FRANKI DILLEN*

EQUIVALENCE THEOREMS IN AFFINE DIFFERENTIAL GEOMETRY

ABSTRACT. In this paper we establish an affine equivalence theorem for affine submanifolds of the real affine space with arbitrary codimension. Next, this theorem is used to prove the classical congruence theorem for submanifolds of the Euclidean space, and to prove some results on affine hypersurfaces of the real affine space.

AMS SUBJECT CLASSIFICATION (1980). 53A 15.

1. INTRODUCTION

Some of the most investigated problems in differential geometry are the 'congruence problems' in (pseudo-)Riemannian differential geometry and the 'equivalence problems' in affine differential geometry. These can be formulated generally as follows: 'given two isometric (resp. affine) immersions f and f' of a (pseudo-)Riemannian (resp. affine) manifold M^n into some (pseudo-) Riemannian (resp. affine) manifold $\tilde{M}^{n+p}$, when f and f' are congruent (resp. equivalent). In the (pseudo-) Riemannian case, $\tilde{M}^{n+p}$ usually is a space of constant curvature and in the affine case, $\tilde{M}^{n+p}$ is the affine space $\mathbb{R}^{n+1}$. Most of the material in the affine case deals with codimension 1, i.e. $p = 1$. In this paper we establish an equivalence theorem for the affine case in arbitrary codimension. We use the formalism developed by K. Nomizu, see for instance [5] and [6], and by K. Nomizu and U. Pinkall, see for instance [7]. As an application of our equivalence theorem, we give an easy proof of some known results in the metric case and in the case of affine hypersurfaces and obtain some new results about affine hypersurfaces.

2. AFFINE IMMERSIONS

Let M^n be an affine manifold with affine connection ∇ and $\tilde{M}^{n+p}$ an affine manifold with affine connection $\tilde{\nabla}$. Let $f: M^n \to \tilde{M}^{n+p}$ be an immersion. We call f an *affine immersion* if there exists a differentiable field of transversal subspaces $N: x \in M^n \mapsto N_x$, i.e. $N_x \subset T_{f(x)}\tilde{M}$, such that

$$(1.1) \qquad T_{f(x)}\tilde{M} = f_*(T_xM) \oplus N_x$$

* Research Assistant of the National Fund for Scientific Research (Belgium).

Geometriae Dedicata **32**: 81–92, 1989.

and

$$(1.2) \qquad \tilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y),$$

for all $X, Y \in \mathfrak{X}(M^n)$ and $h(X, Y) \in N$. Formula (1.2) is called the *formula of Gauss.* We call N_x the *affine normal space* in x and N is called the *affine normal bundle.* A vector field ξ along f taking values in N is called an *affine normal vector field.* We also call M^n an *affine submanifold.* The affine normal space is not unique as there are many possible choices for N that all satisfy (1.1) and (1.2). Nevertheless the subspace O_x of N_x which is defined at each point x by

$$O_x = \operatorname{span}\{h(X, Y) \mid X, Y \in T_x M\},$$

is unique. We call it the *first affine normal space.* Every bundle N that contains O, i.e. $O_x \subset N_x$ for all x, and which satisfies (1.1) is an affine normal bundle. Suppose that we have chosen an affine normal bundle N, that ξ is an affine normal vector field and that $X \in \mathfrak{X}(M^n)$. Then we can decompose $\tilde{\nabla}_X \xi$ into a tangent and a normal part, and we obtain in this way the *formula of Weingarten,*

$$\tilde{\nabla}_X \xi = -f_*(A_\xi X) + \nabla^\perp_X \xi,$$

whereby A_ξ is a (1, 1)-tensor on M^n and $\nabla^\perp_X \xi \subset N$. We call A_ξ the *affine shape operator* of ξ and we can remark that A_ξ is linear in ξ. We call $\nabla^\perp$ the *affine normal connection.* We define the derivative ∇h of h as a (1.3)-tensor with values in N by

$$(\nabla h)(X, Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Because of condition (1.1) we can define a projection t from $T\tilde{M}$ to TM, and a projection n from $T\tilde{M}$ to N. We denote the Riemann–Christoffel-curvature-tensor of M^n, $\tilde{M}^{n+p}$ and $\nabla^\perp$ by R, $\tilde{R}$ and $R^\perp$. There are some fundamental relations between $R, \tilde{R}, R^\perp, h, \ldots$, like the *equation of Gauss,*

$$t(\tilde{R}(f_*X, f_*Y)f_*Z) = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X,$$

the *equations of Codazzi*

$$n(\tilde{R}(f_*X, f_*Y)f_*Z) = (\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z),$$

and

$$t(\tilde{R}(f_*X, f_*Y)\xi) = (\nabla_Y A)_\xi(X) - (\nabla_X A)_\xi(Y),$$

whereby $(\nabla_Y A)_\xi(X)$ is defined by

$$(\nabla_Y A)_\xi(X) = \nabla_Y(A_\xi X) - A_\xi(\nabla_Y X) - (A_{\nabla^\perp_Y \xi})(X),$$

and the *equation of Ricci*

$$n(\tilde{R}(f_*X, f_*Y)\xi) = h(A_\xi X, Y) - h(X, A_\xi Y) + R^\perp(X, Y)\xi,$$

whereby X, Y and $Z \in \mathfrak{X}(M^n)$ and ξ is a vector field along f.

In the case that $\tilde{M}^{n+p}$ is affine flat, the equations of Gauss, Codazzi and Ricci are given by:

$$R(X, Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$
$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z),$$
$$(\nabla_Y A)_\xi(X) = (\nabla_X A)_\xi(Y),$$
$$R^\perp(X, Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y).$$

A point $p \in M^n$ is called *totally geodesic* if $h_p = 0$. If all points of M^n are totally geodesic, then M^n (or f) is called *totally geodesic*. A submanifold is totally geodesic if and only if each geodesic of M^n is also a geodesic of $\tilde{M}^{n+p}$. The only totally geodesic submanifolds of an affine space $\mathbb{R}^{n+1}$ therefore are the affine (linear) subspaces.

If $f: (M^n, \nabla) \to (\tilde{M}^{n+p}, \tilde{\nabla})$ and $f': (M'^n, \nabla') \to (\tilde{M}'^{n+p}, \tilde{\nabla}')$ are affine immersions, then we call f and f' *affine equivalent w.r.t. F and F^N* if there exist affine equivalences F and $\tilde{F}$, i.e. diffeomorphisms $F: (M^n, \nabla) \to (M'^n, \nabla')$ and $\tilde{F}: (\tilde{M}^{n+p}, \tilde{\nabla}) \to (\tilde{M}'^{n+p}, \tilde{\nabla}')$ that preserve the connection, such that $f'(F) = \tilde{F}(f)$, whereby F^N is the normal bundle map defined by $F^N = \tilde{F}_{*|N}$.

3. An Equivalence Theorem

Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, D)$, resp. $f': (M'^n, \nabla') \to (\mathbb{R}^{n+p}, D)$, be affine immersions, whereby D is the usual (flat) affine connection on $\mathbb{R}^{n+1}$, with corresponding second fundamental form h, resp. h', affine shape operator A, resp. A', normal connection $\nabla^\perp$, resp. $\nabla^{\perp'}$ and normal space N, resp. N'. We now investigate under which conditions f and f' are equivalent. In particular, we prove the following theorem.

THEOREM 2.1. *Let $f: (M^n, \nabla) \to (\mathbb{R}^{n+p}, D)$ and $f': (M'^n, \nabla') \to (\mathbb{R}^{n+p}, D)$ be affine immersions. Suppose that the following conditions hold:*

(1) *There exists an affine equivalence $F: (M^n, \nabla) \to (M'^n, \nabla')$.*
(2) *There exists a bundle map $F^N: N \to N'$ which covers F.*
(3) $F^N h(X, Y) = h'(F_*X, F_*Y)$.
(4) $F_*(A_\xi X) = A'_{F^N(\xi)}(F_*X)$.
(5) $F^N(\nabla^\perp_X \xi) = \nabla^{\perp'}_{F_*(X)}(F^N \xi)$.

Then f and f' are equivalent w.r.t. F and F^N.

Proof. We have to find an affine transformation F' of $\mathbb{R}^{n+p}$ such that $F'(f) = f'(F)$. Consider the map $C: M^n \to \mathrm{Gl}(\mathbb{R}, n+p)$, defined by

$$C_p(f_*X) = f'_*(F_*X)$$
$$C_p(\xi) = F^N(\xi),$$

for all $X \in T_pM$ and $\xi \in N_p$, whereby we have identified both $T_{f(p)}\mathbb{R}^{n+p}$ and $T_{f'(p)}\mathbb{R}^{n+p}$ with $\mathbb{R}^{n+p}$. Note that, if V is any vector field along f, then we can define a vector field $C(V)$ along f' by $C(V)(F(p)) = C(V_p)$. We show that C is a constant map, by showing that $DC = 0$, or more precisely that

$$D_{F_*(X)}C(V) - C(D_XV) = 0,$$

for all vector fields V along f and $X \in \mathfrak{X}(M^n)$. It is sufficient to consider two cases: V is affine normal or V is tangent. So we first assume that $V = \xi$ is normal. Then

$$\begin{aligned} D_{F_*(X)}C(\xi) - C(D_X\xi) &= D_{F_*(X)}F^N(\xi) - C(-f_*(A_\xi X) + \nabla^\perp_X\xi) \\ &= -f'_*(A'_{F^N(\xi)}(F_*X)) + \nabla'^\perp_{F_*(X)}F^N\xi \\ &\quad + f'_*(F_*(A_\xi X)) - F^N\nabla^\perp_X\xi = 0, \end{aligned}$$

because of (4) and (5). Next assume that $V = f_*(Y)$. Then

$$\begin{aligned} D_{F_*(X)}C(f_*(Y)) - C(D_Xf_*(Y)) &= D_{F_*(X)}f'_*F_*(Y) - C(f_*(\nabla_XY) + h(X, Y)) \\ &= f'_*(\nabla'_{F_*X}F_*Y) + h'(F_*X, F_*Y) \\ &\quad - f'_*(F_*(\nabla_XY)) - F^Nh(X, Y) = 0, \end{aligned}$$

because of (1) and (3). This means that C is the same linear transformation for every $p \in M^n$. Now consider the map $G = C(f) - f'(F)$ from M^n into $\mathbb{R}^{n+p}$. Then $G_*(X) = C \circ f_* - f'_* \circ F_* = 0$ by the definition of C. Hence G is a constant map and therefore there exists a vector $B \in \mathbb{R}^{n+p}$ such that $B = C(f) - f'(F)$. Or still, $F'(f) = f'(F)$, whereby F' is the affine transformation of $\mathbb{R}^{n+p}$ defined by $F' = C - B$. Hence f and f' are affine equivalent w.r.t. F and F^N. □

3. Applications to Special Cases

In this section we show how some equivalence or congruence results, some of which are known, can be proved by using Theorem 2.1. The following theorem can be found (certainly in the Riemannian case) in most elementary textbooks on differential geometry.

THEOREM 3.1. *Let (M^n, g) and (M'^n, g') be Riemannian manifolds and let*

$f: (M^n, g) \to (\mathbb{R}^{n+p}, k)$ and $f': (M'^n, g') \to (\mathbb{R}^{n+p}, k)$ be isometric immersions, whereby k is any metric on $\mathbb{R}^{n+p}$ which has the usual connection D as a Levi-Civita connection. Suppose that the following conditions hold:

(1) *There exists an isometry $F: (M^n, g) \to (M'^n, g')$.*
(2) *There exists an isometric bundle map $F^N: N \to N'$ which covers F.*
(3) $F^N h(X, Y) = h'(F_* X, F_* Y)$.
(4) $F^N(\nabla^\perp_X \xi) = \nabla'^\perp_{F_*(X)}(F^N \xi)$.

Then f and f' are congruent w.r.t. F and F^N.

Proof. First we show that condition (3) of Theorem 3.1 implies condition (4) of Theorem 2.1. Indeed, since f and f' are isometric immersions and M^n and M'^n are isometric we know that

$$\begin{aligned} g'(A'_{F^N\xi}(F_*X), F_*Y) &= k(h'(F_*X, F_*Y), F^N\xi) = k(F^N h(X, Y), F^N\xi) \\ &= k(h(X, Y), \xi) = g(A_\xi X, Y) = g'(F_* A_\xi X, F_* Y) \end{aligned}$$

for all $X, Y \in T_pM$ and $\xi \in T_p^\perp M$ and therefore we obtain that $A'_{F^N\xi}(F_*X) = F_* A_\xi X$. All the conditions of Theorem 2.1 are therefore satisfied and hence we obtain that f and f' are equivalent w.r.t. F and F^N. Since both F and F^N are isometric, it follows that the equivalence is a congruence. □

From now on we concentrate on affine hypersurfaces of $\mathbb{R}^{n+1}$. So let M^n be an affine hypersurface of $\mathbb{R}^{n+1}$, immersed by f. Then the affine normal space is one-dimensional. Let ξ be a non-vanishing affine normal vector field, and denote the second fundamental form of f by $\mathfrak{h}$. Then there exists a symmetric (0, 2)-tensor field h on M^n such that $\mathfrak{h}(X, Y) = h(X, Y)\xi$ for all tangent vectors X and Y. From now on we call h the second fundamental form of f, instead of $\mathfrak{h}$. We can also define the normal connection form τ of f by $\nabla^\perp_X \xi = \tau(X)\xi$ for all tangent vectors X. We then obtain that

$$R^\perp(X, Y)\xi = (\mathrm{d}\tau)(X, Y)\xi.$$

From now on we write S instead of A_ξ. The equations of Gauss, Codazzi and Ricci then become

$$\begin{aligned} &R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \\ &(\nabla h)(X, Y, Z) + \tau(X)h(Y, Z) = (\nabla h)(Y, X, Z) + \tau(Y)h(X, Z), \\ &(\nabla_X S)(Y) + \tau(X)SY = (\nabla_Y S)(X) + \tau(Y)SX, \\ &h(SX, Y) = h(X, SY) - (d\tau)(X, Y), \end{aligned}$$

whereby ∇h and ∇S are defined by

$$(\nabla h)(X, Y, Z) = Xh(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and

$$(\nabla_X S)(Y) = \nabla_X(SY) - S(\nabla_X Y).$$

Now let $\tilde{\omega}$ be a parallel volume form on $\mathbb{R}^{n+1}$. If M^n is an equiaffine manifold, equipped with a parallel volume form ω, then we call ξ an equiaffine normal vector field if

$$\tilde{\omega}(f_* X_1, f_* X_2, \ldots, f_* X_n, \xi) = \omega(X_1, X_2, \ldots, X_n),$$

for all tangent vector fields $X_1, X_2, \ldots, X_n$. Note that ξ always exists, cf.[7]. Since

$$\nabla\omega = \tau\omega,$$

we know that $\tau = 0$ in case ξ is equiaffine.

One can look at affine differential geometry also from another point of view. Namely, if you consider a submanifold of a Riemannian manifold, then there is in a natural way a Riemannian metric induced on the submanifold. This is not the case for submanifolds of an affine manifold. But if there is a way to chose an affine normal space, then we can induce an affine connection on the submanifold. Of course we would like the normal space and the affine connection to be affine invariants, or at least to be invariant under a subgroup of affine transformations of the surrounding affine space. Here we recall two classical ways to do this for hypersurfaces M^n of the affine space $\mathbb{R}^{n+1}$.

The Equiaffine Normalization

First, we fix some parallel volume form ω on $\mathbb{R}^{n+1}$. Assume that $f\colon M^n \to \mathbb{R}^{n+1}$ is an immersion. For any choice of transversal vector field ξ, we can define on M^n an affine connection ∇ and a bilinear form h by the formula of Gauss. Whether h is non-degenerate or not does not depend on the choice of this transversal vector field. Therefore we call the immersion non-degenerate if there exists a transversal vector field for which the corresponding h is non-degenerate. Let us suppose that f is non-degenerate. In that case, h induces a metric volume form which we shall denote by ω^h. One can also induce a volume form θ on M^n by

$$\theta(X_1, X_2, \ldots, X_n) = \omega(f_* X_1, f_* X_2, \ldots, f_* X_n, \xi).$$

Then it is proved that there exists up to sign a unique choice of ξ such that

(i) $\omega_h = \theta$,
(ii) θ is parallel with respect to $\nabla, \nabla\theta = 0$.

It is immediately clear that the immersion defined this way is indeed an affine immersion and ∇ is called the *canonical connection* and ξ is called the *canonical affine normal vector field*. Note that ξ is equiaffine w.r.t. θ. The second fundamental form associated with ξ is often called the *affine metric* of M. The canonical connection, as well as the canonical affine normal vector field and the affine metric are equiaffine invariants, i.e. invariants under the group $\mathrm{Sl}(\mathbb{R}, n+1) \otimes \mathbb{R}^{n+1}$ of affine transformations of $\mathbb{R}^{n+1}$ that preserve $\tilde{\omega}$. If we apply a general affine transformation to $\mathbb{R}^{n+1}$, then we change the volume form of $\mathbb{R}^{n+1}$ and we obtain again a canonical affine normal vector field. We call ξ canonical if it is canonical w.r.t. some volume form of $\mathbb{R}^{n+1}$. If ξ_1 and ξ_2 are both canonical, then $\xi_1 = c\xi_2$ for some constant $c \in \mathbb{R}$, see, for instance, [2]. Hence they determine the same affine connection ∇. The canonical connection is therefore an affine invariant, as well as the line determined by the canonical affine normal vector field at each point. For more details, I refer to [5] and [6]. In [4] one can find another treatment of the general affine case, and in [3] a similar study is made of the non-degenerate complex hypersurfaces of the complex affine space.

The Centroaffine Normalization

Let O be the origin of $\mathbb{R}^{n+1}$. Assume that $f\colon M^n \to \mathbb{R}^{n+1}$ is an immersion, and suppose that for every $p \in M^n$ the position vector $x(p) = \mathbf{Of(p)}$ is transversal to M^n. Then we can define x as affine normal vector field along f. The induced connection ∇ is called the centroaffine connection on M^n. The centroaffine connection is a centroaffine invariant, i.e. ∇ is invariant under the group $\mathrm{Gl}(\mathbb{R}, n+1)$ of affine transformations that leave O fixed.

THEOREM 3.2. *Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ and $f'\colon (M'^n, \nabla') \to (\mathbb{R}^{n+1}, D)$ be affine immersions. Let ξ and ξ' be affine normal vector fields along f and f'. Suppose that the following conditions hold:*

(1) *There exists an affine equivalence $F\colon (M^n, \nabla) \to (M'^n, \nabla')$.*
(2) $h(X, Y) = h'(F_*X, F_*Y)$.
(3) $\tau(X) = \tau'(F_*X)$.
(4) $\mathrm{rk}(h) = \mathrm{rk}(h') > 1$.

Then f and f' are equivalent w.r.t. F.

Proof. Let us define a bundle map $F^N\colon N \to N'$ by $F^N(\xi) = \xi'$. If we can show that (2) and (4) imply condition (4) of Theorem 2.1, then the theorem is proved. From the equation of Gauss and the fact that F is an affine equivalence, it follows that

$$(3.1)\qquad \begin{aligned} h(Y,Z)F_*(SX) - h(X,Z)F_*(SY) &= F_*(R(X,Y)Z) \\ &= R'(F_*X, F_*Y)F_*Z \\ &= h'(F_*Y, F_*Z)S'(F_*X) \\ &\quad - h'(F_*X, F_*Z)S'(F_*Y) \\ &= h(Y,Z)S'(F_*X) \\ &\quad - h(X,Z)S'(F_*Y), \end{aligned}$$

for all $X, Y, Z \in \mathscr{X}(M^n)$. Now let $X \in T_pM$. We then can choose Y, Z such that $h(X, Z) = 0$ and $h(Y, Z) \neq 0$. Then (3.1) implies that

$$F_*(SX) = S'(F_*X),$$

which proves the theorem. □

Condition (4) in Theorem 3.2 is necessary, as is illustrated by the following example.

EXAMPLE. We construct two affine immersions f and f' of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ which satisfy conditions (1), (2) and (3) of Theorem 3.2 but which are not equivalent.

Let f be defined by

$$f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}\colon (u_1, u_2, \ldots, u_n) \mapsto (u_1, u_2, \ldots, u_n, \tfrac{1}{2}u_1^2).$$

Let us define a transversal vector field along f by $\xi(p) = (0, 0, \ldots, 0, 1)_p$ for all $p \in \mathbb{R}^n$. Then f is an affine immersion (whereby both $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ are equipped with their usual connection) with affine normal vector field ξ. Indeed, since

$$D_{\partial/\partial u_i} f_*\left(\frac{\partial}{\partial u_k}\right) = \delta_{ik}\delta_{i1}\xi,$$

it follows immediately that $\nabla_{\partial/\partial u_i}(\partial/\partial u_k) = 0$, such that f is an affine immersion and $h(\partial/\partial u_i, \partial/\partial u_k) = \delta_{ik}\delta_{i1}$. Or, in other words, $h(\partial/\partial u_i, \partial/\partial u_k) = 0$ if $i \neq 1$ or $k \neq 1$, and $h(\partial/\partial u_1, \partial/\partial u_1) = 1$. Hence $\mathrm{rk}(h) = 1$. Finally, since ξ is parallel it follows easily that $S = 0$ and $\tau = 0$.

Next, let f' be defined by

$$f'\colon \mathbb{R}^n \to \mathbb{R}^{n+1}\colon (u_1, u_2, \ldots, u_n) \mapsto (\cos(u_1), \sin(u_1), u_2, \ldots, u_n).$$

We define a transversal vector field ξ' along f' by $\xi'(u_1, u_2, \ldots, u_n) = -(\cos(u_1), \sin(u_1), 0, \ldots, 0)$. Then f' is an affine immersion (whereby both $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ are again equipped with their usual connection) with affine normal vector field ξ'. Indeed,

$$D_{\partial/\partial u_i} f'_*\left(\frac{\partial}{\partial u_k}\right) = \delta_{ik}\delta_{i1}\,\xi,$$

it follows immediately that $\nabla_{\partial/\partial u_i}(\partial/\partial u_k) = 0$, such that f' is an affine immersion and that $h'(\partial/\partial u_i, \partial/\partial u_k) = \delta_{ik}\delta_{i1}$. So we again have that $h'(\partial/\partial u_i, \partial/\partial u_k) = 0$ if $i \neq 1$ or $k \neq 1$, and $h'(\partial/\partial u_1, \partial/\partial u_1) = 1$. Hence $\mathrm{rk}(h') = 1$. Moreover, it is clear that $S'(\partial/\partial u_i) = 0$ if $i \neq 1$ and $S'(\partial/\partial u_1) = \partial/\partial u_1$ and that $\tau' = 0$.

In conclusion: f and f' satisfy (1), take F the identity transformation, (2) and (3). But they cannot be equivalent, for instance since S is zero and S' is not.

Of course, if $h = h' = 0$, then f and f' are equivalent since they are both totally geodesic. In case h is non-degenerate, we have the following theorem, which generalizes the equivalence theorem in, for instance, [4].

THEOREM 3.3. *Let $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ and $f'\colon (M'^n, \nabla') \to (\mathbb{R}^{n+1}, D)$ be affine immersions. Let ξ and ξ' be affine normal vector fields along f and f'. Suppose that the following conditions hold:*

(1) *There exists a diffeomorphism $F\colon (M^n, \nabla) \to (M'^n, \nabla')$.*
(2) $h(X, Y) = h'(F_* X, F_* Y)$.
(3) $(\nabla h)(X, Y, Z) = (\nabla' h')(F_* X, F_* Y, F_* Z)$.
(4) $\tau(X) = \tau'(F_* X)$.
(5) *h and h' are non-degenerate.*

Then F is an affine equivalence and f and f' are equivalent w.r.t. F.

Proof. If we want to apply Theorem 2.3, then we have to show that (2), (3) and (5) imply that F is an affine equivalence. This means that we have to show that $F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y$ for all tangent vector fields X and Y. So let X, Y and Z be any tangent vector field defined in the neighbourhood of p. Then

$$\begin{aligned}(\nabla h)(X, Y, Z) &= Xh(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\\ &= (\nabla' h')(F_* X, F_* Y, F_* Z)\\ &= (F_* X)h'(F_* Y, F_* Z) - h'(\nabla_{F_* X} F_* Y, F_* Z)\\ &\quad - h'(F_* Y, \nabla_{F_* X} F_* Z)\\ &= Xh(Y, Z) - h'(\nabla_{F_* X} F_* Y, F_* Z)\\ &\quad - h'(F_* Y, \nabla_{F_* X} F_* Z),\end{aligned}$$

which implies that

$$(3.2)\; h'(F_*(\nabla_X Y) - \nabla_{F_* X} F_* Y, F_* Z) + h'(F_* Y, F_*(\nabla_X Z) - \nabla_{F_* X} F_* Z) = 0.$$

We can define a symmetric tensor field T on M^n by

$$F_* T(X, Y) = F_*(\nabla_X Y) - \nabla_{F_*X} F_* Y.$$

Then (3.2) reads as

$$h(T(X, Y), Z) + h(Y, T(X, Z)) = 0. \tag{3.3}$$

From (3.3) it follows that

$$\begin{aligned} 0 &= h(T(X, Y), Z) + h(Y, T(X, Z)) \\ &\quad + h(T(Y, X), Z) + h(X, T(Y, Z)) \\ &\quad - h(T(Z, Y), X) - h(Y, T(Z, X)) \\ &= 2h(T(X, Y), Z), \end{aligned}$$

such that $T(X, Y) = 0$, since h is non-degenerate. This means that F is an affine equivalence.

The following theorem generalizes a theorem, proved by U. Simon in [8] for locally strongly convex hypersurfaces.

THEOREM 3.4. *Let $f: M^n \to (\mathbb{R}^{n+1}, O, D)$ and $f': M^n \to (\mathbb{R}^{n+1}, O, D)$ be immersions $n > 1$, where O is the origin of $\mathbb{R}^{n+1}$. Let ∇ and ∇' be the affine connections induced on M^n by centroaffine normalization. Suppose that the following condition holds:*

$$\nabla = \nabla',$$

then there exists a centroaffine transformation C of $\mathbb{R}^{n+1}$ such that $f' = C(f)$.

Proof. We define a normal bundle map F^N by $F^N: N_p \to N'_p: x(f(p)) \mapsto x(f'(p))$, whereby x is the position vector in $\mathbb{R}^{n+1}$ w.r.t. O. Since both S and S' are $-I$, we already have that $S = S'$. Also $\tau = \tau' = 0$. Since $\nabla = \nabla'$, we know that $R = R'$. Hence we obtain

$$\begin{aligned} h(X, Z)Y - h(Y, Z)X &= R(X, Y)Z \\ &= R'(X, Y)Z = h'(X, Z)Y - h'(Y, Z)X, \end{aligned}$$

and hence

$$(h(X, Z) - h'(X, Z))Y - (h(Y, Z) - h'(Y, Z))X = 0.$$

By taking X and Y linearly independent, we obtain that $h = h'$. All conditions of Theorem 2.1 are satisfied and we can conclude that f and f' are equivalent w.r.t. the identity transformation of M^n, i.e. $f' = C(f)$. Now, we only have to show that C is linear (has no translation part). This is easy, since $C(f(p)) = f'(p)$, and by the construction of C in the proof of Theorem 2.1, it follows that

$C_*(f(p)) = F^N(f(p)) = f'(p)$. We can always suppose that $f(M^n)$ is not contained in a linear subspace of $\mathbb{R}^{n+1}$. Hence $C = C_*$, and C is linear. □

The next theorem is a classical one. It can be found in a lot of textbooks on affine differential geometry, for instance in [1].

THEOREM 3.5. *Let* $f: M^n \to (\mathbb{R}^{n+1}, \omega, D)$ *and* $f': M^n \to (\mathbb{R}^{n+1}, \omega, D)$ *be non-degenerate immersions, where* ω *is a parallel volume form of* $\mathbb{R}^{n+1}$. *Let* ∇ *and* ∇' *be the affine connections induced on* M^n *be equiaffine normalization. Suppose that the following condition holds.*

(1) $h = h'$,
(2) $\nabla h = \nabla' h'$

then $\nabla = \nabla'$ *and there exists an equiaffine transformation* C *of* R^{n+1} *such that* $f' = C(f)$.

Proof. If we apply Theorem 3.3 to this case, then we immediately obtain that the two connections coincide and that there exists an affine transformation C such that $f' = C(f)$. Note that $C(\xi) = \xi'$, where ξ and ξ' are the canonical affine normals of f, resp. f'. We show that C is equiaffine. We denote the induced volume form by f (resp. f') on M^n by θ (resp. θ'), and the metric volume form, associated to h by θ^h. Since $h = h'$, we obtain of course that the metric volume form associated to h' coincides with θ^h. Since both f and f' are canonical w.r.t. ω, we obtain that $\theta = \theta^h = \theta'$. Hence it follows that

$$\begin{aligned}(C^*\omega)(f_*X_1, \ldots, f_*X_n, \xi) &= \omega(C_*f_*X_1, \ldots, C_*f_*X_n, C_*\xi)\\ &= \omega(f'_*X_1, \ldots, f'_*X_n, \xi')\\ &= \theta'(X_1, \ldots, X_n) = \theta(X_1, \ldots, X_n)\\ &= \omega(f_*X_1, \ldots, f_*X_n, \xi),\end{aligned}$$

for all tangent vectors $X_1, \ldots, X_n$. Hence $C^*\omega = \omega$ and thus C is equiaffine. □

REFERENCES

1. Blaschke, W., *Vorlesungen über Differentialgeometrie, II Affine Differentialgeometrie*, Springer, Berlin, 1923.
2. Dillen, F., and Vrancken, L., 'Canonical Affine Connection on Complex hypersurfaces of the Complex Affine Space', in *Geometry and Topology of Submanifolds*, World Scientific, Singapore, 1989, pp. 89–111.
3. Dillen, F., Verstraelen, L. and Vrancken, L., 'Complex Affine Differential Geometry' (to appear in *Atti Acad. Peloritana*).
4. Gigena, S., 'General Affine Geometry of Hypersurfaces I', Preprint.

5. Nomizu, K., 'On Completeness in Affine Differential Geometry', *Geom. Dedicata* **20** (1986), 43–49.
6. Nomizu, K., 'Introduction to Affine Differential Geometry', Preprint MPI/88–37.
7. Nomizu, K. and Pinkall, U., 'Cubic Form Theorems for Affine Immersions', *Results in Math.* **13** (1988), 338–362.
8. Simon, U., 'The Fundamental Theorem in Affine Hypersurface Theory', *Geom. Dedicata* **26** (1988), 125–137.

Author's address:

Franki Dillen,
Katholieke Universiteit Leuven,
Departement Wiskunde,
Celestijnenlaan 200 B,
B-3030 Leuven,
Belgium.

(Received February 21, 1989)