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THE FERMAT PROBLEM IN MINKOWSKI SPACES

1. Introduction

The Fermat problem, in its simplest form, asks for a point in the Euclidean plane minimizing the sum of the distances to three given points. This problem is often referred to as 'Steiner's problem'. The reader will find in Kuhn ([9], [10]) an excellent history of the problem, including an explanation of why the problem should properly be termed the Fermat problem. Cockayne [2] has studied the generalization of the problem to more general metric spaces, including Minkowski spaces (finite dimensional real normed linear spaces). An analysis in the case of particular norms is given by Francis [5] and Hanan [7]. A discussion of the problem for Minkowski planes is also given by Melzak [12].

We concern ourselves here with the case where the ambient space is an n-dimensional Minkowski space whose unit ball is smooth and rotund. The solution of the Fermat problem in this case (that of finding a point that minimizes the sum of the distances to $n + 1$ given points) has an interesting geometric characterization involving certain simplices, which we call 'special simplices'. It turns out to be no more difficult to treat the 'k-point Fermat problem', which we consider in Section 3. In this case the solution is characterized in terms of 'special polyhedra' – those convex polyhedra such that the sum of the distances to the facets from any interior point is constant (see Theorem 2). The proof of Theorem 3 provides an extension to Minkowski spaces of an approach to the Fermat problem attributed first to Torricelli.

In Section 4 we consider various characteristic properties of special simplices. For example, a simplex in a Minkowski space is special when its Minkowskian incenter coincides with its centroid, or if its Minkowskian altitudes are all equal. A Euclidean simplex is special if and only if its facets have equal area. Theorem 7 in Section 4 generalizes an old theorem observed by Vecten, Fassbender, and others (see Kuhn [10]).

We introduce a generalization of the classical 'reflection principle' of Heron in Section 5 and show how it may be applied to the Fermat problem in Minkowski planes. We also show the connection with the 'critical angles' introduced by Cockayne [2].

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Section 6 points out the relationship between special triangles in a Minkowski plane and equilateral triangles in the dual space.

In Section 2 we introduce our notation and some background material from the theory of convex sets.

2. Preliminary results

We shall assume as background the material from the theory of convex sets found in Bonnesen and Fenchel [1] or Eggleston [4]. To establish our notation, we begin by briefly reviewing the basic results we need.

Let E^n be the Cartesian space R^n equipped with the usual Euclidean norm $|\cdot|$. By a *convex body* in E^n we mean a compact convex set having nonempty interior. We shall denote the *boundary* of a convex body K by bd K. If $K \subset E^n$ is a convex body having the origin o as an interior point, we let $f(K, x)$ and $h(K, x)$, $x \in E^n$ denote, respectively, the *gauge function* and the *support function* of K. The *polar dual* of K, denoted K^0, is characterized by the property that

$$(1) \qquad f(K^0, x) = h(K, x) \quad \text{and} \quad h(K^0, x) = f(K, x), \qquad x \in E^n.$$

In case u is a unit vector, $|u| = 1$, then $h(K, u)$ is just the distance from the origin to the supporting hyperplane of K with outward normal vector u.

Assume now that both $f(K, \cdot)$ and $h(K, \cdot)$ are continuously differentiable on $E^n \sim \{o\}$. Then the supporting hyperplane of K with outward unit normal vector v meets bd K in exactly one point, which we denote by $x(K, v)$. Letting grad h denote the gradient of h, we have (Bonnesen–Fenchel [1, p. 26]),

$$(2) \qquad \text{grad } h(K, v) = x(K, v), \qquad |v| = 1.$$

From (1) and (2) we have

$$(3) \qquad \text{grad } f(K, v) = \text{grad } h(K^0, v) = x(K^0, v) = |x(K^0, v)|\, u,$$

where $u = x(K^0, v)/|x(K^0, v)|$. With $r = |x(K^0, v)|$, note that since $ru = x(K^0, v) \in \text{bd } K^0$ we have

$$(4) \qquad rh(K, u) = rf(K^0, u) = f(K^0, ru) = 1,$$

where we have used the homogeneity of $f(K^0, \cdot)$. From (3) and (4) we obtain

$$(5) \qquad \text{grad } f(K, v) = u/h(K, u).$$

From the polar reciprocal relationship of K and K^0, it is easy to see that if u and v are unit vectors, then $u = x(K^0, v)/|x(K^0, v)|$ if and only if

$v = x(K, u)/|x(K, u)|$. Consequently, in (5), v is a unit vector such that the ray λv, $0 \leqslant \lambda < \infty$, intersects bd K in that point $x(K, u)$ where the supporting hyperplane has outward unit normal vector u. Now $\operatorname{grad} f(K, \cdot)$ is homogeneous of degree 0, since $f(K, \cdot)$ is homogeneous of degree 1, so $\operatorname{grad} f(K, \lambda y) = \operatorname{grad} f(K, y)$ if $\lambda > 0$, $y \in E^n \sim \{o\}$. In particular we have from (5), if $x \neq o$,

$$(6) \qquad \operatorname{grad} f(K, x) = u/h(K, u),$$

where u is the outward unit normal vector to the supporting hyperplane of K at the point where the ray **ox** intersects bd K.

We now apply these results to Minkowski spaces. An n-dimensional Minkowski space M^n may be viewed as the Cartesian space R^n equipped with a norm $\|\cdot\|$. The unit ball $Q = \{x \in M^n : \|x\| \leqslant 1\}$ for this norm is a centrally symmetric convex body with center at the origin. For the gauge function of Q we have simply

$$(7) \qquad f(Q, x) = \|x\|, \qquad x \in M^n.$$

The polar dual Q^0 is the unit ball of the dual space of M^n.

We shall assume from now on that both $f(Q, \cdot)$ and $h(Q, \cdot)$ are continuously differentiable (this is equivalent to assuming that Q is smooth and rotund; see Bonnesen–Fenchel [1]). Then from (6) we see that

$$(8) \qquad \operatorname{grad} f(Q, x) = u/h, \qquad x \neq o,$$

where u is the outward unit normal to the supporting hyperplane of Q at the point where the ray **ox** intersects bd Q, and h is the Euclidean distance from o to this hyperplane.

We shall be applying a slightly more general version of (8). Let $x_0 \in M^n$ and let $g(x) = f(Q, x - x_0) = \|x - x_0\|$, $x \in M^n$. Then

$$(9) \qquad \operatorname{grad} g(x) = u/h, \qquad x \neq x_0,$$

where u is the outward unit normal to the supporting hyperplane of $x_0 + Q$ at the point where the ray $\mathbf{x_0 x}$ intersects $\operatorname{bd}(x_0 + Q)$, and h is the Euclidean distance from x_0 to this hyperplane. From the central symmetry of Q we see also that $-u$ is the outward unit normal to the supporting hyperplane of $x + Q$ at the point where the ray $\mathbf{xx_0}$ intersects $\operatorname{bd}(x + Q)$, and h is the Euclidean distance from x to this hyperplane.

One of the simplest extremum problems in M^n is that of determining the minimum distance from a given point x_0 to a given hyperplane H not containing x_0. In other words, one wants to find min $\|x - x_0\|$, with x ranging over H. It is obvious that the minimum value λ exists and that H is the supporting hyperplane of $x_0 + \lambda Q$ at the minimizing point z. By our assumptions on Q we have $(x_0 + \lambda Q) \cap H = \{z\}$. We have that the line through x_0 and z is *Q-orthogonal* to H. In general, we say that a line l in M^n is Q-orthogonal to a hyperplane H if a supporting hyperplane of Q parallel to H intersects bd Q in a point x such that the line determined by o and x is parallel to l.

It is instructive, for our later purposes, to note that the preceding result is consistent with what one obtains using the method of Lagrange multipliers. For if we let $g(x) = \|x - x_0\| = f(Q, x - x_0)$, $x \in M^n$, then grad $g(x) = u/h$, as in (9). By the method of Lagrange multipliers, if $g(z) = \lambda$ is the minimum value of $g(x)$, with $x \in H$, then grad $g(z)$ is perpendicular (in the Euclidean sense) to H. Since grad $g(z) = u/h$, it follows that H is the supporting hyperplane of $x_0 + \lambda Q$ at z.

3. The Fermat problem in M^n

Let M^n be an n-dimensional Minkowski space whose unit ball Q satisfies the smoothness restrictions mentioned in the previous section. Let $x_1, x_2, \ldots, x_k$ be k given points in M^n. The problem of determining a point z minimizing the sum of the distances to the given points we shall call the *k-point Fermat problem*. In other words, with $g(x) = \Sigma_{i=1}^k \|x - x_i\|$, we want to minimize $g(x)$, $x \in M^n$. Note that g is a convex function, being the sum of convex functions, and for each $\alpha > 0$ there exists a ball B centered at o such that $g(x) > \alpha$ for all $x \notin B$. It follows that g attains a minimum value. If $g(z)$ is the minimum value of g and $z \neq x_1, x_2, \ldots, x_k$, we are interested in giving a geometric characterization of z.

Assuming $g(z) = \min\{g(x) : x \in M^n\}$ and $z \neq x_1, x_2, \ldots, x_k$, we have grad $g(z) = o$. From the remarks following (9), this gives

$$\sum_{i=1}^{k} (u_i/h_i) = o, \tag{10}$$

where u_i is the outward unit normal vector to the supporting hyperplane of $z + Q$ at the point where the ray $\mathbf{zx}_i$ intersects bd$(z + Q)$, and h_i is the Euclidean distance from z to this hyperplane.

To clarify the geometric meaning of relation (10), we introduce the following definition.

DEFINITION. Let P be a convex polyhedron with k facets having outward unit normal vectors $u_1, u_2, \ldots, u_k$. We shall say that P is a *special polyhedron* if $\Sigma_{i=1}^k (u_i/h(Q, u_i)) = o$.

Obviously any convex polyhedron whose facets are parallel to the facets of a special polyhedron is again a special polyhedron.

Since $h_i = h(Q, u_i)$ in (10), we obtain the following theorem concerning the solution of the k point Fermat problem in M^n.

THEOREM 1. *Let z minimize the sum of the distances to k given points x_1, $x_2, \ldots, x_k$ in M^n, and suppose $z \neq x_1, x_2, \ldots, x_k$. Let P be the convex polyhedron whose facets are determined by the supporting hyperplanes of $z + Q$ at the points where the rays $\mathbf{zx}_i$ intersect* $\mathrm{bd}(z+Q)$. *Then P is a special polyhedron.*

Note that the theorem implies that a polyhedron having bounding hyperplanes $H_1, \ldots, H_k$ such that $\mathbf{zx}_i$ is Q-orthogonal to H_i, $i = 1, \ldots, k$, is a special polyhedron.

The following characteristic property of special polyhedra generalizes a result proved by Gál [6]. See also [13].

THEOREM 2. *A convex polyhedron P in M^n is a special polyhedron if and only if the sum of the Minkowskian distances to the hyperplanes determining the facets of P is the same for all points inside P.*

Proof. Since the properties in question are translation invariant, we may assume that o is interior to P. Let $H_1, H_2, \ldots, H_k$ be the hyperplanes determining the facets of P and $u_1, u_2, \ldots, u_k$ their respective outward unit normal vectors. Let $x_i = x(Q, u_i)$, as defined in Section 2, and let r_i be the Euclidean distance from o to x_i. Let $h_i = h(Q, u_i)$. For any point $x \in P$, let d_i be the Euclidean distance from x to the point where the ray originating at x and parallel to $\mathbf{ox}_i$ intersects H_i. Then the Minkowskian distance from x to H_i is d_i/r_i. Denote by p_i the Euclidean distance from x to H_i. From similar triangles we obtain $d_i/r_i = p_i/h_i$. Thus the sum of the Minkowskian distances from x to the bounding hyperplanes of P is

$$(11) \qquad \sum_{i=1}^{k} (d_i/r_i) = \sum_{i=1}^{k} (p_i/h_i).$$

But $h_i = x_i \cdot u_i$ and $p_i = (y_i - x_i) \cdot u_i$, where $y_i = (\mathbf{ox}_i) \cap H_i$. Since $y_i = \lambda_i x_i$, where λ_i is the Minkowskian distance from o to H_i, we have $p_i = \lambda_i h_i - x \cdot u_i$. Using (11), this gives

$$(12) \qquad \sum_{i=1}^{k} (d_i/r_i) = \sum_{i=1}^{k} \lambda_i - x \sum_{i=1}^{k} (u_i/h_i) = s - x \sum_{i=1}^{k} (u_i/h_i),$$

where $s = \Sigma_{i=1}^{k} \lambda_i$ is the sum of the Minkowskian distances from o to the bounding hyperplanes of P. Consequently, the sum of the Minkowskian distances from x to the bounding hyperplanes of P is a constant $s - \lambda$ if and only if $x \cdot \Sigma\ (u_i/h_i)$ is a constant λ for all $x \in P$. The choice $x = o$ shows that this constant could only be $\lambda = 0$, in which case $\Sigma\ (u_i/h_i)$ would necessarily have to equal o. Thus the sum is constant if and only if P is a special polyhedron, as we wanted to prove. □

Theorem 2 enables us to show that the necessary condition for a minimum given in Theorem 1 is also sufficient.

THEOREM 3. *Let $x_1, x_2, \ldots, x_k$ be given points in M^n and $z \in M^n$, with $z \neq x_1, x_2, \ldots, x_k$ and z not on any line determined by points of $\{x_1, \ldots, x_k\}$. Let P be the convex polyhedron whose facets are determined by the supporting hyperplanes of $z + Q$ at the points where the rays $\mathbf{zx}_i$ intersect* $\mathrm{bd}(z + Q)$. *If P is a special polyhedron, then z minimizes the sum of the distances to $x_1, x_2, \ldots, x_k$.*

Proof. Let H_i be the hyperplane containing x_i such that $\mathbf{zx}_i$ is Q-orthogonal to H_i. Then $H_1, \ldots, H_k$ bound a convex polyhedron P^* with facets parallel to corresponding facets of P; hence P^* is a special polyhedron. For $x \in M^n$ and H any hyperplane, let $d(x, H)$ denote the Minkowskian distance from x to H. Then, using Theorem 2, for any $x \in P^*$ we have

$$(13) \qquad \sum_{i=1}^{k} \|x - x_i\| \geqslant \sum_{i=1}^{k} d(x, H_i) = \sum_{i=1}^{k} d(z, H_i) = \sum_{i=1}^{k} \|z - x_i\|.$$

This shows that z minimizes $\Sigma\ \|x - x_i\|$ for x ranging over P^*. Now the considerations leading to (12) show that in fact the sum of the *signed* Minkowskian distances from any $y \in M^n$ to the bounding hyperplanes of P^* is constant. Thus

$$\sum d(y, H_i) \geqslant \sum \pm\, d(y, H_i) = \sum d(x, H_i),$$

if $y \notin P^*$ and $x \in P^*$. It follows that z minimizes $\Sigma\ \|x - x_i\|$ for x ranging over M^n, as we wanted to prove. □

In the special case where $M^n = E^n$, so Q is the ordinary Euclidean unit ball, a convex polyhedron P is special if and only if $\Sigma\ u_i = o$, where the u_i are the outward unit normals to the facets. We then obtain from Theorems 1 and 3 the well-known result (see Kuhn [9]) for the k-point Fermat problem in E^n. Namely, suppose $x_1, \ldots, x_k$ are given points in E^n, and $z \neq x_1, \ldots, x_k$. Then $\Sigma\ |x - x_i|$ is minimized for $x = z$ if and only if $\Sigma\ u_i = o$, where u_i is the unit vector in the direction of $\mathbf{zx}_i$, $i = 1, 2, \ldots, k$.

The proof of Theorem 3 is a generalization of Torricelli's treatment of the classical Fermat problem for three points in the Euclidean plane. See Honsberger [8] for an excellent elementary account of this.

4. Special simplices

Let S be a simplex in M^n. The *Q-insphere* of S is the largest homothetic copy of Q contained in S. If $x + rQ$ is the Q-insphere of S, then x is the *Q-incenter* and r the *Q-inradius* of S.

By Theorem 2, S is a special simplex precisely when the sum of the Minkowskian distances to its facets is the same for all points in S. The next theorem gives a characterization of special simplices in terms of the Q-insphere.

THEOREM 4. *A simplex $S \subset M^n$ is a special simplex if and only if its Q-incenter coincides with its centroid.*

Proof. Let $H_0, H_1, \ldots, H_n$ be the hyperplanes determining the facets of S and $u_0, u_1, \ldots, u_n$ their respective outward unit normal vectors. Let x be the Q-incenter of S and p_i the Euclidean distance from x to H_i. Note that $p_i = rh(Q, u_i)$, $i = 0, \ldots, n$, where r is the Q-inradius of S. For each i, let a_i be the vertex of S opposite H_i, and set $b_i = a_i - x$. Then we have $b_i \cdot u_j = p_j$, if $i \neq j$, with $i, j = 0, \ldots, n$. It follows that

$$(14) \qquad \left(\sum_{i=0}^{n} b_i\right) \cdot (u_j/p_j) = n + (b_j \cdot u_j)/p_j = b_j \cdot \sum_{i=0}^{n} (u_i/p_i).$$

Since $p_i = rh(Q, u_i)$, we have $\Sigma\, u_i/p_i = o$ if and only if $\Sigma\, (u_i/h(Q, u_i)) = o$, which by (14) holds exactly when $\Sigma\, b_i = o$. But $\Sigma\, b_i = o$ if and only if $x = (\Sigma\, a_i)/(n + 1) =$ the centroid of S. Thus we have our required result that S is a special simplex if and only if the Q-incenter x coincides with the centroid. □

Let $H_0, H_1, \ldots, H_n$ be the hyperplanes determining the facets of a simplex $S \subset M^n$. Let a_i be the vertex of S opposite H_i. Then $d(a_i, H_i)$ is a *Q-altitude* of S, where $d(x, H)$ denotes the Minkowskian distance from $x \in M^n$ to the hyperplane H. We have the following characterization of special simplices in terms of their Q-altitudes.

THEOREM 5. *A simplex $S \subset M^n$ is a special simplex if and only if all its Q-altitudes are equal.*

Proof. It clearly suffices to prove the theorem in the case where the Q-insphere of S is Q itself. With the notation as above, let h_i be the Euclidean distance from the vertex a_i of S to the supporting hyperplane H_i, and let p_i be the Euclidean distance from the Q-incenter o to H_i. Then the ith Q-altitude of S is $d(a_i, H_i) = h_i/p_i$. Let V be the volume of S and V_i the volume of the simplex whose vertices are x and those vertices a_j of S with $j \neq i$. Then $d(a_i, H_i) = h_i/p_i = V/V_i$. It follows that the Q-altitudes of S are equal if and only if $V_0 = V_1 = \cdots = V_n$, which is equivalent to saying that the barycentric coordinates of o relative to S are equal, which happens precisely when o is the centroid of S. Thus, by Theorem 4, the Q-altitudes of S are equal if and only if S is a special simplex, as we wanted to prove. □

Easy examples show that, for $n \geqslant 3$, special simplices in E^n need not be regular simplices. In E^3 they are the so-called isosceles tetrahedra (see Court [3]). The following is a simple characterization of special simplices in Euclidean spaces.

THEOREM 6. *A simplex in E^n is a special simplex if and only if all its facets have the same area.*

Proof. Let A_i be the area of the facet of S opposite the vertex a_i and let h_i be the Euclidean distance from a_i to the hyperplane determining that facet. Since $h_i A_i = nV$, where V is the volume of S, we see that $A_0 = A_1 = \cdots = A_n$ if and only if $h_0 = h_1 = \cdots = h_n$. Thus the required result follows from Theorem 5 with $Q =$ the Euclidean unit ball. □

It is, of course, true that any convex polytope in E^n whose facets have equal area is a special polyhedron, although the converse is not true. It might be worth remarking that if a convex polytope has facets of areas $A_1, \ldots, A_k$ with outward unit normal vectors $u_1, \ldots, u_k$ respectively, then (see Bonnesen–Fenchel [1, pp. 118–119])

$$\sum_{i=1}^{k} A_i u_i = o.$$

Thus for any $j = 1, 2, \ldots, k$ we have

$$A_j \sum_{i=1}^{k} u_i = \sum_{i=1}^{k} (A_j - A_i) u_i. \tag{15}$$

If P is a simplex, then $\{u_i : i \neq j\}$ is a linearly independent set of vectors, so the right-hand side of (15) is o if and only if $A_i = A_j$ for all i. It follows then that $\Sigma\, u_i = o$ if and only if all facets of the simplex have equal area, giving another proof of Theorem 6.

The following theorem gives a generalization of a result noticed by Vecten, Fassbender, and others (see Kuhn [10]) to n-dimensional Minkowski spaces.

THEOREM 7. *Let z minimize the sum of the distances to $n + 1$ given points $x_0, x_1, \ldots, x_n$ in M^n, with $z \neq x_0, x_1, \ldots, x_n$. Suppose the hyperplanes H_0, $H_1, \ldots, H_n$ which pass through $x_0, x_1, \ldots, x_n$ respectively, and such that each ray $\mathbf{zx}_i$ is Q-orthogonal to H_i, bound a simplex S. Then S has the maximum Q-altitude among all special simplices with facets passing through $x_0, x_1, \ldots$, x_n.*

Proof. By the remark following Theorem 1, S is a special simplex which, by Theorems 2 and 5, has all its Q-altitudes equal to $\Sigma\ \|z - x_i\|$. For any other special simplex with facets passing through $x_0, \ldots, x_n$, the sum of the Minkowskian distances from z to its bounding hyperplanes is at most $\Sigma\ \|z - x_i\|$, so its Q-altitudes are at most this large. This completes the proof. □

5. Heron's problem and the Fermat problem in M^2

We outline in this section an approach to the Fermat problem in Minkowski planes that proceeds from the solution of 'Heron's problem', that of minimizing the length of a path joining two points on one side of a line and intersecting the line.

Let M^2 be a Minkowski plane whose unit disk Q satisfies the smoothness restrictions set forth in Section 2. Let L be a straight line in M^2 and x_1, x_2 any two given points lying in one of the open halfplanes determined by L. The analogue of Heron's problem is to determine $\min(\|x - x_1\| + \|x - x_2\|)$ with x varying over L. Since $g(x) = \|x - x_1\| + \|x - x_2\|$ is convex on L, and unbounded as x tends to ∞ in either direction along L, a minimum value $g(z)$ exists. The method of Lagrange multipliers implies that grad $g(z)$ is perpendicular (in the Euclidean sense) to L. The relation (9) then gives

$$(16) \qquad (u_1/h_1) + (u_2/h_2) = \lambda u,$$

where λ is real, u is the unit normal vector of L pointing into the halfplane occupied by x_1 and x_2, u_i is the unit normal vector to $z + Q$ at the point where $\mathbf{zx}_i$ intersects $\mathrm{bd}(z + Q)$, and $h_i = h(Q, u_i)$.

Equation (16) implies that the component of $(u_1/h_1) + (u_2/h_2)$ parallel to L is zero. If L_i is the (by our assumptions on Q) unique supporting line of $z + Q$ at the point where $\mathbf{zx}_i$ intersects $\mathrm{bd}(z + Q)$, and $y_i = L_i \cap L$, then it is straightforward to check that z is the midpoint of the line segment joining y_1 to y_2. Thus we have the following analogue of the 'reflection property' in relation to Heron's problem.

THEOREM 8. *Let x_1, x_2 belong to one of the open halfplanes determined by a line L in M^2, and let $z \in L$ minimize $\|x - x_1\| + \|x - x_2\|$ as x ranges over L. Then z is the midpoint of the line segment joining $L_1 \cap L$ to $L_2 \cap L$, where L_i is the supporting line of $z + Q$ at the point where the ray $\mathbf{zx}_i$ intersects* $\text{bd}(z + Q)$.

Note that if Q is the usual Euclidean unit disk, then Theorem 8 gives the usual 'reflection property', namely that the rays $\mathbf{zx}_1$ and $\mathbf{zx}_2$ make equal angles with L. Of course, this also follows directly from (16), since $h_1 = h_2$ in the Euclidean case. Indeed, we obtain the usual reflection property whenever L is perpendicular to a symmetry axis of Q.

We give now a sketch of how Theorem 8 leads to the solution of the three-point Steiner problem in M^2; that is, Theorem 1 in case $k = 3$ and $n = 2$. Let x_1, x_2, x_3 be noncollinear points in M^2 and suppose z minimizes $\sum \|x - x_i\|$ as x ranges over M^2. Suppose $z \neq x_1, x_2, x_3$. Let C be a Minkowski circle centered at x_3 and passing through z. In other words, $C = \text{bd}(x_3 + \lambda Q)$, where λ is chosen so that $z \in C$. Let $\sigma = \|z - x_1\| + \|z - x_2\|$ and let E be the Minkowski ellipse $E = \{x : \|x - x_1\| + \|x - x_2\| = \sigma\}$. The convexity of the function $g(x) = \|x - x_1\| + \|x - x_2\|$, $x \in M^2$, implies that E bounds a convex set, and it is not difficult to check that the interiors of the convex sets bounded by C and E are disjoint (otherwise the minimizing property of z is contradicted). Thus there is a separating line L passing through z, with C in one halfplane and E in the other. Since L is tangent to E at z, the minimum of $\|x - x_1\| + \|x - x_2\|$ with x varying over L occurs when $x = z$. Consequently, we may apply Theorem 7 to x_1, x_2, z, and L. Hence, if L_i is the supporting line of $z + Q$ at the point where $\mathbf{zx}_i$ intersects $\text{bd}(z + Q)$, $i = 1, 2$, then z is the midpoint of the line segment joining $L_1 \cap L$ to $L_2 \cap L$. If L_3 is the supporting line of $z + Q$ at the point where $\mathbf{zx}_3$ intersects $\text{bd}(z + Q)$, it is easy to see that L_3 is parallel to L. Thus, if T is the triangle bounded by L_1, L_2, L_3, we see that z lies on the median of T through $L_1 \cap L_2$. Since the preceding argument could have been carried out for any pair of $\{x_1, x_2, x_3\}$, we conclude that z lies on all the medians of T; that is, z is the centroid of T. Since z is also the Q-incenter of T, we finally have the result we want, namely that T is a special triangle.

If z minimizes the sum of the distances to x_1, x_2, x_3 in M^2, and $z \neq x_1$, x_2, x_3, then the angle formed by the rays $\mathbf{zx}_1$ and $\mathbf{zx}_2$ is a *critical angle* as defined by Cockayne [2], as are the other two angles around z. We can verify that Cockayne's criterion for critical angles holds, using the properties of special triangles as follows.

Let T be the special triangle bounded by the supporting lines L_1, L_2, L_3 of $z + Q$ as in the preceding paragraph. Let $y_i = \mathbf{zx}_i \cap L_i$, $i = 1, 2, 3$. We want to show that if y is strictly between the rays $\mathbf{zx}_1$ and $\mathbf{zx}_2$ and inside T, then

$$\|y - y_1\| + \|y - y_2\| + \|y - z\| > \|z - y_1\| + \|z - y_2\|. \tag{17}$$

If $d(y, L_i)$ is the Minkowskian distance from y to L_i, we have

$$\begin{aligned} \|y - y_1\| + \|y - y_2\| + \|y - z\| + \|z - y_3\| & \\ \geqslant \|y - y_1\| + \|y - y_2\| + \|y - y_3\| & \\ > \sum_{i=1}^{3} d(y, L_i) = \sum_{i=1}^{3} d(z, L_i) = \sum_{i=1}^{3} \|z - y_i\|. & \end{aligned} \tag{18}$$

The required inequality (17) now follows by cancellation of the term $\|z - y_3\|$ in (18).

6. Special triangles

By Theorem 6, the special triangles in E^2 are exactly the equilateral triangles, but this is not the case in general Minkowski planes. However, there is a certain correspondence between special triangles in M^2 and equilateral triangles in the dual space.

Let Q be the unit disk of M^2 and T a special triangle circumscribed about Q. Let L_1, L_2, L_3 be the supporting lines of Q that determine the sides of T and u_1, u_2, u_3 their respective outward unit normal vectors. Then $\Sigma\ (u_i/h(Q, u_i)) = o$. The unit disk for the dual space of M^2 is Q^0, the polar dual of Q. From the discussion in Section 2 one sees that for any unit vector u we have $u/h(Q, u) = x(Q^0, v)$, for some v. Thus corresponding to u_1, u_2, u_3 are three points $y_i = x(Q^0, v_i) \in \text{bd } Q^0$, $i = 1, 2, 3$, such that

$$y_1 + y_2 + y_3 = o.$$

The points y_1, $-y_3$, y_2, $-y_1$, y_3, $-y_2$ are the vertices of the affine image of a Euclidean regular hexagon H inscribed in Q^0 (see Laugwitz [11]), which is partitioned into six equilateral triangles (in the metric determined by Q^0) by the rays from o through the vertices.

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