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ISOMETRIC IMMERSIONS WITH HOMOTHETICAL GAUSS MAP

ABSTRACT. Let $f: M \to \mathbb{R}$ be an isometric immersion of an m-dimensional Riemannian manifold M into the n-dimensional Euclidean space. Its Gauss map $g: M \to G_m(\mathbb{R}^n)$ into the Grassmannian $G_m(\mathbb{R}^n)$ is defined by assigning to every point of M its tangent space, considered as a vector subspace of $\mathbb{R}^n$. The third fundamental form b of f is the pull-back of the canonical Riemannian metric on $G_m(\mathbb{R}^n)$ via g. In this article we derive a complete classification of all those f (with flat normal bundle) for which the Gauss map g is homothetical; i.e. b is a constant multiple of the Riemannian metric on M. Using these results we furthermore classify all those f (with flat normal bundle) for which the third fundamental form b is parallel w.r.t. the Levi–Civita connection on M.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Since the very beginning of differential geometry the Gauss map has played an important role in surface theory. A natural generalization of this classical map for an isometric immersion $f: M \to \mathbb{R}^n$ of an m-dimensional Riemannian manifold into the n-dimensional Euclidean space is defined by assigning to every point p of M its tangent space $f_* T_p M$, considered as a vector subspace of $\mathbb{R}^n$. The Gauss map $g: M \to G_m(\mathbb{R}^n)$ into the Grassmannian $G_m(\mathbb{R}^n)$ obtained in this way has been extensively studied, and a beautiful survey on results concerning g and on alternative definitions of the Gauss map of f can be found in [11]. In this paper we will mainly consider the pull back b of the canonical Riemannian metric on $G_m(\mathbb{R}^n)$ via g, which is called *the third fundamental form of f*. It is very natural to pose the following problems:

1. *Find all f for which the Gauss map g is homothetical* (*i.e. b is a constant multiple of the Riemannian metric on M*).

and, more generally:

2. *Find all f for which the third fundamental form b is parallel.*

There are important examples of isometric immersions having the desired properties.

Firstly, if M is a compact connected Riemannian homogeneous space G/K such that the isotropy representation of K is irreducible, then for any $i \in \mathbb{N}_+$ the so called ith standard immersion corresponding to the ith positive eigenvalue of the Laplacian on M has homothetical Gauss map, see [2], [10],

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[18] and Chapter 4, §§5–6, of [1] for the definition and concrete examples.

Secondly, all isometric immersions $f: M \to \mathbb{R}^n$ with a parallel second fundamental form have a parallel third fundamental form. Using this fact Ferus could prove a decomposition theorem for these immersions in his article [4] which was an important step towards their classification in his subsequent papers [5], [6].

In view of the complicated geometric and algebraic structure of the above examples it seems too ambitious to try to solve problems 1 and 2 in their general form. However, under the assumption of a flat normal bundle, we obtain complete answers by the following theorems:

THEOREM 1. *Let M be an m-dimensional simply connected complete Riemannian manifold and let $f: M \to \mathbb{R}^n$ be an isometric immersion. Then:*

(a) $b = 0$ *if and only if* $M = \mathbb{R}^m$ *and* f *is an isometry of* $\mathbb{R}^m$ *onto an* m*-dimensional affine subspace of* $\mathbb{R}^n$.

(b) *Let* f *have a flat normal bundle. Then* $b = (1/r^2)\langle \cdot, \cdot \rangle$ *with* $r \in \mathbb{R}_+$ *if and only if there exist numbers* k_0, $k_1 \in \mathbb{N}$ *such that* M *is a Riemannian product* $S^{m_1}(r) \times \cdots \times S^{m_{k_0}}(r) \times \mathbb{R} \times \cdots \times \mathbb{R}$ *of Euclidean spheres* $S^{m_i}(r) := \{p \in \mathbb{R}^{m_i+1} \mid \|p\| = r\}$ *with* $m_i \geq 2$ *and* k_1 *Euclidean lines* $\mathbb{R}$ *and* f *is a Riemannian product of isometric immersions* $f_1, \ldots, f_{k_0+k_1}$, *where* f_i *is*
   (i) *for* $i = 1, \ldots, k_0$, *a canonical imbedding of* $S^{m_i}(r)$ *into some Euclidean space* $\mathbb{R}^{n_i}$, *and*
   (ii) *for* $i = k_0 + 1, \ldots, k_0 + k_1$, *the arclength parametrization of a curve in some Euclidean space* $\mathbb{R}^{n_i}$ *which has constant curvature* $1/r$.

THEOREM 2. *Let M be an m-dimensional simply connected complete Riemannian manifold and let $f: M \to \mathbb{R}^n$ be an isometric immersion. Then b is parallel and the normal bundle of f is flat if and only if M is a Riemannian product of simply connected Riemannian manifolds $M_1, \ldots, M_k$ and f is a Riemannian product of isometric immersions $f_i: M_i \to \mathbb{R}^{n_i}$, $i = 1, \ldots, k$, with homothetical Gauss maps and flat normal bundles.*

The proofs of these theorems will be given in Section 4.

REMARK. We make the assumption that 'M is simply connected and complete' in the theorems solely to use the global version of the de Rham decomposition theorem. Without it the theorems hold in an appropriate 'local' formulation. For isometric immersions into real space forms of non-null curvature analogous results are valid.

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2. Notations

All manifolds, vector bundles, curves, etc., are supposed to be of class C^∞ unless otherwise explicitly stated. The set of locally defined sections of a vector bundle E is denoted by $\Gamma(E)$.

Furthermore, let M be a connected Riemannian manifold of dimension $m \geqslant 1$ with Levi–Civita connection ∇ and curvature tensor R, and $f: M \to \mathbb{R}^n$ an isometric immersion into the n-dimensional Euclidean space. $\nu(f)$ denotes the normal bundle of f, $\pi: \nu(f) \to M$ the bundle projection, $h: TM \times_M TM \to \nu(f)$ resp. $A: \nu(f) \to \mathrm{End}(TM)$ the normal resp. tangential version of the second fundamental form of f and b the third fundamental form of f. Due to [7] and [9],

$$(1) \qquad b(v, w) = \sum_{i=1}^{m} \langle h(v, e_i), h(w, e_i)\rangle$$

for all $p \in M$, all $v, w \in T_pM$ and every orthonormal basis $(e_1, \ldots, e_m)$ of T_pM. Under the use of the mean curvature normal H of f and of the Ricci tensor Ric of M the Gauss equation leads to the invariant description

$$b(v, w) = m\langle h(v, w), H\rangle - \mathrm{Ric}(v, w).$$

Suppose M is the Riemannian product of Riemannian manifolds $M_1, \ldots, M_k$, $f_i: M_i \to \mathbb{R}^{n_i}$ are isometric immersions $(i = 1, \ldots, k)$ and $\psi: \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^n$ is an isometry. Then $f = \psi \circ (f_1 \times \cdots \times f_k)$ is called *a Riemannian product of* $f_1, \ldots, f_k$. Denoting the second and third fundamental form of f_i by h_i and b_i, respectively, we get in this situation:

$$(2) \qquad h(v, w) = \psi_*(h_1(v_1, w_1), \ldots, h_k(v_k, w_k))$$

and

$$(3) \qquad b(v, w) = \sum_{i=1}^{k} b_i(v_i, w_i)$$

for all $p = (p_1, \ldots, p_k) \in M$ and all $v = (v_1, \ldots, v_k)$, $w = (w_1, \ldots, w_k) \in T_pM \cong \oplus T_{p_i}M_i$; furthermore,

$$(4) \qquad \nu(f) \text{ is flat} \Leftrightarrow \nu(f_i) \text{ is flat for all } i \in \{1, \ldots, k\},$$

where $\nu(f)$ is called *flat*, if the curvature form of the canonical covariant derivative of $\nu(f)$ vanishes.

3. THE GEOMETRY OF ISOMETRIC IMMERSIONS WITH FLAT NORMAL BUNDLE

As in the classical theory of surfaces, any isometric immersion $f\colon M \to \mathbb{R}^n$ can be studied from the viewpoint of principal curvatures. If f has a flat normal bundle this investigation leads to strong results, which yield the main tools for the proof of our theorems.

In this case, for every $p \in M$ there exist an integer $s = s(p) \in \{1, \ldots, m\}$ and a uniquely determined subset $\mathscr{H}_p = \{\eta_1, \ldots, \eta_s\}$ of $\nu_p(f)$ with s elements such that T_pM is the orthogonal sum of the *non-trivial* vector subspaces

$$E(\eta_i) := \{v \in T_pM \mid h(v, w) = \langle v, w\rangle \eta_i \text{ for all } w \in T_pM\},$$
$$i = 1, \ldots, s.$$

Therefore, if $T_pM \to E(\eta_i)$, $v \mapsto v^i$ denotes the orthogonal projection, the second fundamental form of f has the simple representations

$$(5) \qquad h(v, w) = \sum_{i=1}^{s} \langle v^i, w^i\rangle \eta_i \quad \text{for all } v, w \in T_pM$$

and, equivalently,

$$(6) \qquad A_\zeta v = \sum_{i=1}^{s} \langle \eta_i, \zeta\rangle v^i \quad \text{for all } v \in T_pM,\ \zeta \in \nu_p(f).$$

From these we see that for every $\zeta \in \nu_p(f)$ the numbers $\langle \eta_1, \zeta\rangle, \ldots, \langle \eta_s, \zeta\rangle$ are the eigenvalues of A_ζ; the eigenspace corresponding to $\langle \eta_i, \zeta\rangle$ is the orthogonal sum of those $E(\eta_j)$ with $\langle \eta_j, \zeta\rangle = \langle \eta_i, \zeta\rangle$.

For that reason $\eta \in \mathscr{H}_p$ is called *a principal normal of f at p*, the linear form $\langle \eta, \cdot\rangle\colon \nu_p(f) \to \mathbb{R}$ *a principal curvature* and $v \in E(\eta)$ *a principal vector*.

The function $M \to \{1, \ldots, m\}$, $p \mapsto \#\mathscr{H}_p$ (:= number of the principal normals of f at p) is lower-semicontinuous. Let G_s denote the open kernel of $\{p \in M \mid \#\mathscr{H}_p = s\}$; then $\bigcup_{s=1}^{m} G_s$ is open and dense in M (see [14, Prop. 1] for a proof of the above statements).

The following proposition describes the global structure of the set of principal normals, for abbreviation we set $\mathscr{H}_U := \bigcup_{p \in U} \mathscr{H}_p$ for any subset U of M.

PROPOSITION. (a) *For every $s \in \{1, \ldots, m\}$ with $G_s \neq \emptyset$ the set $\mathscr{H}_s := \mathscr{H}_{G_s}$ is an m-dimensional regular submanifold of $\nu(f)$, and $\pi|\mathscr{H}_s\colon \mathscr{H}_s \to G_s$ is an s-fold covering. Therefore, for every simply connected open subset U of G_s there are sections $\eta_1, \ldots, \eta_s \in \Gamma(\nu(f))$ defined on U such that $\mathscr{H}_p = \{\eta_1(p), \ldots, \eta_s(p)\}$ for all $p \in U$. Furthermore, $E_i := (E(\eta_i(p))_{p \in U}$, $i = 1, \ldots,$*

s, are C^∞-vector subbundles of $TM \mid U$.

(b) *$\pi|\mathscr{H}_M: \mathscr{H}_M \to M$ is a proper map, i.e. $\mathscr{H}_U$ is compact for every compact subset U of M.*

(c) *$\mathscr{H}_M$ is a closed subset of $\nu(f)$.*

(d) *For every C^∞-curve $c: [0, 1] \to M$ with $c([0, 1]) \subset G_s$ and $c(1) \in \partial G_s$ ($s \in \{1, \dots, m\}$) and every initial value $\eta \in \mathscr{H}_{c(0)}$ there is a unique continuous lift $\tilde{c}: [0, 1] \to \mathscr{H}_M$ (i.e. $\tilde{c}(0) = \eta$, $\pi \circ \tilde{c} = c$).*

REMARK. The 'principal vector bundles' E_i in (a) are always integrable, and their integral manifolds ('curvature surfaces') are totally umbilical submanifolds of M. For a proof of these facts – which we do not use in this note – and more information about the geometry of E_i see [12], [14], [15], [16].

Proof. For (a): For every $p_0 \in G_s$ there are a neighbourhood V of p_0 in G_s and sections $\zeta_1, \dots, \zeta_s: V \to G_s$ such that $\mathscr{H}_p = \{\zeta_1(p), \dots, \zeta_s(p)\}$ for all $p \in V$, cf. [14, Prop. 1c]. With the help of such sections we can transfer the C^∞-structure on M to a C^∞-structure on $\mathscr{H}_s$ with respect to which $\mathscr{H}_s$ is a regular submanifold of $\nu(f)$ and $\pi|\mathscr{H}_s$ is an s-fold covering of G_s.

Let U be a simply connected open subset of G_s and $\eta_1, \dots, \eta_s: U \to \mathscr{H}_s$ sections such that $\mathscr{H}_p = \{\eta_1(p) \dots, \eta_s(p)\}$ for all $p \in U$. As E_i is the kernel of the vector bundle homomorphism $\bar{h}_i$ from $TM|U$ into $\mathrm{Hom}(TM|U, \nu(f)|U)$ defined by

$$\bar{h}_i(v)(w) := h(v, w) - \langle v, w\rangle \eta_i(p) \quad \text{for all } p \in U \text{ and all } v, w \in T_pM,$$

the function $p \mapsto \dim E_i(p)$ is upper-semicontinuous on U. Since T_pM is the orthogonal sum of the $E_i(p)$, $i = 1, \dots, s$, for every $p \in U$, it follows that the dimensions of $E_1, \dots, E_s$ are constant on U which means that $E_1, \dots, E_s$ are C^∞-vector-subbundles of $TM|U$.

For (b): First we prove a simple characterization of the principal vectors: For every $p \in M$ and every $v \in T_pM$

$$v \in \mathscr{E}_p := \bigcup_{\eta \in \mathscr{H}_p} E(\eta) \tag{7}$$

is equivalent to

$$h(v, w) = 0 \quad \text{for all } w \in T_pM \text{ with } \langle v, w\rangle = 0. \tag{8}$$

(8) has the advantage of being formulated without the help of the principal normals.

'(7) $\Rightarrow$ (8)' is trivial.

For '(8) $\Rightarrow$ (7)': The case $\mathscr{H}_p = \{\eta\}$ is trivial.

Now let $\#\mathscr{H}_p \geqslant 2$ and $v \in T_pM$ with (8). From the assumption that (7) is wrong follows the existence of $\eta_i, \eta_j \in \mathscr{H}_p$, $\eta_i \neq \eta_j$, such that $v^i \neq 0$, $v^j \neq 0$. The vector $w := \langle v^j, v^j \rangle v^i - \langle v^i, v^i \rangle v^j$ is orthogonal to v, but (5) leads to

$$h(v, w) = \langle v^i, v^i \rangle \langle v^j, v^j \rangle (\eta_i - \eta_j) \neq 0$$

which contradicts (8). Thus '(8) $\Rightarrow$ (7)' is shown.

Now let U be a compact subset of M and let $T^1M := \{v \in TM \mid \langle v, v \rangle = 1\}$ the unit tangential bundle of M. Because of (5) the set $\mathscr{H}_U$ is the image of $T^1M \cap \bigcup_{p\in U} \mathscr{E}_p$ under the map $T^1M \to \nu(f)$, $v \mapsto h(v, v)$. Hence for the proof of (b) it suffices to show:

(9) $$T^1M \cap \bigcup_{p\in U} \mathscr{E}_p \text{ is compact.}$$

For (9): The fibres of the bundle projection $\tau: T^1M \to M$ are the spheres $T^1_pM := \{v \in T_pM \mid \langle v, v \rangle = 1\}$, the vector subbundle $\mathscr{V} := \operatorname{kernel} \tau_*$ is consequently characterized by

(10) $$\mathscr{V}_e = T_eT^1_{\tau(e)}M = \{W \in T_eT_{\tau(e)}M \mid \langle e, KW \rangle = 0\} \quad \text{for all } e \in T^1M,$$

where $K: TTM \to TM$ is the connection map of ∇ (in particular $K \mid T_eT_{\tau(e)}M$ is the canonical identification of $T_eT_{\tau(e)}M$ with $T_{\tau(e)}M$); cf. [3]. If we define the section $\tilde{h}$ of $\operatorname{Hom}(\mathscr{V}, \tau^*\nu(f))$ by

$$\tilde{h}_e(W) := h(e, KW) \quad \text{for all } e \in T^1M \text{ and all } W \in \mathscr{V}_e,$$

then the subset $T^1M \cap \bigcup_{p\in M} \mathscr{E}_p$ of T^1M is, on account of (8) and (10), the pre-image of the 'null-section' $\{0_e \in \operatorname{Hom}(\mathscr{V}_e, \nu_{\tau(e)}(f)) \mid e \in T^1M\}$ of $\operatorname{Hom}(\mathscr{V}, \tau^*\nu(f))$ under $\tilde{h}$ and as such closed in T^1M. Since τ is a proper map, $\tau^{-1}(U)$ is compact, so

$$T^1M \cap \bigcup_{p\in U} \mathscr{E}_p = \pi^{-1}(U) \cap \left(T^1M \cap \bigcup_{p\in M} \mathscr{E}_p \right)$$

is compact which shows (9).

For (c): (c) follows easily from (b).

For (d): Given c and η as prescribed there exists, according to (a), a unique C^∞-lift $\bar{c}: [0, 1] \to \mathscr{H}_s$ of $c|[0, 1]$ with respect to $\pi|\mathscr{H}_s$ such that $\bar{c}(0) = \eta$. It is enough to show that $\lim_{t\to 1} \bar{c}(t)$ exists in $\mathscr{H}_M$, because then the continuous extension $\tilde{c}: [0, 1] \to \mathscr{H}_M$ of $\bar{c}$ has the required properties. Applying (b) one proves easily with an argumentation similar to the one in [13, p. 15, Prop. 1.4], that the limit set $\omega(\bar{c}) := \{\zeta \in \nu(f) \mid \text{There is a sequence } (t_n) \in [0, 1]^{\mathbb{N}} \text{ with } t_n \to 1 \text{ and } \bar{c}(t_n) \to \zeta\}$ is not empty and connected. Since $\omega(\bar{c}) \subset \mathscr{H}_{c(1)}$ according to (c), $\omega(\bar{c})$ consists of one element $\bar{\eta} \in \mathscr{H}_{c(1)}$, i.e. $\lim_{t\to 1} \bar{c}(t) = \bar{\eta}$. □

4. THE PROOFS OF THE THEOREMS

Proof of Theorem 1. For (a): (a) is trivial.

For (b): The 'if' part: The third fundamental forms of all f_i described in (i) and (ii) are constant multiples of the corresponding Riemannian metric with the factor $1/r^2$, from (2) and the definition of the product metric on M it follows that $b = (1/r^2)\langle\cdot, \cdot\rangle$.

The 'only if' part: Assume there is $r \in \mathbb{R}_+$ such that $b = (1/r^2)\langle\cdot, \cdot\rangle$. With the notation from Section 3 we get, according to (1) and (5),

$$(11) \qquad \|\eta\| = \frac{1}{r} \quad \text{for all } p \in M \text{ and all } \eta \in \mathscr{H}_p.$$

Define $k := \max_{p\in M} \#\mathscr{H}_p$ and G_k as in Section 3.

If $k = 1$, then $G_k = M$, and f is because of (5) a totally umbilical immersion of which the mean curvature normal has constant length (see (11)), so in the case $m = 1$ we get $M = \mathbb{R}$, and $f: \mathbb{R} \to \mathbb{R}^n$ describes a curve parametrized by arclength which has constant curvature $1/r$, and in the case $m \geqslant 2$ we obtain $M = S^m(r)$, and f is a canonical imbedding of $S^m(r)$ into $\mathbb{R}^n$.

In the following we assume $k \geqslant 2$.

(12) *Let U be a simply connected open subset of G_k.*

Consider the principal normal fields $\eta_1, \ldots, \eta_k$ defined on U and the corresponding principal vector bundles $E_1, \ldots, E_k$ as in part (a) of the proposition.

(13) *Then $E_1, \ldots, E_k$ are parallel.*

For (13): Fix $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, k\}\backslash\{i\}$, $Y \in \Gamma(E_i)$, and $X \in \Gamma(TM \mid U)$. To prove (13) we must show

$$(14) \qquad (\nabla_X Y)^j = 0.$$

Since (14) is linear in X, we can assume $X \in \Gamma(E_l)$ for $l \in \{1, \ldots, k\}$.

For (14): For $\zeta \in \Gamma(\nu(f)|U)$ it follows from the Codazzi equation and (6) that

$$\begin{aligned}(15) \qquad 0 &= (\nabla_X A_\zeta)(Y) - A_{D_X\zeta} Y - ((\nabla_Y A_\zeta)(X) - A_{D_Y\zeta} X)\\ &= \langle D_X\eta_i, \zeta\rangle Y - \sum_{t=1}^{k} \langle\eta_t - \eta_i, \zeta\rangle(\nabla_X Y)^t\\ &\quad -\left(\langle D_Y\eta_l, \zeta\rangle X - \sum_{t=1}^{k} \langle\eta_t - \eta_l, \zeta\rangle(\nabla_Y X)^t\right)\end{aligned}$$

For $l = j$ we get from (15):

$$\langle \eta_j - \eta_i, \zeta \rangle (\nabla_X Y)^j = -\langle D_Y \eta_j, \zeta \rangle X.$$

Setting $\zeta = \eta_j$ into this equation yields $(\nabla_X Y)^j = 0$ by (11) and the Ricci identity (note $\langle \eta_j - \eta_i, \eta_j \rangle \neq 0$ because of (11)).

For $l \in \{1, \ldots, k\} \setminus \{i, j\}$ we obtain from (15):

$$(16) \qquad \langle \eta_j - \eta_i, \zeta \rangle (\nabla_X Y)^j = \langle \eta_j - \eta_l, \zeta \rangle (\nabla_Y X)^j.$$

Now for all $p \in U$ there is a normal vector $\xi \in v_p(f)$ such that $\langle \eta_j(p) - \eta_i(p), \xi \rangle \neq 0$ and $\langle \eta_j(p) - \eta_l(p), \xi \rangle = 0$ (otherwise $\eta_i(p)$, $\eta_j(p)$ and $\eta_l(p)$ would be three different points on a straight line in $v_p(f)$ lying at the same time on the sphere $\{\eta \in v_p(f) \mid \|\eta\| = 1/r\}$, which is impossible). Setting $\zeta_p = \xi$ into (16) yields $(\nabla_X Y)^j = 0$, again.

For $l = i$, at last, we have for every $Z \in \Gamma(E_j)$:

$$\langle (\nabla_X Y)^j, Z \rangle = -\langle Y, (\nabla_X Z)^i \rangle = 0$$

(for $(\nabla_X Z)^i = 0$ by the first part of the proof of (14)), hence $(\nabla_X Y)^j = 0$ is valid.

Thus $E_1, \ldots, E_k$ are parallel. By this fact, the Gauss equation and (5) it follows that for all $i, j \in \{1, \ldots, k\}$, $i \neq j$, and all $X \in \Gamma(E_i)$, $Y \in \Gamma(E_j)$ with $\|X\| = \|Y\| = 1$:

$$\begin{aligned} 0 &= \langle R(X, Y)Y, X \rangle \\ &= \langle h(X, X), h(Y, Y) \rangle - \langle h(X, Y), h(X, Y) \rangle \\ &= \langle \eta_i, \eta_j \rangle. \end{aligned}$$

From this we see that the principal normals of f at any point $p \in G_k$ are mutually orthogonal.

Now $G_k = M$ will be shown.

Assume $G_k \neq M$. Since G_k is not empty and open and M is connected, ∂G_k is not empty. Therefore there exists a C^∞-curve $c: [0, 1] \to M$ such that $c([0, 1]) \subset G_k$, $c(1) \in \partial G_k$, and for any of the pairwise distinct initial values $\eta_1, \ldots, \eta_k \in \mathscr{H}_{c(0)}$ there is a continuous lift $\tilde{c}_i: [0, 1] \to \mathscr{H}_M$ of c; see Proposition (d). For continuity reasons the principal normals $\tilde{c}_1(1), \ldots, \tilde{c}_k(1) \in \mathscr{H}_{c(1)}$ are mutually orthogonal, and by (11) they all are pairwise distinct, i.e. $\#\mathscr{H}_{c(1)} = k$ $(:= \max_{p \in M} \#\mathscr{H}_p)$. But, as $\#\mathscr{H}$ is lower-semicontinuous, $G_k = \{p \in M \mid \#\mathscr{H}_p = k\}$ holds, so $c(1) \in \partial G_k \cap G_k = \emptyset$, which is absurd.

This proves $G_k = M$; therefore in (12) we can especially choose $U := G_k = M$. Then, by (13) and de Rham's theorem, M is the Riemannian product of the integral manifolds $M_1, \ldots, M_k$ of $E_1, \ldots, E_k$ through one point $p_0 \in M$, which are all simply connected and complete. As $h(X, Y) = 0$ for all $x \in \Gamma(E_i)$, $Y \in \Gamma(E_j)$ with $i, j \in \{1, \ldots, k\}$, $i \neq j$ (see (5)), f is according to the wellknown lemma of Moore ([8, p. 163]) a Riemannian product of isometric immersions $f_i: M_i \to \mathbb{R}^{n_i}$, $i = 1, \ldots, m$. From (2) and (11) follows that $f_1, \ldots, f_k$ are totally umbilical immersions with mean curvature normals of constant length, after an appropriate renumbering we obtain (i) resp. (ii) for $f_1, \ldots, f_k$. □

REMARK. We realize by the last lines of the above proof that the simply connectedness and the completeness of M are only needed for the global decomposition of M.

Proof of Theorem 2. The 'if' part: Let f_i, $i = 1, \ldots, k$, be as described in the theorem. Then b_i is (being a constant multiple of the Riemannian metric on M_i) parallel w.r.t. the Levi-Civita connection ∇^i of M_i. Since $\nabla_X Y = (\nabla^1_X Y_1, \ldots, \nabla^k_X Y_k)$ holds for all X, $Y = (Y_1, \ldots, Y_k) \in \Gamma(TM)$, we obtain using (3): $\nabla b = 0$.

The 'only if' part: Suppose $\nu(f)$ is flat and b is parallel. Let B be the field of self-adjoint endomorphisms on TM characterized by $b(v, w) = \langle Bv, w \rangle$ for all $p \in M$ and all $v, w \in T_pM$. As B is parallel, the eigenvalues of B are constant and the eigenspace bundles $F_i := \mathrm{kernel}(B - \lambda_i I)$ corresponding to the mutually distinct eigenvalues λ_i, $i = 1, \ldots, k$, are parallel. For that reason M is the Riemannian product of the integral manifolds $M_1, \ldots, M_k$ of $F_1, \ldots, F_k$ through one point $p_0 \in M$, which are all simply connected and complete, see [17, Satz 1]. For every point $p \in M$ and every orthonormal basis $(\xi_1, \ldots, \xi_{n-m})$ of $\nu_p(f)$ the formula $B_p = \sum_{i=1}^{n-m} A^2_{\xi_i}$ is valid (see (1)); on account of the flatness of $\nu(f)$ according to the Ricci equation B_p commutes with A_ζ for all $\zeta \in \nu_p(f)$ which implies $h(X, Y) = 0$ for all $X \in \Gamma(F_i)$, $Y \in \Gamma(F_j)$ with $i, j \in \{1, \ldots, k\}$, $i \neq j$. Moore's lemma [8, p. 163], and (3), (4) now prove the assertion on f. □

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