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GEOMETRIC RELATIONS AMONG VORONOI DIAGRAMS

ABSTRACT. Two general classes of Voronoi diagrams are introduced and, along with their modifications to higher order, are shown to be geometrically related. This geometric background, on the one hand, serves to analyse the size and combinatorial structure and, on the other, implies general and efficient methods of construction for various important types of Voronoi diagrams considered in the literature.

1. INTRODUCTION

Let G denote a finite subset of the d-dimensional Cartesian space R^d for $d \geqslant 1$, and let f be a function from $R^d \times G$ to R. The points in G (which are called *generators*, or sources or sites) and (the *distance function*) f impose a subdivision on R^d in a very natural way: For $p, q \in G$, let

$$\mathrm{dom}(p, q) = \{x \in R^d \mid f(x, p) < f(x, q)\}$$

be the *dominance* of p over q, and define

$$\mathrm{reg}(p) = \bigcap_{q \in G - \{p\}} \mathrm{dom}(p, q)$$

as the *region* of p among G (with respect to f). The set of all reg(p) for $p \in G$, along with the components that describe the boundary of reg(p) in an explicit manner, induce a diagram in R^d that is probably best known in the fields of discrete and computational geometry under the name *Voronoi diagram* of G and f (here $V(G, f)$ for short).

Up to now, many different types of Voronoi diagrams have been investigated from both the geometric and the algorithmic point of view. The purpose of this paper is to point out that a particular type of Voronoi diagram is very general in the sense that it can be brought, geometrically, into connection with various other types. This diagram is $V(\Gamma, \phi)$, for $\Gamma \subset R^d$ some set of generators, and for

$$\phi(x, p) = (x - p)^2 + \omega(p), \quad \omega(p) \in R$$

the *power function* (or the Laguerre distance). $V(\Gamma, \phi)$ has appeared in the literature under the names *power diagram* [3] (which we shall adopt and abbreviate by PD), Dirichlet cell complex [15], sectional Dirichlet tessellation [2], Laguerre–Voronoi diagram [12], and others. The generality inherent in the concept of PDs is already reflected by the following two phenomena: PDs

*Research supported by the Austrian Fond zur Foerderung der wissenschaftlichen Forschung.

Geometriae Dedicata **27** (1988), 65–75

(in R^d) are intimately related to such central concepts in discrete and computational geometry as convex hulls and hyperplane arrangements (in R^{d+1}), see [3]. Moreover, many cell complexes in R^d which *a priori* are not defined via the notation of Voronoi diagram (e.g. hyperplane arrangements, or simple complexes for $d \geqslant 3$) can be interpreted as PDs, see [4].

The linkage of PDs to various types of Voronoi diagrams is of particular interest. In the following sections, three general classes of Voronoi diagrams are introduced and shown to be geometrically related to PDs. For each class, several examples are given along with transforms that relate them to PDs. Due to the assertion below, PDs in R^d can be constructed efficiently, and there exist tight upper bounds on their size. This provides a general, dimension-independent construction method for Voronoi diagrams which has many practical applications in the low-dimensional case. The high-dimensional instances give rise to various problems in discrete geometry as well as in complexity theory.

PROPOSITION 1. *Let* $\text{size}_d(n)$ *denote the maximal number of faces of the convex hull CH of n points in* R^d, *and let* $\text{time}_d(n)$ *be the time complexity of constructing CH.*

[16]: $\text{size}_d(n) = O(n^{\lfloor d/2 \rfloor})$.
[17], [18]: $\text{time}_d(n) = O(\max\{\text{size}_d(n), n \log n\})$ *for d even, and* $\text{time}_d(n) = O(n^2 + \text{size}_d(n) \log n)$ *for d odd.*
[3], [12]: *For* $\Gamma \subset R^d$ *and* $|\Gamma| = n$, *a power diagram* $V(\Gamma, \phi)$ *realizes at most* $\text{size}_{d+1}(n)$ *components and can be constructed in* $O(\text{time}_{d+1}(n))$ *time and optimal* $O(\text{size}_{d+1}(n))$ *space.*

By construction of a diagram we mean the computation of a data structure which reflects the combinatorial structure realized by the individual components of the diagram. Since we are mainly concerned with geometrical properties of diagrams, we refrain from any implementation details for which we refer, e.g., to [3] and [12].

2. Affine Voronoi Diagrams

Let $V(G, f)$ be a Voronoi diagram in R^d. For two distinct generators $p, q \in G$, their *separator* is defined by

$$\text{sep}(p, q) = R^d - (\text{dom}(p, q) \cup \text{dom}(q, p)).$$

We shall term $V(G, f)$ *affine* if $\text{sep}(p, q)$ is a hyperplane of R^d, for any distinct $p, q \in G$. Intuitively speaking, affine diagrams are just those Voronoi diagrams whose regions are convex polyhedra. Whether $V(G, f)$ is affine only depends on the properties of the distance function f. Moreover, if $V(G, f)$ is affine and F is a strictly monotone function on the range of f, then $V(G, F \circ f)$ is also affine. The following general assertion can be stated:

THEOREM 1. *For any affine Voronoi diagram $V(G, f)$ in R^d there exist a set Γ of generators and a power function ϕ such that*

$$V(\Gamma, \phi) = V(G, f).$$

The result follows from the property $\text{sep}(p, q) \cap \text{sep}(q, r) \subseteq \text{sep}(p, r)$, that trivially holds for any f and distinct p, q, $r \in G$, and that is necessary and sufficient for a set of hyperplanes of R^d to be defined by the power function [3]. The remainder of this section will show that, via a quite different approach, Γ and ϕ can be calculated directly from G and f, in time $O(|\Gamma|)$, for any particular type of affine Voronoi diagram considered in the literature so far.

Let the *general quadratic-form distance Q* be defined by

$$Q(x, p) = (x - p)^T M(x - p) + w(p),$$

for $w(p) \in R$ and M a real and (w.l.o.g.) symmetric $(d \times d)$ -matrix. Unified treatments of the $V(G, Q)$ type of diagram are proposed in [9] by means of arrangements and in [11] by exploiting a transform that maps $V(G, Q)$ into a PD. The following approach is preferable because of its simplicity. Since, for distinct $p, q \in G$, their separator

$$\begin{aligned}\text{sep}(p, q) = \{x \in R^d \mid 2(q - p)^T Mx \\ = q^T Mq - p^T Mp + w(p) - w(q)\}\end{aligned}$$

is a hyperplane, we deduce from Theorem 1 that $V(G, Q)$ actually is a power diagram. In fact, it coincides with $V(\Gamma, \phi)$, for

$$\Gamma = \{\pi \mid \pi = Mp, p \in G\},$$

$$\phi(x, \pi) = (x - \pi)^2 + \omega(\pi),$$

$$\omega(\pi) = -\pi^2 + p^T Mp + w(p).$$

To keep this paper short, elementary analytic proofs are omitted throughout. Observe at this point that the power function ϕ depends only on the constants ω if Γ is fixed. Hence, $V(\Gamma, \phi)$ is determined if Γ and ω are. We may require that matrix M be non-singular. If M contains only $d - i$ independent rows $(1 \leqslant i \leqslant d - 1)$ then $\text{sep}(p, q)$ is parallel to the same i coordinate axes for all $p, q \in G$, $p \neq q$. In this case, $V(G, Q)$ is essentially the diagram in the lower-dimensional space R^{d-i}.

The two most prominent representatives of diagrams definable in R^d via the general quadratic-form distance Q are the closest-point Euclidean Voronoi diagram [19] ($M = I$, the identity matrix; $w(p) = 0, \forall p \in G$) and its furthest-point counterpart [19] ($M = -I$; $w(p) = 0, \forall p \in G$). For $M = I$ and arbitrary $w(p) \in R$, the power diagram ([12], [3]) is obtained. In the light of Theorem 1, this leads us to the following conclusion:

COROLLARY 1. *The class of diagrams definable by Q coincides with the class of power diagrams and thus with the class of affine Voronoi diagrams.*

This means that PDs are the simplest type of diagrams definable by Q that are universal in this class. PDs are straightforward generalizations of (the most fundamental) closest-point Euclidean Voronoi diagrams which themselves, however, are not universal in the above class. Though of minor importance, two additional types of diagrams in that class should be mentioned. For $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $w(p) = 0$, $Q(x, p)$ describes the area of the axis-parallel rectangle with diagonal vertices p and x. $V(G, Q)$ can now be used to find the largest empty axis-parallel rectangle among G [8]. Something of a mixture of the Euclidean closest- and furthest-point Voronoi diagram in R^d is induced by the Jordan matrix $M = \text{diag}[\sigma_1, \ldots, \sigma_d]$, for $\sigma_i \in \{-1, 1\}$ and $w(p) = 0$. This type is mentioned (but without applications) in [12] and [9].

From Corollary 1 it is obvious that the classes of diagrams created by Q and by the *affine distance*

$$A(x, p) = p^T x + w(p), \quad w(p) \in R,$$

respectively, are identical (see also [9]). Note that a power diagram $V(\Gamma, \phi) = V(G, A)$ (whose existence follows from Corollary 1) is given by

$$\Gamma = \{\pi \mid \pi = -\tfrac{p}{2}, p \in G\}, \qquad \omega(\pi) = -\pi^2 + w(p).$$

In [1] and [12] affine Voronoi diagrams of the form $V(G, Q) \cap h$, for a set G of generators and a particular hyperplane h in R^{d+1}, have received some attention. Let Γ and ϕ be such that the power diagram $V(\Gamma, \phi)$ coincides with $V(G, Q)$, and let π' denote the orthogonal projection of $\pi \in \Gamma$ onto h (h may be arbitrary). If we choose $\Gamma' = \{\pi' \mid \pi \in \Gamma\}$

and ϕ' such that

$$\omega'(\pi') = \omega(\pi) - (\pi - \pi')^2$$

then $V(\Gamma', \phi') = V(G, Q) \cap h$ holds.

This closes our consideration of affine Voronoi diagrams. In conclusion, all types of diagrams discussed above realize the same number of polyhedra as power diagrams, and can be constructed within the complexity bounds that hold for power diagrams (cf. Proposition 1). This unifies several previously known results on affine diagrams.

3. Affinely transformable Voronoi diagrams

One is tempted to believe that affine diagrams are the only Voronoi diagrams that are geometrically related to power diagrams. It is the purpose of this section to characterize the $V(G, f)$ type of diagrams that represent projected sections of PDs.

Let h_0 be the hyperplane $x_{d+1} = 0$ of R^{d+1}. proj(S) denotes the vertical (i.e. parallel to the x_{d+1}-axis) projection of $S \subset R^{d+1}$ onto h_0 and aff(S) denotes the affine hull of S. For some strictly increasing function $F: R \to R$ and distinct generators $p, q \in G$, we define

$$\mathrm{cone}_F(p) = \{(x, x_{d+1}) | x \in h_0, x_{d+1} = F(f(x, p))\}$$

and

$$\alpha_F(p, q) = \mathrm{aff}(\mathrm{cone}_F(p) \cap \mathrm{cone}_F(q)).$$

A diagram $V(G, f)$ in h_0 is termed *affinely transformable* if there exists an F such that $\alpha_F(p, q)$ is a hyperplane of R^{d+1} for any distinct $p, q \in G$. Moreover, we say that $V(G, f)$ can be *embedded* into a Voronoi diagram V in R^{d+1} if V realizes a region C for each $p \in G$ such that $\mathrm{proj}(C \cap \mathrm{cone}_F(p)) = \mathrm{reg}(p)$, for some F. By means of the terminology introduced, the following important connection can be established.

THEOREM 2. *A Voronoi diagram $V(G, f)$ in h_0 is affinely transformable iff it can be embedded into a power diagram in R^{d+1}.*

Proof. For $x \in h_0$ and $p \in G$, let x_p be the vertical projection of x onto $\mathrm{cone}_F(p)$. By definition of dom(p, q), $x \in \mathrm{dom}(p, q)$ is equivalent to $f(x, p) < f(x, q)$ and, since F is strictly increasing, to $F(f(x, p)) < F(f(x, q))$ which means that x_p is below (i.e. has smaller x_{d+1}-coordinate than) x_q. The definition of $\alpha_F(p, q)$ and the fact that it is a hyperplane of R^{d+1} now imply that $x_p \in \mathrm{hsp}(p, q)$ iff $x \in \mathrm{dom}(p, q)$, where hsp(p, q) is a fixed (open) halfspace of R^{d+1} bounded by $\alpha_F(p, q)$. Recall that $\mathrm{reg}(p) = \bigcap_{p \neq q} \mathrm{dom}(p, q)$ and consider the convex polyhedron $C = \bigcap_{p \neq q} \mathrm{hsp}(p, q)$. Then $x \in \mathrm{reg}(p)$ holds iff $x_p \in C$; that is, $\mathrm{reg}(p) = \mathrm{proj}(C \cap \mathrm{cone}_F(p))$.

In particular, $x \in \mathrm{sep}(p, q)$ is equivalent to x_p $(=x_q) \in \alpha_F(p, q)$, for $x \in h_0$. From $\mathrm{sep}(p, q) \cap \mathrm{sep}(q, r) \subseteq \mathrm{sep}(p, r)$ we now deduce $\alpha_F(p, q) \cap \alpha_F(q, r) \subseteq \alpha_F(p, r)$ which tells us that these hyperplanes are the separators for some power diagram V in R^{d+1} (cf. Theorem 1). Consequently, polyhedron C is one of the regions of V, which proves that $V(G, f)$ can be embedded into V. □

Theorem 2 shows that the regions of an affinely transformable diagram $V(G, f)$ can be obtained by projecting certain sections of the polyhedra of a power diagram $V(\Gamma, \phi)$ in one dimension higher. (Note, however, that not all polyhedra may contribute to regions of $V(G, f)$.) Provided that a suitable function F is known, that $\text{cone}_F(p)$ is computationally simple (in the sense that its intersection with a polyhedron C can be computed in time proportional to the number of faces of C), and that Γ and ϕ are available, the complexity of constructing $V(\Gamma, \phi)$ is an upper bound for that of $V(G, f)$. All three conditions are met if f is one of the following distance functions.

Let $p, q \in G \subset h_0$ and $x \in h_0$. The *additive Euclidean distance a* is given by

$$a(x, p) = |x - p| - w(p), \quad w(p) \geqslant 0.$$

Its separators, $\text{sep}(p, q)$, are hyperboloids in h_0 with rotational axes through p and q. If we choose $F(a) = a$ then $\text{cone}_F(p)$ is described by the equation $x_{d+1} = |x - p| - w(p)$, and thus is a cone with vertical rotational axis and apex $(p, -w(p))$. It is an easy analytic exercise to prove that $\alpha_F(p, q)$ is a hyperplane of R^{d+1} and that $V(G, a)$ can be embedded into the power diagram $V(\Gamma, \phi)$, for

$$\Gamma = \{\pi = (p, w(p)) \mid p \in G\}, \qquad \omega(\pi) = -2w(p)^2.$$

The planar instances of $V(G, a)$ have been treated, for example, in [2] from the mathematical, and in [14], [20], [10] from the algorithmic standpoint. The above relationship between $V(G, a)$ and PDs yields the only known construction method in higher dimensions. It is particularly efficient in R^3, where $V(G, a)$ can be shown to contain $\Theta(|G|^2)$ components and is constructed in $O(|G|^2 \log|G|)$ time.

The second class of diagrams considered here is generated by the *multiplicative Euclidean distance m*, for

$$m(x, p) = \frac{|x - p|}{w(p)}, \quad w(p) > 0,$$

which yields spheres in h_0 as separators. Setting $F(m) = 2m^2$ yields the equation $x_{d+1} = 2|x - p|^2/w(p)^2$ for $\text{cone}_F(p)$, which describes a paraboloid with apex $(p, 0)$ and focus $(p, w(p)^2/4)$. As a matter of fact, $V(G, m)$ in h_0 can be embedded into the power diagram $V(\Gamma, \phi)$ in R^{d+1}, if we take

$$\Gamma = \left\{\pi = \left(p, \frac{w(p)^2}{2}\right) \mid p \in G\right\}, \qquad \omega(\pi) = -\frac{w(p)^4}{4}.$$

Among others, [5], [2] [7], and [3] consider the one-, two-, and higher-dimensional instances of diagrams of the type $V(G, m)$, respectively. The algorithmic results obtained there via different approaches all are achieved by our general method of construction. In the most interesting cases of $G \subset R^1$, $G \subset R^2$ (and $G \subset R^3$), the method can be shown to be optimal to within a constant factor (and the factor $\log|G|$), respectively.

4. Generalizing to Higher Order

The well-known concept of modifying a Voronoi diagram $V(G, f)$ in R^d to higher order [19] involves generating its regions by subsets rather than by elements of G. For $S \subset G$, its *region* is defined by

$$\operatorname{reg}(S) = \bigcap_{p \in S,\, q \in G - S} \operatorname{dom}(p, q).$$

The subdivision of R^d induced by $\operatorname{reg}(S)$, for all $S \subset G$ with $|S| = k$, is commonly called the *order-k Voronoi diagram* of G (with respect to f), for short, k-$V(G, f)$. A very interesting and important fact is that the class of power diagrams in R^d is closed under order-k modification.

THEOREM 3. *For any order-k power diagram k-$V(G, f)$ in R^d there exist a set Γ_k of generators and a power function ϕ_k such that*

$$V(\Gamma_k, \phi_k) = k\text{-}V(G, f).$$

The existence of Γ_k and ϕ_k follows from a result in [4]. In the proof below (that is essentially distinct and shorter), Γ_k and ϕ_k are actually constructed.

Proof. We define Γ_k and ϕ_k as follows:

$$\Gamma_k = \left\{\pi_S = \sum_{p \in S} p \,|\, S \subset G, |S| = k\right\},$$

$$\omega_k(\pi_S) = \sum_{p \in S} (p^2 + w(p)) - \pi_S{}^2.$$

Let $T = (S - \{p\}) \cup \{q\}$, for $p \in S$ and $q \in G - S$. $\operatorname{dom}(p, q)$ denotes the dominance of p over q in $V(G, f)$, and $\operatorname{dom}_k(\pi_S, \pi_T)$ denotes the dominance of π_S over π_T in $V(\Gamma_k, \phi_k)$. By simple analytic calculations, $\operatorname{dom}(p, q) = \operatorname{dom}_k(\pi_S, \pi_T)$ holds. From

$$\begin{aligned}\operatorname{reg}(S) &= \bigcap_{p \in S,\, q \in G - S} \operatorname{dom}(p, q) \\ &= \bigcap_{T = (S - \{p\}) \cup \{q\}} \operatorname{dom}(p, q)\end{aligned}$$

we thus deduce

$$\operatorname{reg}(S) = \bigcap_{T = (S - \{p\}) \cup \{q\}} \operatorname{dom}_k(\pi_S, \pi_T)$$

which is identical to

$$\bigcap_{T \subset G, |T| = k} \operatorname{dom}_k(\pi_S, \pi_T) = \operatorname{reg}_k(\pi_S),$$

since only the former dominances contribute to the intersection. In other words, the region of S in k-$V(G, f)$ coincides with the region of π_S in $V(\Gamma_k, \phi_k)$ for each $S \subset G$ with $|S| = k$. This proves k-$V(G, f) = V(\Gamma_k, \phi_k)$. □

Let us now consider the consequences of Theorem 3. A serious shortcoming of the set $\Gamma_k \subset R^d$ of generators constructed in its proof is that Γ_k may contain a large number (in fact, $\Theta(|G|^k)$) of *redundant* generators, i.e. generators which define an empty polyhedron in $V(\Gamma_k, \phi_k) = k$-$V(G, f)$. Hence its algorithmic application becomes attractive only if we bypass the calculation of redundant generators in Γ_k. W.l.o.g., assume that G contains no redundant generator. One possible approach (which is detailed for the special case of Euclidean closest-point Voronoi diagrams in [6]) relies on the following:

OBSERVATION 1. *Let $\pi_S, \pi_T \in \Gamma_k$ be non-redundant. $\pi_{S \cup T}$ is non-redundant in Γ_{k+1} iff the intersection g of the closures of $\mathrm{reg}_k(\pi_S)$ and $\mathrm{reg}_k(\pi_T)$ has dimension $d - 1$.*

Proof. Observe first that g has dimension $d - 1$ iff S is of the form $(T - \{p\}) \cup \{q\}$, such that $\mathrm{aff}(g)$ is the hyperplane $\mathrm{sep}(p, q)$ of R^d. This is equivalent to the existence of a point $x \in g$ that satisfies

$$f(x, p) = f(x, q) < f(x, r) < f(x, t),$$

for all $r \in S \cap T$ and all $t \in G - (S \cap T)$. Because of $(S \cap T) \cup \{p, q\} = S \cup T$, the above can be rewritten as

$$\exists x \in R^d: f(x, r) < f(x, t), \quad \forall r \in S \cup T, \quad \forall t \in G - (S \cup T),$$

which just means that $x \in \mathrm{reg}_{k+1}(S \cup T) \neq \emptyset$ such that $\pi_{S \cup T}$ is non-redundant in Γ_{k+1}. □

Using Observation 1 and the formulas in Theorem 3, Γ_k and ϕ_k can be obtained in time proportional to the size of $V(\Gamma_{k-1}, \phi_{k-1})$ provided this diagram has been constructed. As is easily verified, we may take the input data G and f for Γ_1 and ϕ_1, respectively. This implies an iterative method for calculating Γ_k and ϕ_k whose time complexity is the same as for constructing all diagrams $V(\Gamma_i, \phi_i)$, for $i = 1, \ldots, k - 1$. It is well known that the maximal size of an order-i Voronoi diagram for n generators (and thus the cardinality of Γ_i) is increasing with i. Hence, if k is considered as a constant (as is done in many practical applications), the time needed for calculating Γ_k and ϕ_k is dominated by the time needed for constructing $V(\Gamma_k, \phi_k) = k$-$V(G, f)$ from Γ_k and ϕ_k. In conjunction with Proposition 1, this implies the following efficient result:

COROLLARY 2. *Let $m = R(n, k, d)$ denote the maximal number of regions realized by an order-k power diagram of n generators in R^d. If $k = O(1)$ then k-$V(G, f)$ can be constructed in $O(\mathrm{time}_{d+1}(m))$ time and $O(\mathrm{size}_{d+1}(m))$ space.*

Our investigations in Section 2 immediately imply that any affine order-k Voronoi diagram in R^d with $k = O(1)$ can be constructed within the bounds in Corollary 2. Note that the same bounds are also valid for affine order-$(n - k)$ diagrams with $k = O(1)$, since these diagrams are order-k diagrams for the same set of generators and the matrix M in the quadratic-form distance changed to $-M$. In addition, results for affinely transformable order-k diagrams can be obtained. Modifying the notion of $\text{cone}_F(p)$ introduced in Section 3 to order k by defining

$$\text{cone}_F(S) = \{(x, x_{d+1}) | x \in h_0, x_{d+1} = \max_{p \in S} F(f(x, p))\},$$

for $S \subset G$, $|S| = k$, yields the following order-k version of Theorem 2 (that can be proved in a similar manner).

THEOREM 2'. *$V(G, f)$ in h_0 is affinely transformable iff k-$V(G, f)$ can be embedded into an order-k power diagram k-$V(\Gamma, \phi)$ in R^{d+1}.*

Note that Γ can be chosen to contain no redundant generator iff G does (which we shall now assume). Thus, if $\text{cone}_F(S)$ is 'computationally simple' and Γ and ϕ are available then k-$V(G, f)$ (and thus also $(n - k)$-$V(G, f)$) can be constructed in $O(\text{time}_{d+2}(R(n, k, d + 1)))$ time by Corollary 2, for $|G| = n$ and $k = O(1)$.

This implies the only known method for constructing the order-k modifications of the diagrams $V(G, a)$ and $V(G, m)$ treated in Section 3. Its efficiency is difficult to analyse since not much is known on the size of order-k diagrams. While $R(n, k, d)$ is in $\Omega(n^{\lceil d/2 \rceil})$ and in $O(n^{d+1})$ ([3], [9]), it can be shown that the size of k-$V(G, a)$ (k-$V(G, m)$) is in $\Omega(n)$ and $O(n^3)$ ($\Omega(n^2)$ and $O(n^3)$) for $h_0 = R^2$ and in $\Omega(n^2)$ and $O(n^4)$ ($\Omega(n^2)$ and $O(n^4)$) for $h_0 = R^3$.

5. Concluding remarks

We have introduced the general classes of affine and affinely transformable Voronoi diagrams and have shown that they, as well as their order-k modifications, are related to a class that, in some sense, is universal among Voronoi diagrams: power diagrams. Power diagrams have been thoroughly investigated in the past, from both the mathematical and the algorithmic standpoint. Hence many results carry over to the above classes which include various important types of Voronoi diagrams.

Several questions are raised by our investigations. Which other types of Voronoi diagrams are, for example, affinely transformable? Do the diagrams induced by the L_p-metric (see, e.g., [13]) fit into this class? It can be shown that the diagrams induced by $f(x, p) = (|x - p|/w_1(p)) - w_2(p)$, $w_1(p)$, $w_2(p) > 0$, (which is a 'mixture' of the additive and multiplicative Euclidean distance) do not. Concerning order-k power diagrams, a fast method for (directly) calculating the non-redundant generators in Γ_k would considerably speed up the algorithms proposed for affine and affinely transformable order-k diagrams. In addition, only relatively weak upper and lower bounds on the size of order-k PDs are known for $2 \leqslant k \leqslant n-2$ and $d \geqslant 3$ (see, e.g., [3], [9]). However, as the efforts of many researchers have shown, establishing 'good' bounds seems to be very complicated.

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(Received, December 9, 1986; revised version, June 15, 1987)