\succsim_{\operatorname{conv}}$ does. If we try to enlarge the domain of the ranking $\succsim_{\# \wedge \operatorname{conv}}$, we have to make dominance decisions. In Properties 4.4. and 4.5. we characterize special types of situations, for which the dominance decisions are uniquely defined. One could also think of dominance properties where the direction of the dominance is not uniquely defined but varies with the number of opportunities joining a given set A, or with the distances of the alternatives $z_1,\ldots,z_m$ to the set A. But those more refined properties would need a justification that could only be based on the special characteristic of a concrete choice problem.

We now characterize the ranking rules we have defined in this section uniquely by their properties.

Let D be the set of pairs $(A, B) \in Z \times Z$ such that A and B can be compared by $\succsim_{\operatorname{conv}}$, i.e. $A \succsim_{\operatorname{conv}} B$ or $B \succsim_{\operatorname{conv}} A$ holds.

Theorem 4.1. *There is only one binary relation $\succsim$ on D such that $\succsim$ is reflexive and transitive and fulfills the Properties*
3.1. (Indifference between Shifted Situations),
3.2. (Indifference between Reflected Situations),
3.4. (Weak Monotonicity),

4.1. (Weak Exchange Monotonicity) and
4.3. (Non-Dominance).

This relation is $\gtrsim_{\# \wedge \text{conv}}$.

The proof of Theorem 4.1 can be found in the Appendix.

Theorem 4.2. *There is only one binary relation* $\gtrsim$ *on* D *such that* $\gtrsim$ *is reflexive and transitive and fulfills the Properties*
3.1. (Indifference between Shifted Situations),
3.2. (Indifference between Reflected Situations),
3.4. (Weak Monotonicity),
4.2. (Strict Exchange Monotonicity) and
4.4. (Dominance of #).

This is the relation $\gtrsim_{\text{lex}(\#,\text{conv})}$.

The proof of Theorem 4.2 can be found in the Appendix.

Theorem 4.3. *There is only one binary relation* $\gtrsim$ *on* D *such that* $\gtrsim$ *is reflexive and transitive and fulfills the Properties*
3.1. (Indifference between Shifted Situations),
3.2. (Indifference between Reflected Situations),
3.4. (Weak Monotonicity),
4.1. (Weak Exchange Monotonicity) and
4.5. (Dominance of the Inclusion of Convex Hulls).

This is the relation $\gtrsim_{\text{lex}(\text{conv},\#)}$.

The proof of Theorem 4.3. can be found in the Appendix.

The essential differences in the Theorems are to be found in the dominance properties. There are some additional differences concerning the properties of Exchange Monotonicity. In Theorem 4.2. we have to require Strict Exchange Monotonicity, otherwise $\gtrsim_{\#}$ would also fulfill the set of requirements of Theorem 4.2. Whereas in the other theorems, it suffices to require Weak Exchange Monotonicity. $\gtrsim_{\text{conv}}$ does not fulfill the requirements of Theorem 4.3, because the last statement of the property of Dominance of the Inclusion of Convex Hulls does not hold for this relation. In Theorem 4.1 and 4.3 Weak Exchange Monotonicity and the respective dominance property imply Strict Exchange Monotonicity. We omit the proof of the independence of the properties in these cases.

From our motivation of the dominance properties it follows that the three relations we have uniquely characterized are the only relations that handle the conflict between number of elements and inclusion of convex hulls in a global way.

5. Concluding remarks

All rankings of opportunity sets we discussed in the previous sections have the following property:

For all $A, B \in Z: A \subseteq B \Rightarrow A \lesssim B$.

This property is implied by Weak Monotonicity and reflexivity. Therefore, all binary relations we consider are enlargements of the binary relation defined by the inclusion of sets. This property means that a larger set B of commodity bundles, characteristic n-tuples, n-tuples of primary goods or n-tuples of functionings guarantees a degree of freedom of choice that is at least as great or even greater than that of a smaller set $A \subseteq B$. This consequence is not acceptable without any contradiction. Considering the ranking of capability sets in terms of well-being freedom, Gaertner [2] criticizes the property of monotonicity of freedom of choice with respect to the inclusion of sets. "A larger set of n-tuples of functionings is not necessarily tantamount to a preferable set (sometimes even a smaller set could be a better set when information gathering and processing become too costly)." Assuming the property above we have to neglect costs of information gathering, more generalized that means the costs to enjoy freedom of choice. We have to allow the individuals instead to omit the analysis of certain options in order to reduce the largeness of their set of opportunities. Usually this kind of decisions belongs to the possible actions of an individual or a society. In this case, the larger set B is at least as good in terms of freedom of choice as a set $A \subseteq B$. Nevertheless, it remains an open question how to define properties of rankings of freedom of choice that do not implicitely lead to an enlargement of the ranking defined by the inclusion of opportunity sets.

6. Appendix

Proof of Theorem 3.1. We assume that a relation $\succsim$ fulfills the assumptions in Theorem 3.1.

In order to prove Theorem 3.1, we assume that there is a finite sequence of shiftings and reflections $f_1, \ldots, f_m$ such that $C = f_1 \circ \ldots \circ f_m(B)$ and $\operatorname{conv}(C) \subseteq \operatorname{conv}(A)$ for some $A, B \in Z$. Then we have to show that $\operatorname{conv}(C) = \operatorname{conv}(A)$ implies $A \sim B$ and $\operatorname{conv}(C) \subsetneqq \operatorname{conv}(A)$ implies $A \succ B$.

Since B is finite, C is finite.

Let $\tilde{C}$ be the smallest subset of C such that $\operatorname{conv}(\tilde{C}) = \operatorname{conv}(C)$. $\tilde{C}$ is uniquely defined. From Property 3.3 and transitivity it follows that $\tilde{C} \sim C$. Analogously, we define $\tilde{A}$ and it follows $\tilde{A} \sim A$.

Case 1. $\operatorname{conv}(C) = \operatorname{conv}(A)$.

Since $\operatorname{conv}(\tilde{C}) = \operatorname{conv}(C) = \operatorname{conv}(A) = \operatorname{conv}(\tilde{A})$ and elements of $\tilde{C}$ or $\tilde{A}$ cannot be generated by other elements of $\tilde{C}$ resp. $\tilde{A}$, it follows that $\tilde{C} = \tilde{A}$. This implies $\tilde{C} \sim \tilde{A}$ by reflexitivity. Hence $C \sim A$ by transitivity. Applications of Properties 3.1. and 3.2. imply $C \sim B$ and by transitivity $A \sim B$.

Case 2. $\operatorname{conv}(C) \subsetneqq \operatorname{conv}(A)$.

Then there exists $\tilde{a} \in \tilde{A}$ with $\tilde{a} \notin \tilde{C}$. This implies $\tilde{C} \cup \{\tilde{a}\} \succ \tilde{C} \sim C$ because of Property 3.3.

Including the other elements of A into $C \cup \{\tilde{a}\}$ yields by the application of Property 3.4.

$$\tilde{C} \cup \tilde{A} \succsim \tilde{C} \cup \{\tilde{a}\} \succ \tilde{C} \sim C \ .$$

Since $\operatorname{conv}(\tilde{C} \cup \tilde{A}) = \operatorname{conv}(\tilde{A}) = \operatorname{conv}(A)$, Property 3.3. and transitivity yield $\tilde{A} \sim \tilde{C} \cup \tilde{A}$.

This implies

$$A \sim \tilde{A} \sim \tilde{C} \cup \tilde{A} \succ C \quad \text{(by transitivity)}$$

and therefore $A \succ C \sim B$ (by Properties 3.1. and 3.2. and transitivity) and the desired result $A \succ B$ (by transitivity). This result together with Lemma 3.1 completes the proof of Theorem 3.1. □

Proof of the independence of properties 3.1., 3.2. and 3.3. An example of a ranking on D that is reflexive, transitive and fulfills 3.2. and 3.3., but not 3.1.:
Let a be an element of X. We define

$$Z_a = \{\{x\} \mid x \in X, x = a \quad \textit{or} \quad x = f(a) \text{ where } f \text{ is any sequence of reflections with respect to any component}\} .$$

Z_a is the set of all singletons containing elements in X that are equal to a or images of a under any of the considered sequences of reflections. We define the following ranking:

$$\forall \{x\}, \{y\} \in Z_a \quad \{x\} \sim \{y\} ,$$

$$\forall A \in Z \backslash Z_a, \{x\} \in Z_a \quad A \succ \{x\} ,$$

$$\forall A, B \in Z \backslash Z_a \quad A \succsim B \Leftrightarrow A \succsim_{\text{conv}} B .$$

An example of a ranking on D that is reflexive, transitive and fulfills 3.1., 3.3., but not 3.2.:

We assume that there are two points a, b in X that differ in a coordinate i and in at least one other coordinate and such that their images under the reflection belonging to coordinate i are also in X. Otherwise we would not need Property 3.2. in the characterization of $\succsim_{\text{conv}}$.

Let f_i be the reflection with respect to coordinate i. We define $A = \{a, b\}$ and $B = \{f_i(a), f_i(b)\}$ and a ranking by

$$A \succ B ,$$

$$\forall C \in Z \backslash \{B\} \; C \succsim A \Leftrightarrow C \succsim_{\text{conv}} A , \quad A \succsim C \Leftrightarrow A \succsim_{\text{conv}} C ,$$

$$\forall C \in Z \backslash \{A\} \; C \succsim B \Leftrightarrow C \succsim_{\text{conv}} B , \quad B \succsim C \Leftrightarrow B \succsim_{\text{conv}} C ,$$

$$\forall C, D \in Z \backslash \{A, B\} \; C \succsim D \Leftrightarrow C \succsim_{\text{conv}} D .$$

An example of a ranking on D that is reflexive, transitive and fulfills 3.1., 3.2., but not 3.3. is $\succsim_{\#}$ □

Proof of Theorem 4.1. We have to show the following statement.
If $\succsim$ is a binary relation on D with the properties in Theorem 4.1, then

(i) $A \succsim_{\#} B$ and $A \succsim_{\text{conv}} B$ implies $A \succsim B$ and
(ii) $\neg A \succsim_{\#} B$ or $\neg A \succsim_{\text{conv}} B$ implies $\neg A \succsim B$.

Proof of (i). $A \succsim_{\text{conv}} B$ implies the existence of a finite sequence of shiftings and reflections $f_1, \ldots, f_m$ such that $C = f_1 \circ \ldots \circ f_m(B)$ and

$$\text{conv}(C) \subseteq \text{conv}(A) .$$

$B \sim C$ holds by Properties 3.1. and 3.2.

$$\# B = \# C \leq \# A \quad \text{by assumption} .$$

If $\# B = \# A = 1$, then $A \succsim B$ by Property 3.1.

If $\# B=1$, $\# A>1$, then we can choose $C=\{x\}$ with $x \in A$. Weak Monotonicity implies $A \succsim C \sim B$, and by transitivity $A \succsim B$.

If $\# B \geq 2$ we enlarge C by a number of $k=\# A-\# B$ alternatives $z_1, \ldots, z_k \in \operatorname{conv}(C)$ with $z_1, \ldots, z_k \notin C \cup A$. If $k=0$, we define $\{z_1, \ldots, z_k\}=\emptyset$.

$\operatorname{conv}(C \cup\{z_1, \ldots, z_k\})=\operatorname{conv}(C) \subseteq \operatorname{conv}(A)$ and $\#(C \cup\{z_1, \ldots, z_k\})=\# A$. Applying Weak Exchange Monotonicity to the set $C \cup\{z_1, \ldots, z_k\}$ and replacing all elements of $C \cup\{z_1, \ldots, z_k\}$ by elements of A yields

$$C \cup\{z_1, \ldots, z_k\} \precsim A .$$

Since we know by Weak Monotonicity that $C \cup\{z_1, \ldots, z_k\} \succsim C$, we have shown that $B \sim C \precsim C \cup\{z_1, \ldots, z_k\} \precsim A$ holds; this implies $B \precsim A$ by transitivity.

Proof of (ii). If $\neg A \succsim_{\text{conv}} B$ holds because A and B are not comparable, then this implies $(A, B) \notin D$ and $(B, A) \notin D$. Therefore $\neg A \succsim_{\text{conv}} B$ means $A \prec_{\text{conv}} B$.

Let C be the set defined as usual.

First we consider the case $\# A>\# B=\# C$ and $A \prec_{\text{conv}} B$, i.e. $\operatorname{conv}(A) \subsetneq \operatorname{conv}(C)$. We have to show that $\neg A \succsim B$ is true. Let x be an element of $\operatorname{conv}(A) \backslash A$. Then by Non-Dominance $\neg A \cup\{x\} \succsim(A \backslash A) \cup C$ holds and $(A \backslash A) \cup C=C$, hence $\neg A \cup\{x\} \succsim C$ by transitivity and reflexivity. Assume $A \succsim B$, this means $A \succsim C$ and $A \cup\{x\} \succsim A \succsim C$ by transitivity and Weak Monotonicity. This is a contradiction. Therefore $\neg A \succsim B$ holds.

Second we consider the case $\# A<\# B=\# C$ and $B \prec_{\text{conv}} A$, i.e. $\operatorname{conv}(C) \subseteq \operatorname{conv}(A)$. We have to show that $\neg A \succsim B$ holds.

If $\# B=2$, then $\# A=1$ and we can choose $C=\{x, y\}$ with $x \neq y$ and $A=\{x\}$. Then $C \succ A$ follows from the last statement of the property of Non-Dominance: $\{x\} \cup\{y\} \succ(\{x\} \backslash\{x\}) \cup\{x\}$. This implies $B \succ A$.

If $\# B \geq 3$, let x be an element of C such that $\operatorname{conv}(C \backslash\{x\}) \subsetneq \operatorname{conv}(C) \subset \operatorname{conv}(A)$ and y an element of $\operatorname{conv}(C \backslash\{x\}) \backslash(C \backslash\{x\})$. Then by Non-Dominance

$$\neg(C \backslash\{x\}) \cup\{y\} \precsim((C \backslash\{x\}) \backslash(C \backslash\{x\})) \cup A \quad \text{holds and}$$

$$((C \backslash\{x\}) \backslash(C \backslash\{x\})) \cup A=A .$$

If we assume $A \succsim B$, this implies $A \succsim C \succsim(C \backslash\{x\}) \cup\{y\}$. The second relation follows from the application of Weak Exchange Monotonicity to $C=(((C \backslash\{x\}) \cup\{y\}) \backslash\{y\}) \cup\{x\}$.

From the contradiction it follows that $\neg A \succsim B$ holds.

Therefore, it remains the case ($\# A \leq \# B$ and $A \prec_{\text{conv}} B$).

Let the set C be defined as above. In this case $1 \leq \# A \leq \# C$ and $\operatorname{conv}(A) \subsetneq \operatorname{conv}(C)$ holds.

This implies the existence of an element $z \in C$ with $z \notin \operatorname{conv}(A)$ and $z \notin \operatorname{conv}(C \backslash\{z\})$. We choose $m=\# C-\# A$ elements from $C \backslash\{z\}$. Let us call them $w_1, \ldots, w_m$. If $m=0$ then we define $\{w_1, \ldots, w_m\}=\emptyset$.

Now we compare for $x \notin A \cup\{w_1, \ldots, w_m\}$, $x \in \operatorname{conv}(A \cup\{w_1, \ldots, w_m\})$

$$(A \cup\{w_1, \ldots, w_m\}) \cup\{x\} \quad \text{with}$$

$$((A \cup\{w_1, \ldots, w_m\}) \backslash(A \cup\{w_1, \ldots, w_m\})) \cup C=C .$$

We know that $A \cup\{w_1, \ldots, w_m\} \subseteq \operatorname{conv}(C)$ and $z \notin \operatorname{conv}(A \cup\{w_1, \ldots, w_m\})$ hold. Therefore we can apply Non-Dominance.

This implies $\neg A \cup\{w_1, \ldots, w_m\} \cup\{x\} \succsim C$.

If we assume $A \succsim B$, it follows from Weak Monotonicity that

$A \cup \{w_1,\ldots,w_m\} \cup \{y\} \succsim A \succsim B \sim C$, and by transitivity

$A \cup \{w_1,\ldots,w_m\} \cup \{y\} \succsim C$. This contradiction implies $\neg A \succsim B$. □

Proof of Theorem 4.2. We have to show

(i) $A \succ_{\#} B$ or $(A \sim_{\#} B$ and $A \succsim_{\text{conv}} B)$ implies $A \succsim B$

and

(ii) $\neg A \succ_{\#} B$ and $(\neg A \sim_{\#} B$ or $\neg A \succsim_{\text{conv}} B)$ implies $\neg A \succsim B$.

Proof of (i). First we consider the case $(A \succsim_{\#} B$ and $A \succsim_{\text{conv}} B)$. Analogously to the proof of (i) in Theorem 4.1, where we do not use any dominance-property, it follows that $A \succsim B$ holds.

Now we consider the remaining case $\# A > \# B$ and $A \prec_{\text{conv}} B$. We choose C as usual. Then $\# A > \# C$ and $\text{conv}(A) \subsetneqq \text{conv}(C)$ hold.

If $\# A = 2$, then $\# C = 1$ and $\text{conv}(A) \subseteq \text{conv}(C)$ is not possible. Therefore $\# A \geqq 3$ holds. Let x be an element of A, then there exists an element $y \in \text{conv}(A \setminus \{x\}) \setminus \text{A}$.

Dominance of $\#$ implies

$$(A \setminus \{x\}) \cup \{y\} \succ ((A \setminus \{x\}) \setminus (A \setminus \{x\})) \cup C = C$$

and

$$A = (((A \setminus \{x\}) \cup \{y\}) \setminus \{y\}) \cup \{x\} \succsim (A | \setminus \{x\}) \cup \{y\}$$

by Weak Exchange Monotonicity.

Transitivity yields $A \succ C$ and therefore $A \succ B$ implying $A \succsim B$.

Proof of (ii). First we consider the case $\# A \leq \# B$ and $A \prec_{\text{conv}} B$. We have to show $B \succ A$. We choose C as usual such that $\# A \leq C$ and $\text{conv}(A) \subsetneqq \text{conv}(C)$. If $\# A = 1$ we can choose C such that $A \subseteq C$. Then by Dominance of $\# A \cup (C \setminus A) \succ (A \setminus A) \cup A$, and hence $C \succ A$ follows, and because of $B \sim C$, $B \succ A$ holds. If $\# A \geq 2$, we enlarge A by $k = \# C - \# A$ elements $\{x_1,\ldots,x_k\} \subseteq \text{conv}(A) \setminus A$.

Then $\text{conv}(A \cup \{x_1,\ldots,x_k\}) \subsetneqq \text{conv}(C)$ holds. Applying Strict Exchange Monotonicity to $A \cup \{x_1,\ldots,x_k\}$ and C yields

$$A \cup \{x_1,\ldots,x_k\} \prec (A \cup \{x_1,\ldots,x_k\}) \setminus (A \cup \{x_1,\ldots,x_k\}) \cup C = C \ .$$

Weak Monotonicity implies $A \cup \{x_1,\ldots,x_k\} \succsim A$ and by transitivity $C \succ A$ follows, and because of $B \sim C$ the desired result $B \succ A$ holds.

It remains the case $\# A < \# B$ and $A \succsim_{\text{conv}}$ B.

We have to show $B \succ A$. In this case we apply the second part of the proof of (i) with reversed roles of A and B. (Notice that in this part of the proof of (i) the strict comparison $A \prec_{\text{conv}} B$ is not used.) We obtain $B \succ A$. □

Proof of Theorem 4.3. We have to show

(i) $A \succ_{\text{conv}} B$ or $(A \sim_{\text{conv}} B$ and $A \succsim_{\#} B)$ implies $A \succsim B$

and

(ii) $\neg A \succ_{\text{conv}} B$ and $(\neg A \sim_{\text{conv}} B$ or $A \succsim_{\#} B)$ implies $\neg A \succsim B$.

Proof of (i). The case ($A \succsim_{\text{conv}} B$ and $\# A \geq \# B$) can be handled like the analogous case in Theorem 4.1 and 4.2. In this case $A \succsim B$ holds.

We now have to consider the remaining case $\# A < \# B$ and $A \succ_{\text{conv}} B$. We choose C as usual such that $\# A < \# B = \# C$ and $\text{conv}(C) \subsetneqq \text{conv}(A)$. Let x be an element of $\text{conv}(C) \backslash C$. Then Dominance of the Inclusion of Convex Hulls and Weak Monotonicity imply $C \precsim C \cup \{x\} \prec C \backslash C \cup A = A$. Transitivity leads to $A \succ B$, and therefore $A \succsim B$ holds.

Proof of (ii). If $A \prec_{\text{conv}} B$ and $\# A \leq \# B$ hold, we choose C as usual such that $\# C = \# B$ and $\text{conv}(A) \subsetneqq \text{conv}(C)$.

If $\# A = 1$, then $A \prec_{\text{conv}} B$ implies $\# B \geq 2$. In this case we can choose C such that $A \subseteq C$. Then $A \cup (C \backslash A) \succ (A \backslash A) \cup A$ because of the last statement in the property of Dominance of the Convex Hulls. Hence $C \succ A$ and therefore $B \succ A$ holds.

If $\# A \geq 2$, we define $k = \# C - \# A$ and choose a subset $\{w_1, \ldots, w_k, x\} \subset \text{conv}(A) \backslash A$. If $k = 0$, $\{w_1, \ldots, w_k, x\}$ is defined to be $\{x\}$. Then $A \precsim A \cup \{w_1, \ldots, w_k\} \cup \{x\} \prec (A \cup \{w_1, \ldots, w_k\}) \backslash (A \cup \{w_1, \ldots, w_k\}) \cup C$ holds by Dominance of the Inclusion of Convex Hulls, and by Weak Monotonicity we obtain $A \precsim A \cup \{w_1, \ldots, w_k\} \cup \{x\} \prec C \sim B$, and hence by transitivity $A \prec B$ holds.

If $A \prec_{\text{conv}} B$ and $\# A > \# B$ and C is defined as usual, we apply the proof of the second part of (i) with reversed roles of A and B, and we obtain $A \prec B$. □

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