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TWO COUNTEREXAMPLES CONCERNING TRANSVERSALS FOR CONVEX SUBSETS OF THE PLANE

A family $\mathscr{F}$ of subsets of the plane is said to have property T if the family admits a common transversal, that is, if there is a straight line which intersects every member of $\mathscr{F}$. The family $\mathscr{F}$ is said to have property $T(n)$ if every n-membered subfamily of $\mathscr{F}$ has property T.

The interest in common transversals stems from a more general question raised by Vincensini (cf. [1]), which, in terms of transversals for plane convex sets asks if there is a number k such that $T(k)$ implies T. For arbitrary families $\mathscr{F}$ of compact convex sets such a 'stabbing number' does not exist (see the examples cited in [3]). However, with additional restrictions on the relative positions or shapes of the members of $\mathscr{F}$, several positive results have been obtained. The earliest, due to Santaló, is that $T(6)$ implies T for families of parallelograms with parallel edges (cf. [3]). In the special case where $\mathscr{F}$ consists of disjoint translate of a parallelogram, Grünbaum [2] has shown that $T(5)$ implies T. Two conjectures [1, 2] that have been outstanding for some time are that $T(5)$ implies T for disjoint families of congruent squares, and that $T(6)$ implies T for disjoint families of congruent compact convex sets. (By a disjoint family it is meant a family whose members are mutually disjoint.)

In this note we provide counterexamples to both conjectures.

1. *Given any natural number k, $k > 5$, there exists a disjoint family $\mathscr{F}$ consisting of k congruent squares such that $\mathscr{F}$ has property $T(5)$ but does not have property $T(6)$.*

To construct a family for the case where $k = 6$, four congruent squares and two transversals are placed in the configuration of Figure 1. (If a family $\mathscr{F}$ with more than six squares is desired, one may simply begin with squares S_1 and S_2 at a greater distance from each other.)

Then, straight lines L_1 and L_2 are drawn with L_1 passing through the vertices P_3 and P_4 of squares S_3 and S_2 respectively, and with L_2 passing through the vertices P_1 and P_2 of S_1 and S_3 respectively. Neither L_1 nor L_2 is a transversal for the family of four squares of Figure 1.

Next, two additional congruent squares, S_5 and S_6 are placed so that the following conditions are fulfilled:

S_5 (resp., S_6) has one vertex on the line L_2 (L_1) between the squares S_3 and S_4, S_5 (S_6) is disjoint from L_5 (L_1) and has an edge parallel to L_5 (L_1). When

Geometriae Dedicata **9** (1980) 461–465. 0046–5755/80/0094–0461 $00.75

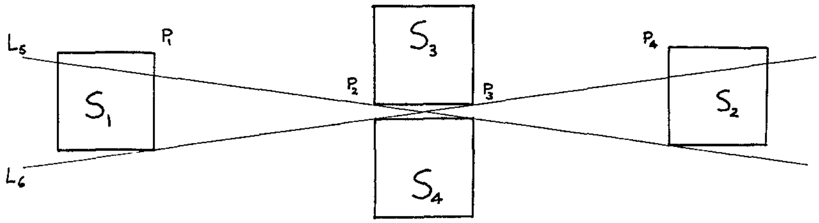

Fig. 1.

S_1 and S_2 are far enough apart, this construction will always be possible. The insertion of S_5 is shown in Figure 2. With all six squares in position, the configuration is roughly as depicted in Figure 3. For the correct configuration in regions near the vertices P_1 and P_2, Figure 2 should be examined.

The configuration that has been constructed, of six squares $S_1, S_2, \ldots, S_6$, and four straight lines L_1, L_2, L_5, L_6, has the property that L_i, $i = 1, 2, 5, 6$ misses the square S_i but intersects the remaining five squares. It is also clear that there are horizontal straight lines, L_3 and L_4 such that L_3 (resp., L_4) misses S_3 (S_4) but intersects the remaining five squares. Thus, the family $\{S_1, S_2, \ldots, S_6\}$ has property $T(5)$.

To see that it does not have property T, note that if L is a straight line whose direction lies strictly within the acute angle formed by L_2 and L_5 then L misses either S_3 or S_5 (see Figure 2). If L is a straight line whose direction lies strictly within the acute angle formed by L_5 and L_6, then L misses either

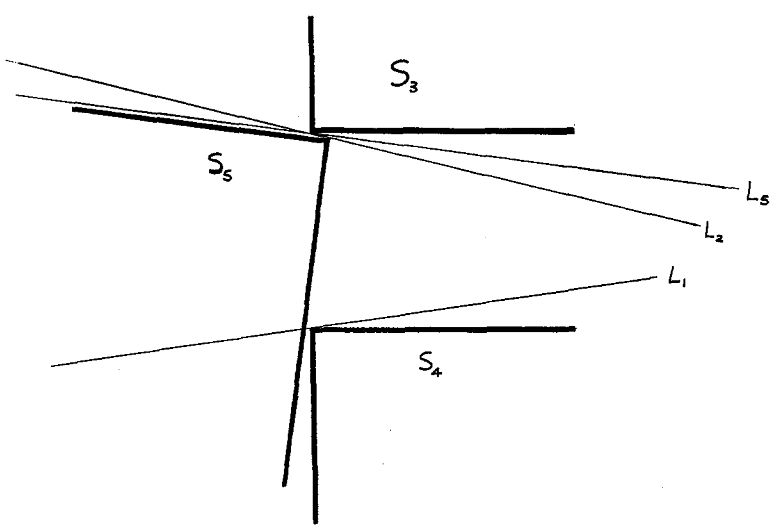

Fig. 2.

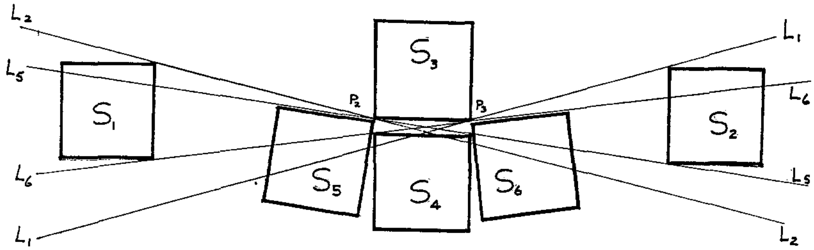

Fig. 3.

S_3 or S_4. If L is a straight line whose direction lies within the acute angle formed by L_6 and L_1, then L misses either S_3 or S_6 (this is the 'mirror image' of the situation involving L_2 and L_5). Also, it is clear that if L is a straight line whose direction lies within the obtuse angle formed by L_2 and L_1, then L misses either S_1 or S_2. Finally, any line that is parallel to either L_1, L_2, L_5, or L_6 must miss one of the sets. (There are straight lines parallel to both L_1 and L_2 which strictly separate S_1 and S_2. Some straight line parallel to L_5 strictly separates S_3 and S_5 and some straight line parallel to L_6 strictly separates S_3 and S_6). We have shown that any straight line in the plane misses at least one of the sets, that is, $\mathscr{F}$ does not have property T.

As indicated earlier, if a larger family of disjoint congruent squares is desired, the construction may be repeated with S_1 and S_2 farther apart. Then, additional squares, each a parallel translate of S_1, may be inserted between S_1 and S_5 and between S_6 and S_2. The resulting family will have property $T(5)$, but will fail to have property $T(6)$.

2. *If $\mathscr{F}$ is a finite family of compact sets $A_1, A_2, \ldots, A_n$ in the plane, which does not possess property T, then there exists a real number δ, $\delta > 0$, such that the family $A_1 + \delta B^2, A_2 + \delta B^2, \ldots, A_n + \delta B^2$ also lacks property T (B^2 the closed unit disc).*

The proof is by standard compactness arguments.

3. *Given any natural number $n \geqslant 3$ there exists a disjoint family $\mathscr{F}$ of n congruent rectangles such that $\mathscr{F}$ has property $T(n-1)$ but fails to have property T.*

We shall first show how to construct a disjoint family of n congruent straight line segments. Through a point P in the plane pass $n - 1$ straight lines which make equal angles with each other (that is, the angles between successive lines is $\pi/(n-1)$). Treating the $n - 1$ straight lines as $2(n-1)$ rays emanating from P, label one of the rays R_0 and, continuing in a counterclockwise direction, label the successive rays $R_1, R_2, R_3, \ldots, R_{n-1}$. Then give R_{n-2} and R_{n-1} additional labels Q_1 and Q_2 respectively, and, continuing

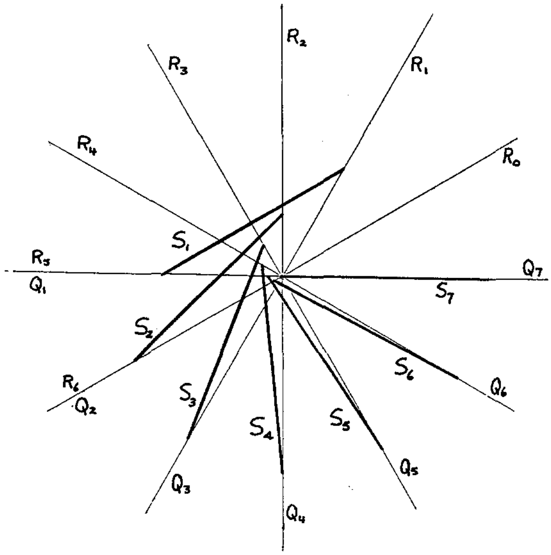

Fig. 4.

in a counterclockwise direction, label the remaining rays $Q_3, Q_4, \ldots, Q_n$. Then all rays have been labelled, with two rays having double labels. Also, for $i = 1, 2, \ldots, n-1$, the rays R_i and Q_i form an angle of $(n-3)\pi/(n-1)$. Let S_1 be a straight line segment with endpoints on R_1 and Q_1, with S_1 parallel to R_0. Having constructed S_i, $1 \leqslant i < n-1$, the segment S_{i+1} is constructed so that it is equal in length to S_1, has one endpoint on R_{i+1} between $S_i \cap R_{i+1}$ and P, and the other endpoint on Q_{i+1}. The last straight line segment, S_n, is constructed so that it is contained in the ray Q_n and has one endpoint at P. Figure 4 illustrates the configuration when $n = 7$.

With this construction, the straight line which contains the ray Q_i, $i = 2, 3, \ldots, n$, intersects all segments except the segment S_{i-1}. Also, the straight line passing through $S_1 \cap Q_1$ and $S_{n-1} \cap R_{n-1}$ intersects all segments except S_n, and so the family $\mathscr{F} = \{S_1, S_2, \ldots, S_n\}$ has property $T(n-1)$.

To see that $\mathscr{F}$ does not have property T, let L be any straight line in the plane. Then, for some $i = 1, 2, \ldots, n-1$, L is either parallel to R_i or L has a direction which lies strictly within the acute angle formed by R_i and R_{i-1}. But, for $i = 1, 2, \ldots, n-2$, such a straight line L would have a translate which separates S_{i+1} and S_n, and if $i = n-1$, some translate of L would separate S_1 and S_n. That is, L cannot intersect each set S_i, $i = 1, 2, \ldots, n$, showing that $\mathscr{F}$ does not have property T.

By the previous lemma it is now clear that there also exists a family of n congruent rectangles which has property $T(n-1)$ but fails to have property T.

Finally, we would mention that in [1] and [2] it was also conjectured that $T(5)$ implies T for disjoint translates of an arbitrary convex body. It appears that neither of our counterexamples can be modified to produce a counterexample for this conjecture.

This research was supported in part by NSERC Grant A-8761.

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(Received October 27, 1976)