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GEOMINIMAL SURFACE AREA

1. Introduction

Minkowski geometry, relative differential geometry and affine differential geometry belong, essentially, to the same area of geometry but their developments have been independent of each other. A connecting link is the concept of geominimal surface area which is investigated in this paper.

In order to describe the setting for this theory, we first review some results of L. A. Santaló [27]. Let K be a convex body (compact convex set with interior points) with volume $|K|$ in n-dimensional Euclidean space E^n. For $p \in \operatorname{int} K$, let $\sigma_p(K, u)$ be the supporting function of K with respect to p. The volume $I(K, p)$ of the polar reciprocal of K with respect to the unit sphere centered at p is given by

$$(1.1) \qquad I(K, p) = \frac{1}{n} \int_{\Omega} \sigma_p^{-n}(K, u) \, d\omega_u,$$

where Ω is the unit sphere in E^n centered at the origin z. There exists a unique point $s(K)$, which we will call the Santaló point of K, such that

$$(1.2) \qquad I(K, s(K)) = \min_{p \in \operatorname{int} K} I(K, p) = I_m(K).$$

If π_r denotes the r-dimensional volume of the r-dimensional unit ball, then

$$(1.3) \qquad I_m(K)\,|K| \leqslant \pi_n^2,$$

with equality if and only if K is an ellipsoid. The product $I_m(K)|K|$ is an affine invariant of K. If K has an interior point at the origin z, then a necessary and sufficient condition that $s(K) = z$ is

$$(1.4) \qquad \int_{\Omega} u\sigma^{-n-1}(u) \, d\omega = 0,$$

where $\sigma(u) = \sigma_z(K, u)$.

Now, let $\mathcal{T}^n$ denote the set of all convex bodies T, with supporting function $\sigma(u)$, such that $s(T) = z$ and

$$(1.5) \qquad I_m(T) = \frac{1}{n} \int_{\Omega} \sigma^{-n}(u) \, d\omega = \pi_n.$$

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Then, for each convex body K, there is a unique $T \in \mathscr{T}^n$ which is homothetic to K. From (1.3), if $T \in \mathscr{T}^n$, then

$$(1.6) \qquad |T| \leqslant \pi_n,$$

with equality if and only if T is an ellipsoid.

The relative surface area $A(K, T)$ of a convex body K with respect to T for $T \in \mathscr{T}^n$ is defined in relative differential geometry by

$$(1.7) \qquad A(K, T) = \int_{\partial K} \sigma(u)\, \mathrm{d}S_u = nV(K, \dots, K, T) = nV_1(K, T)$$

where $\sigma(u) = \sigma_z(T, u)$ and $\mathrm{d}S_u$ is the Euclidean surface area element of ∂K at a point with outer normal u. From Minkowski's inequality for mixed volumes, we obtain the isoperimetric inequality

$$(1.8) \qquad A^n(K, T) \geqslant n^n |T|\, |K|^{n-1},$$

where the equality holds if and only if K is homothetic to T.

Busemann's definition of surface area in Minkowski spaces can be expressed by (1.7) for a solution T of the Minkowski isoperimetric problem. Let M be the unit Minkowski ball with center at z and let $|M(u)|$ be the $(n-1)$-dimensional volume of the section $M(u)$ of M determined by the hyperplane through z normal to u. Then $\sigma(u) = \pi_{n-1}/|M(u)|$ is the supporting function (on Ω) of a solution T to the Minkowski isoperimetric problem. The constraint (1.5) provides a normalization of volume for the Minkowski space. For $n=2$, this normalization yields $|M| = \pi$. For $n \geqslant 3$, this yields $|M| \geqslant \pi_n$ with equality if and only if M is an ellipsoid (see [21], p. 1535]). Thus, surface area in Minkowski spaces may be studied by restricting attention to a suitable subset of $\mathscr{T}^n$.

Additional details of the motivation for the above setting are given in [24].

Throughout this paper $\mathscr{S}^n$ will denote a *nonempty* subset of $\mathscr{T}^n$. If K is a convex body, we define the geominimal surface area of K relative to $\mathscr{S}^n$ by

$$(1.9) \qquad G(K, \mathscr{S}^n) = \inf\{A(K, T) \mid T \in \mathscr{S}^n\}.$$

The basic theory concerning geominimal surface area is developed in Section 2. In Section 3 a close connection is established between this theory and affine differential geometry. There an analogue to surfaces of constant curvature plays a dominant role.

2. The basic theory

We first observe that $G(K, \mathscr{S}^n)$ enjoys the usual properties of a surface area. Let $\mathscr{K}^n$ denote the set of all convex bodies in E^n.

(2.1) LEMMA. *Let* $K, K_1, K_2 \in \mathcal{K}^n$.

a. $G(\lambda K + a, \mathcal{S}^n) = \lambda^{n-1} G(K, \mathcal{S}^n)$ *for* $\lambda > 0$, $a \in E^n$.

b. *If* $K_1 \subset K_2$, *then* $G(K_1, \mathcal{S}^n) \leqslant G(K_2, \mathcal{S}^n)$.

c. *If* $\alpha > 0$, $\beta > 0$, *then*

$$G^{1/n-1}(\alpha K_1 + \beta K_2, \mathcal{S}^n) \geqslant \alpha G^{1/n-1}(K_1, \mathcal{S}^n) + \beta G^{1/n-1}(K_2, \mathcal{S}^n).$$

Proof. Both (a) and (b) follow from the Definition (1.9) and the properties of mixed volumes [3, pp. 40–41]. The General Brünn-Minkowski Theorem [5, p. 49] implies that $A^{1/n-1}((1-\varrho) K_1 + \varrho K_2, T)$ is a concave function of ϱ for $0 \leqslant \varrho \leqslant 1$. Consequently the infimum $G^{1/n-1}((1-\varrho) K_1 + \varrho K_2, \mathcal{S}^n)$ is a concave function of ϱ for $0 \leqslant \varrho \leqslant 1$. Property (c) now follows from (a). This completes the proof.

Throughout this paper the topology on a family of nonempty compact convex sets is that induced by the Hausdorff metric. A point $p(K) \in E^n$, defined for all $K \in \mathcal{K}^n$, is called an affine-invariant point of K (compare [11, p. 238]), if for any (nonsingular) affine map $\mathfrak{g}$ of E^n onto itself, $\mathfrak{g}(p(K)) = p(\mathfrak{g}(K))$, and $p(K)$ depends continuously on K.

(2.2) LEMMA. *The Santaló point* $s(K)$ *of* $K \in \mathcal{K}^n$ *is an affine-invariant point of* K.

Proof. Consider (1.1. An affinity $\mathfrak{g}$ may be expressed as an affinity $\mathfrak{h}$, leaving p fixed, followed by a translation. The transformation rule for polar reciprocals [23, p. 236] implies

$$|K|\, I(K, p) = |\mathfrak{h}(K)|\, I(p, \mathfrak{h}(K)) = |\mathfrak{g}(K)|\, I(\mathfrak{g}(p), \mathfrak{g}(K)).$$

As p ranges over $\operatorname{int} K$, $\mathfrak{g}(p)$ ranges over $\operatorname{int}\mathfrak{g}(K)$. Since the Santaló point is unique, $\mathfrak{g}(s(K)) = s(\mathfrak{g}(K))$. Next we prove the continuity.

For $u \in \Omega$, $\sigma_p(K, u)$ is the distance from p to the supporting plane to K in the direction u. Consequently $\sigma_{(1-\lambda)p+\lambda q}(K, u) = (1-\lambda)\,\sigma_p(K, u) + \lambda \sigma_q(K, u)$ for $p, q \in \operatorname{int} K$ and $0 \leqslant \lambda \leqslant 1$. Since $f(x) = x^{-n}$ for $x > 0$ is a strictly convex function, we obtain, using the preceding equality, $I(K, (1-\lambda) p + \lambda q) \leqslant (1-\lambda)\, I(K, p) + \lambda I(K, q)$. Moreover, for $0 < \lambda < 1$, the equality sign holds if and only if $\sigma_p(K, u) = \sigma_q(K, u)$ for all $u \in \Omega$. The latter condition implies $p = q$ and, consequently, $I(K, p)$ is a strictly convex function of p for $p \in \operatorname{int} K$. Set $C(\lambda) = \{p \in \operatorname{int} K \mid I(K, p) \leqslant \lambda\}$. Since $I(K, p) \to \infty$ as $p \to \partial K$, $C(\lambda)$ is a convex body for all $\lambda > I_m(K)$ and $C(\lambda) \to s(K)$ as $\lambda \to I_m(K)$. Let $U \subset K$ be an open ball of radius ε centered at $s(K)$. Choose $\lambda > I_m(K)$ so that $C(\lambda) \subset U$ and let $p \in C(\lambda)$. Suppose $K_i \to K$. Then, for all sufficiently large i, $p \in \operatorname{int} K_i$. Since $\sigma_p(K_i, u)$ converges uniformly to $\sigma_p(K, u)$ on Ω, $I(K_i, p) \to I(K, p)$. Moreover, since $C(\lambda)$ is compact, $I(K_i, p)$ converges uniformly to

$I(K,p)$ on $C(\lambda)$. Consequently, $I(K_i,p)$ has a relative minimum in $\operatorname{int} C(\lambda)$ for all sufficiently large i. But a relative minimum of $I(K_i,p)$ is an absolute minimum. Thus $s(K_i)\in \operatorname{int} C(\lambda)\subset U$ for all sufficiently large i. This completes the proof.

A metric space R is finitely compact if every bounded infinite subset has an accumulation point in R. The Blaschke Selection Theorem implies that the set $\mathscr{C}^n$ of all nonempty compact convex sets is a finitely compact metric space.

(2.3) LEMMA. *$\mathscr{T}^n$ is a finitely compact metric space.*

Proof. Since $\mathscr{C}^n$ is finitely compact, we need only show that $\mathscr{T}^n$ is a closed subset of $\mathscr{C}^n$. Suppose $T_i\to C\in\mathscr{C}^n$, where each $T_i\in\mathscr{T}^n$. Bambah [1] calculated a lower bound, $I_m(K)\,|K|\geqslant 4^n/(n!)^2$, for $K\in\mathscr{K}^n$. Since $I_m(T_i)=\pi_n$, we have $C\in\mathscr{K}^n$. Lemma (2.2) implies that $s(C)=z$ and, since $I_m(K)$ is continuous, $I_m(C)=\pi_n$.

(2.4) LEMMA. *Let $K\in\mathscr{K}^n$, $\{K_\alpha\}\subset\mathscr{K}^n$, where each K_α contains a translate of K. If $\{A(K_\alpha, T_\alpha)\}$ is bounded for the family $\mathscr{S}^n=\{T_\alpha\}$, then $\mathscr{S}^n$ is a bounded subset of $\mathscr{T}^n$.*

Proof. Let $M>A(K_\alpha,T_\alpha)$ for all α. By hypothesis, each K_α contains a ball B_α with fixed radius $r>0$. Let $D_\alpha L_\alpha$ be a maximal segment in T_α, where D_α is the diameter of T_α and L_α is a segment of unit length. Then $M\geqslant nV_1(K_\alpha,T_\alpha)\geqslant nV_1(B_\alpha, D_\alpha L_\alpha)=D_\alpha r^{n-1}\pi_{n-1}$. Thus $\mathscr{S}^n$ is bounded.

(2.5) THEOREM. *If $\mathscr{S}^n$ is closed and $K\in\mathscr{K}^n$, then there exists $T\in\mathscr{S}^n$ such that $G(K,\mathscr{S}^n)=A(K,T)$.*

Proof. Let $\{T_i\}\subset\mathscr{S}^n$ satisfy $A(K,T_i)\to G(K,\mathscr{S}^n)$. By (2.4), with $K_\alpha=K$, the sequence $\{T_i\}$ is bounded. From the hypothesis and (2.3), there exists a subsequence of $\{T_i\}$ which converges to an element $T\in\mathscr{S}^n$. The continuity of mixed volumes implies $A(K,T)=G(K,\mathscr{S}^n)$.

(2.6) THEOREM. *$G(K,\mathscr{S}^n)$ is a continuous function of K on $\mathscr{K}^n$.*

Proof. We may assume that $\mathscr{S}^n$ is closed, since $\operatorname{cl}(\mathscr{S}^n)\subset\mathscr{T}^n$ and $G(K,\mathscr{S}^n)=G(K,\operatorname{cl}(\mathscr{S}^n))$. Since $A(K,T)$, for $T\in\mathscr{S}^n$, is a continuous function of K, $G(K,\mathscr{S}^n)$ is upper semicontinuous. Let $K_i\to K$ and set $k=\liminf G(K_i,\mathscr{S}^n)$. By (2.5) there exists $T_i\in\mathscr{S}^n$ such that $A(K_i,T_i)=G(K_i,\mathscr{S}^n)$. Let $B\subset\operatorname{int}K$ be a (closed) ball. For all i sufficiently large, $K_i\supset B$. Since $\{A(K_i,T_i)\}$ is bounded, Lemma (2.4) implies that $\{T_i\}$ is bounded. By taking a subsequence of a subsequence, but without changing notation, we have $A(K_i,T_i)\to k$ and $T_i\to T\in\mathscr{S}^n$. Consequently $k=A(K,T)\geqslant G(K,\mathscr{S}^n)$, and $G(K,\mathscr{S}^n)$ is lower semicontinuous.

(2.7) *Remark*. We may extend the definitions of $A(K, T)$ and $G(K, \mathscr{S}^n)$ to $K \in \mathscr{C}^n$. However, unlike $A(K, T)$, $G(K, \mathscr{S}^n)$ need not be continuous on $\mathscr{C}^n$. For example, in E^2 let $4d^2 (=8\pi^{-1})$ be the area of a parallelogram in $\mathscr{T}^2$. The set $\mathscr{S}^2$ of all parallelograms in $\mathscr{T}^2$ with a side of length $2d$ on the line $y=d$, is a closed subset of $\mathscr{T}^2$. Let $\{L_i\}$ be a sequence of segments, each of length a and positive slope, which converge to a segment L parallel to the x-axis. Then, for $T \in \mathscr{S}^2$, $A(L, T)=2ad=G(L, \mathscr{S}^2)$ but $G(L_i, \mathscr{S}^2) \to 0$.

(2.8) THEOREM. *Let $\mathscr{S}^n$ have the property that if $T_1, T_2 \in \mathscr{S}^n$, then there exist $\alpha>0$, $\beta>0$ such that $\mathscr{S}^n$ contains some element homothetic to $\alpha T_1+\beta T_2$. For $K \in \mathscr{K}^n$, there is not more than one $T \in \mathscr{S}^n$ such that $G(K, \mathscr{S}^n)=A(K, T)$.*

Proof. Suppose $T_1, T_2 \in \mathscr{S}^n$ such that $G(K, \mathscr{S}^n)=A(K, T_1)=A(K, T_2)$. We may assume that $\alpha T_1+\beta T_2$ is a translate of some $T \in \mathscr{S}^n$. Then $A(K, T)= =nV_1(K, \alpha T_1+\beta T_2)=(\alpha+\beta)\, G(K, \mathscr{S}^n)$. Thus $\alpha+\beta \geqslant 1$. From (1.1) and (1.5), $I(\alpha T_1+\beta T_2, z) \geqslant \pi_n$. Applying Minkowski's inequality for integrals [15, p. 146] to $I^{-1/n}(\alpha T_1+\beta T_2, z)$, we obtain

$$\begin{aligned} \pi_n^{-1/n} &\geqslant I^{-1/n}(\alpha T_1+\beta T_2, z) \geqslant \alpha I^{-1/n}(T_1, z)+\beta I^{-1/n}(T_2, z) \\ &= (\alpha+\beta)\, \pi_n^{-1/n} \geqslant \pi_n^{-1/n}. \end{aligned}$$

But the equality can hold throughout only if $s(\alpha T_1+\beta T_2)=z$ and $\alpha\sigma_1(u)$ is proportional to $\beta\sigma_2(u)$ on Ω, where $\sigma_i(u)$ is the supporting function of T_i. From the definition of $\mathscr{T}^n$, this implies $T_1=T_2$ which completes the proof.

If, for a given $K \in \mathscr{K}^n$, there exists a unique element $T \in \mathscr{S}^n$ such that $G(K, \mathscr{S}^n)=A(K, T)$, then this unique T will be denoted by $T(K, \mathscr{S}^n)$.

(2.9) DEFINITION. The set $\mathscr{S}^n$ is said to be distinguished if for every $K \in \mathscr{K}^n$, the convex body $T(K, \mathscr{S}^n)$ exists.

From (2.5) and (2.8) we observe that $\mathscr{S}^n$ is distinguished if $\mathscr{S}^n$ is closed and satisfies the hypothesis of (2.8). However, a distinguished set need not satisfy the hypothesis of (2.8). For example, the set of all ellipsoids in $\mathscr{T}^n$ is distinguished (see [22]), but the vector sum of two ellipsoids need not be an ellipsoid.

Denote by $\mathscr{T}_0^n$ the set of all centrally symmetric bodies in $\mathscr{T}^n$ and by $\mathscr{H}^n$ the set of all projection bodies (or zonoids) in $\mathscr{T}^n$.

(2.10) COROLLARY. *The sets $\mathscr{T}^n$, $\mathscr{T}_0^n$ and $\mathscr{H}^n$ are distinguished.*

Proof. This follows from (2.5) and (2.8). In particular, we observe that the limit of a sequence of zonoids is a zonoid and the vector sum of zonoids is a zonoid.

(2.11) LEMMA. *Suppose* $\mathscr{S}^n$ *is distinguished and* $\{K_i\} \subset \mathscr{K}^n$, $K \in \mathscr{K}^n$.

a. $G^n(K, \mathscr{S}^n) \geqslant n^n |T(K, \mathscr{S}^n)| \, |K|^{n-1}$ *with equality if and only if* K *and* $T(K, \mathscr{S}^n)$ *are homothetic.*

b. *If* $K_1 \supset K_2$ *and* $G(K_1, \mathscr{S}^n) = G(K_2, \mathscr{S}^n)$, *then* $T(K_1, \mathscr{S}^n) = T(K_2, \mathscr{S}^n)$.

c. *If* $K_i \to K$, *then* $T(K_i, \mathscr{S}^n) \to T(K, \mathscr{S}^n)$.

Proof. (a) This follows directly from the hypothesis and (1.8).

(b) $G(K_1, \mathscr{S}^n) = A(K_1, T(K_1, \mathscr{S}^n)) \geqslant A(K_2, T(K_1, \mathscr{S}^n)) \geqslant G(K_2, \mathscr{S}^n) = G(K_1, \mathscr{S}^n)$. From uniqueness, $T(K_1, \mathscr{S}^n) = T(K_2, \mathscr{S}^n)$.

(c) A consequence of (3.15b) of the next section is that a distinguished set is closed and therefore finitely compact. From the proof of (2.6), the set $\{T(K_i, \mathscr{S}^n)\}$ is bounded. Furthermore, every convergent subsequence of $\{T(K_i, \mathscr{S}^n)\}$ must converge to $T(K, \mathscr{S}^n)$ by uniqueness. Consequently the whole sequence converges to $T(K, \mathscr{S}^n)$. This completes the proof.

Let $\mathfrak{M}$ denote the group of central affine transformations with determinant ± 1. For the orthogonal group, we use the conventional notation $O(n)$. Denote by $\mathfrak{M}(\mathscr{S}^n)$ the subgroup of $\mathfrak{M}$ which leaves $\mathscr{S}^n$ invariant. Finally, denote by $\mathfrak{M}(K)$, $K \in \mathscr{K}^n$, the affine symmetry group that K would have if K is translated so that $s(K) = z$. Thus $\mathfrak{M}(K)$ is always a subgroup of $\mathfrak{M}$. In particular, $\mathfrak{M}(\mathscr{T}^n) = \mathfrak{M}(\mathscr{T}^n_0) = \mathfrak{M}(\mathscr{H}^n) = \mathfrak{M}$, where we observe that the affine image of a zonoid is a zonoid.

(2.12) LEMMA. *Suppose* $\mathscr{S}^n$ *is distinguished and* $K \in \mathscr{K}^n$.

a. $\mathfrak{M}(K) \cap \mathfrak{M}(\mathscr{S}^n) \subset \mathfrak{M}(T(K, \mathscr{S}^n))$.

b. *If* E *is an ellipsoid and* $\mathfrak{M}(E) \subset \mathfrak{M}(\mathscr{S}^n)$, *then* $T(E, \mathscr{S}^n)$ *is homothetic to* E.

Proof. (a) Let $\mathrm{g} \in \mathfrak{M}$. Then by a property of mixed volumes [22, p. 824], $V_1(\mathrm{g}K, \mathrm{g}T) = V_1(K, T)$ and therefore $G(\mathrm{g}K, \mathrm{g}\mathscr{S}^n) = G(K, \mathscr{S}^n)$. Now $\mathrm{g}\mathscr{S}^n$ is also distinguished and thus $\mathrm{g}T(K, \mathscr{S}^n) = T(\mathrm{g}K, \mathrm{g}\mathscr{S}^n)$. Let $\mathrm{g} \in \mathfrak{M}(K) \cap \mathfrak{M}(\mathscr{S}^n)$ and $s(K) = z$. Then $\mathrm{g}K = K$, $\mathrm{g}\mathscr{S}^n = \mathscr{S}^n$ and $\mathrm{g}T(K, \mathscr{S}^n) = T(K, \mathscr{S}^n)$. Thus $\mathrm{g} \in \mathfrak{M}(T(K, \mathscr{S}^n))$.

(b) For $\mathrm{g} \in \mathfrak{M}$ one may verify that $\mathrm{g}\mathfrak{M}(K)\mathrm{g}^{-1} = M(\mathrm{g}K)$ and $\mathrm{g}\mathfrak{M}(\mathscr{S}^n)\mathrm{g}^{-1} = \mathfrak{M}(\mathrm{g}\mathscr{S}^n)$. Now let g transform E into a ball. Then the orthogonal group $O(n) = \mathfrak{M}(\mathrm{g}E) = \mathrm{g}\mathfrak{M}(E)\mathrm{g}^{-1} \subset \mathrm{g}\mathfrak{M}(\mathscr{S}^n)\mathrm{g}^{-1} = \mathfrak{M}(\mathrm{g}\mathscr{S}^n)$. By (a), $O(n) \subset \mathfrak{M}(T(\mathrm{g}E, \mathrm{g}\mathscr{S}^n))$. Consequently $T(\mathrm{g}E, \mathrm{g}\mathscr{S}^n)$ is the unit euclidean ball B_n. Therefore $T(E, \mathscr{S}^n)$ is homothetic to E. This completes the Proof.

We denote by $\mathscr{E}^n$ the set of all ellipsoids in $\mathscr{T}^n$.

(2.13) COROLLARY. *Suppose* $\mathscr{S}^n$ *is distinguished and* $K \in \mathscr{K}^n$.

a. *If* K *is centrally symmetric and* $\mathfrak{M}(\mathscr{S}^n)$ *contains the reflection in the origin* z, *then* $T(K, \mathscr{S}^n)$ *is centrally symmetric.*

b. *If* $O(n) \subset \mathfrak{M}(\mathscr{S}^n)$, *then* $B_n \in \mathscr{S}^n$.

c. *If* $\mathfrak{M}(\mathscr{S}^n) = \mathfrak{M}$, *then* $\mathscr{E}^n \subset \mathscr{S}^n$.

Proof. This is a direct consequence of (2.12). In (b), B_n may be the only element in $\mathscr{S}^n$. In (c), $\mathscr{S}^n$ may coincide with $\mathscr{E}^n$ since in [22] it is shown that $\mathscr{E}^n$ is distinguished.

Let $\mathscr{K}(\mathscr{S}^n)$ denote the set of all convex bodies K such that K is homothetic to some element of $\mathscr{S}^n$.

(2.14) LEMMA. *If* $K \in \mathscr{K}(\mathscr{S}^n)$, *then*

$$n^n |K|^n I_m(K) \geqslant \pi_n G^n(K, \mathscr{S}^n).$$

Proof. We may assume that $s(K) = z$. Then, $T = (\pi_n^{-1} I_m(K))^{1/n} K \in \mathscr{S}^n$. Thus, by (1.8), $A^n(K, T) = n^n |T|\, |K|^{n-1} = n^n \pi_n^{-1} |K|^n I_m(K) \geqslant G^n(K, \mathscr{S}^n)$.

(2.15) THEOREM. *Let* $K \in \mathscr{K}^n$ *and set* $G(K) = G(K, \mathscr{T}^n)$. *Then,*

$$n^n \pi_n |K|^{n-1} \geqslant G^n(K)$$

with equality if and only if K is an ellipsoid.

Proof. The inequality follows from (2.14) and (1.3) where the equality could occur only for an ellipsoid. However, by (2.12b), (1.8) and (1.6), equality does occur if K is an ellipsoid. This completes the proof.

The supporting function $p(u)$ of the projection body P of a convex body K is given by

$$(2.16) \qquad p(u) = \tfrac{1}{2} \int_{\partial K} |u \cdot \tau| \, \mathrm{d}S_\tau = n V_1(K, \mathbf{u}).$$

Projection bodies are, therefore, always centrally symmetric about the origin z.

(2.17) THEOREM. *Let* $K_1, K_2 \in \mathscr{K}^n$ *have projection bodies* P_1, P_2 *respectively.*

a. *If* $P_1 \supset P_2$ *and* $\mathscr{S}^n \subset \mathscr{H}^n$, *then* $G(K_1, \mathscr{S}^n) \geqslant G(K_2, \mathscr{S}^n)$.

b. *If* $P_1 = P_2$ *and* $\mathscr{S}^n \subset \mathscr{T}_0^n$, *then* $G(K_1, \mathscr{S}^n) = G(K_2, \mathscr{S}^n)$. *In addition, If* $\mathscr{S}^n$ *is distinguished, then* $T(K_1, \mathscr{S}^n) = T(K_2, \mathscr{S}^n)$.

Proof. (a) We will show that $A(K_1, T) \geqslant A(K_2, T)$ for all $T \in \mathscr{H}^n$ if and only if $P_1 \supset P_2$. This implies (a). Let Q be a convex body with projection body $T \in \mathscr{H}^n$. By (2.16), the supporting function $\sigma(u)$ of T is given by

$$\sigma(u) = \tfrac{1}{2} \int_{\partial Q} |u \cdot \tau| \, \mathrm{d}S_\tau .$$

Substitution of this integral for $\sigma(u)$ in (1.7) and interchanging the order of integration yields $V_1(K_i, T) = V_1(Q, P_i)$. If $P_1 \supset P_2$, then $A(K_1, T) = nV_1(K_1, T) = nV_1(Q, P_1) \geqslant nV_1(Q, P_2) = nV_1(K_2, T) = A(K_2, T)$. Now suppose $A(K_1, T) \geqslant A(K_2, T)$ for all $T \in \mathscr{H}^n$. By the properties of mixed volumes, the inequality $V_1(K_1, T \geqslant V_1(K_2, T)$ may first be extended in T to all convex bodies in $\mathscr{K}(\mathscr{H}^n)$ and then to all zonoids; in particular, to unit segments. Hence $P_1 \supset P_2$ by (2.16).

(b) We will show that $A(K_1, T) = A(K_2, T)$ for all $T \in \mathscr{T}_0^n$ if and only if $P_1 = P_2$. This implies (b). Suppose $A(K_1, T) = A(K_2, T)$ for all $T \in \mathscr{T}_0^n$. Since $\mathscr{H}^n \subset \mathscr{T}_0^n$, the proof of (a) implies $P_1 = P_2$. Now suppose $P_1 = P_2$. Then $V_1(K_1, K) = V_1(K_2, K)$ for all centrally symmetric compact convex sets K; see [23, Theorem 1] and [28, p. 77]. Consequently, $A(K_1, T) = A(K_2, T)$ for all $T \in \mathscr{T}_0^n$.

(2.18) *Remark*. We observe that $A(K_1, T) = A(K_2, T)$ for all $T \in \mathscr{T}^n$ if and only if K_1 is a translate of K_2. For if the relative surface areas of K_1 and K_2 are always equal, then $V_1(K_1, K) = V_1(K_2, K)$ for all nonempty compact convex sets K. Consequently, by [5, p. 61], the area functions of K_1 and K_2 are equal on all Borel sets of Ω and K_1 must be a translate of K_2.

(2.19) THEOREM. *Let $K \in \mathscr{K}^n$ be a polytope. Then $T(K, \mathscr{T}^n)$ $[T(K, \mathscr{T}_0^n)]$ is a polytope with the property that each facet of $T(K, \mathscr{T}^n)$ $[T(K, \mathscr{T}_0^n)]$ with outer normal u is parallel to a facet of K with outer normal u* [u or $-u$].

Proof. We prove the theorem for $T(K, \mathscr{T}^n)$. The proof for $T(K, \mathscr{T}_0^n)$ is similar. Let T^* be the intersection of all closed supporting half-spaces to $T(K, \mathscr{T}^n)$ whose outer normals coincide with the outer normals to the facets of K. Then $G(K) = nV_1(K, T(K, \mathscr{T}^n)) = nV_1(K, T^*)$. Since $T^* \supset T(K, \mathscr{T}^n)$, $\pi_n = I(T(K, \mathscr{T}^n), z) \geqslant I(T^*, Z) \geqslant I_m(T^*)$, with equality throughout if and only if $T^* = T(K, \mathscr{T}^n)$. A translate of $(\pi_n^{-1} I_m(T^*))^{1/n} T^*$ belongs to $\mathscr{T}^n$. Thus $G(K) \leqslant nV_1(K, (\pi_n^{-1} I_m(T^*))^{1/n} T^*) = (\pi_n^{-1} I_m(T^*))^{1/n} G(K)$. Hence, $T^* = T(K, \mathscr{T}^n)$. This completes the proof.

If $K \in \mathscr{K}^n$ and K is homothetic to $T(K, \mathscr{T}^n)$, then we say that K is *selfminimal*. By (2.12b), the ellipsoids are selfminimal.

(2.20) THEOREM. *If $K \in \mathscr{K}^n$ is an affinely regular polytope, then K is selfminimal.*

Proof. For the definitions of regular and affinely regular polytopes, see [12, pp. 411–412]. Since $\mathfrak{M}(\mathscr{T}^n) = \mathfrak{M}$, we may assume that K is regular and $s(K) = z$. Let u be the outer normal of a facet of the polytope $T(K, \mathscr{T}^n)$ and let v be the outer normal of a facet of K. By Lemma (2.12a), the symmetry of K which interchanges the facets of K with outer normals u and v is also a symmetry of $T(K, \mathscr{T}^n)$. Consequently the facets of K and $T(K, \mathscr{T}^n)$ have the same set of outer normals and the (n-1)-dimensional measures of corresponding facets are proportional. By Minkowski's fundamental theorem [3, pp. 118–119], K and $T(K, \mathscr{T}^n)$ are homothetic.

(2.21) LEMMA. *Let $\mathscr{F}^n$ be a nonempty affine invariant subset of $\mathscr{K}^n$ which is closed relative to $\mathscr{K}^n$. If $f(K)$ is a real-valued continuous function on $\mathscr{F}^n$ which has the same value for all affine transforms of K, then the infimum and supremum of $f(K)$ are attained on $\mathscr{F}^n$.*

Proof. By a result of John [16] (see also Leichweiss [17]), there exists an ellipsoid $E \subset K$ such that the concentric ellipsoid E^* in the ratio n is the Löwner-ellipsoid of K. Thus an affine image K^* of K satisfies $B_n \subset K^* \subset nB_n$. The lemma now follows from Blaschke's Selection Theorem and the properties of a continuous function on a compact set. This completes the proof.

We define,

$$a(\mathscr{F}^n, \mathscr{S}^n) = \inf\{|K|^{1-n} G^n(K, \mathscr{S}^n) \mid K \in \mathscr{F}^n \subset \mathscr{K}^n\},$$
$$b(\mathscr{F}^n, \mathscr{S}^n) = \sup\{|K|^{1-n} G^n(K, \mathscr{S}^n) \mid K \in \mathscr{F}^n \subset \mathscr{K}^n\},$$
$$d(\mathscr{S}^n) = \inf\{|K|\, I_m(K) \mid K \in \mathscr{K}(\mathscr{S}^n)\},$$
$$\mathscr{A}(\mathscr{F}^n, \mathscr{S}^n) = \{K \in \mathscr{F}^n \mid |K|^{1-n} G^n(K, \mathscr{S}^n) = a(\mathscr{F}^n, \mathscr{S}^n)\},$$
$$\mathscr{B}(\mathscr{F}^n, \mathscr{S}^n) = \{K \in \mathscr{F}^n \mid |K|^{1-n} G^n(K, \mathscr{S}^n) = b(\mathscr{F}^n, \mathscr{S}^n)\},$$
$$\mathscr{D}(\mathscr{S}^n) = \{K \in \mathscr{K}(\mathscr{S}^n) \mid |K|\, I_m(K) = d(\mathscr{S}^n)\}.$$

(2.22) LEMMA. *Let $\mathscr{F}^n$ be any nonempty subset of $\mathscr{K}^n$.*

a. *If $\mathscr{F}^n \supset \mathscr{K}(\mathscr{S}^n)$, then $a(\mathscr{F}^n, \mathscr{S}^n) = n^n \pi_n^{-1} d(\mathscr{S}^n)$ and $\mathscr{D}(\mathscr{S}^n) \subset \subset \mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$.*

b. *If $\mathscr{F}^n \supset \mathscr{K}(\mathscr{S}^n)$ and $\mathscr{S}^n$ is closed, then $\mathscr{D}(\mathscr{S}^n) = \mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$ and if $K \in \mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$, then $T(K, \mathscr{S}^n)$ exists and is homothetic to K.*

c. *If $\mathscr{F}^n$ satisfies the hypothesis of (2.21) and $\mathfrak{M}(\mathscr{S}^n) = \mathfrak{M}$, then $\mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$ and $\mathscr{B}(\mathscr{F}^n, \mathscr{S}^n)$ are nonempty.*

d. *If $\mathscr{S}^n$ is closed and $\mathfrak{M}(\mathscr{S}^n) = \mathfrak{M}$, then $\mathscr{D}(\mathscr{S}^n)$ is nonempty.*

Proof. (a) Let $T \in \mathscr{S}^n$ and $K \in \mathscr{F}^n$. From (1.8) and (1.5), $|K|^{1-n} A^n(K, T) \geqslant \geqslant n^n \pi_n^{-1} I_m(T)\, |T| \geqslant n^n \pi_n^{-1} d(\mathscr{S}^n)$. Consequently, $|K|^{1-n} G^n(K, \mathscr{S}^n) \geqslant \geqslant n^n \pi_n^{-1} d(\mathscr{S}^n)$ and thus, $a(\mathscr{F}^n, \mathscr{S}^n) \geqslant n^n \pi_n^{-1} d(\mathscr{S}^n)$. Suppose $K \in \mathscr{K}(\mathscr{S}^n) \subset \subset \mathscr{F}^n$. By (2.14), $n^n \pi_n^{-1} I_m(K)\, |K| \geqslant |K|^{1-n} G^n(K, \mathscr{S}^n) \geqslant a(\mathscr{F}^n, \mathscr{S}^n)$. Thus, $n^n \pi_n^{-1} d(\mathscr{S}^n) \geqslant a(\mathscr{F}^n, \mathscr{S}^n)$. Now, suppose there exists $K \in \mathscr{D}(\mathscr{S}^n)$. Then $K \in \mathscr{K}(\mathscr{S}^n) \subset \mathscr{F}^n$ and $I_m(K)\, |K| = d(\mathscr{S}^n)$. By the argument directly above, $|K|^{1-n} G^n(K, \mathscr{S}^n) = a(\mathscr{F}^n, \mathscr{S}^n)$ and $K \in \mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$.

(b) Suppose $K \in \mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$. By (2.5), there exists $T \in \mathscr{S}^n$ such that $G(K, \mathscr{S}^n) = A(K, T)$. Then $a(\mathscr{F}^n, \mathscr{S}^n) = |K|^{1-n} G^n(K, \mathscr{S}^n) = = |K|^{1-n} A^n(K, T) \geqslant n^n \pi_n^{-1} I_m(T)\, |T| \geqslant n^n \pi_n^{-1} d(\mathscr{S}^n) = a(\mathscr{F}^n, \mathscr{S}^n)$. Since equality must hold throughout, T is homothetic to K (by (1.8)) and T (therefore K) belongs to $\mathscr{D}(\mathscr{S}^n)$.

(c) By the hypothesis, (2.6) and the proof of (2.12a), the function $f(K) = |K|^{1-n} G^n(K, \mathscr{S}^n)$ satisfies the conditions in Lemma (2.21). Consequently, $\mathscr{A}(\mathscr{F}^n, \mathscr{S}^n)$ and $\mathscr{B}(\mathscr{F}^n, \mathscr{S}^n)$ are nonempty.

(d) The proof is similar to (c).

(2.23) *Remark.* The values of $a(\mathscr{F}^n, \mathscr{S}^n)\, b(\mathscr{F}^n, \mathscr{S}^n)$ are known only in a few special cases. From (2.15), we have $b(\mathscr{K}^n, \mathscr{T}^n) = n^n \pi_n$ and $\mathscr{B}(\mathscr{K}^n, \mathscr{T}^n)$ is the set of ellipsoids. Gustin [14] shows that $b(\mathscr{K}^2, \mathscr{E}^2) = 12\sqrt{3}$ and $\mathscr{B}(\mathscr{K}^2, \mathscr{E}^2)$ is the set of triangles. A common conjecture (see [19] and [13, p. 59]) is $d(\mathscr{T}^n) = (n+1)^{n+1}((n!)^{-2}$ and $d(\mathscr{T}_0^n) = 4^n (n!)^{-1}$, where $\mathscr{D}(\mathscr{T}^n)$ contains the n-simplexes and $\mathscr{D}(\mathscr{T}_0^n)$ contains the parallelotopes and crosspolytopes. These conjectures are, at least, consistent with Lemma (2.22) and Theorem (2.20).

(2.24) THEOREM. *Let* $K \in \mathscr{K}^2$. *Then*

$$b(\mathscr{K}^2, \mathscr{T}^2) = 4\pi \geqslant |K|^{-1} G^2(K, \mathscr{T}^2) \geqslant 27\pi^{-1} = a(\mathscr{K}^2, \mathscr{T}^2),$$

$$b(\mathscr{K}^2, \mathscr{T}_0^2) = 54\pi^{-1} \geqslant |K|^{-1} G^2(K, \mathscr{T}_0^2) \geqslant 32\pi^{-1} = a(\mathscr{K}^2, \mathscr{T}_0^2),$$

where $\mathscr{B}(\mathscr{K}^2, \mathscr{T}^2)$ *is the set of ellipses,* $\mathscr{A}(\mathscr{K}^2, \mathscr{T}^2)$ *and* $\mathscr{B}(\mathscr{K}^2, \mathscr{T}_0^2)$ *contain the triangles, and* $\mathscr{A}(\mathscr{K}^2, \mathscr{T}_0^2)$ *contains the parallelograms.*

Proof. Mahler [18] shows that $d(\mathscr{T}^2) = \frac{27}{4}$ and $d(\mathscr{T}_0^2) = 8$ where $\mathscr{D}(\mathscr{T}^2)$ contains the triangles and $\mathscr{D}(\mathscr{T}_0^2)$ contains the parallelograms. The right-hand side of both inequalities now follow from Lemma (2.22). Because of (2.15), it only remains to prove the left-hand side of the second inequality.

Given $K \in \mathscr{K}^2$, Eggleston [8] shows that there exists a triangle K_1 with $|K_1| = |K|$, and the width of K_1 in every direction is greater than or equal to the corresponding width of K. Since $\mathscr{K}^2 = \mathscr{T}_0^2$, Theorem (2.17a) implies $|K_1|^{-1} G^2(K_1, \mathscr{T}_0^2) \geqslant |K|^{-1} G^2(K, \mathscr{T}_0^2)$ and therefore $\mathscr{B}(\mathscr{K}^2, \mathscr{T}_0^2)$ contains the triangles. To calculate $b(\mathscr{K}^2, \mathscr{T}_0^2)$, let $H = \frac{1}{2}K_1 + \frac{1}{2}(-K_1)$. Then H is an affinely regular hexagon and has the same projection body as K_1. By Theorem (2.17b), $G(H, \mathscr{T}_0^2) = G(K_1, \mathscr{T}_0^2)$ and $T(H, \mathscr{T}_0^2) = T(K_1, \mathscr{T}_0^2)$. But by Theorem (2.20), H is selfminimal. Consequently, $|K_1|^{-1} G^2(K_1, \mathscr{T}_0^2) = (3/2)\, |H|^{-1} G^2(H, \mathscr{T}_0^2) = 6\pi^{-1} |H|\, I_m(H) = 54\pi^{-1}$.

3. Surfaces of constant curvature

We define the relative surface area ΔA of a neighborhood ΔS of a point p of a convex surface ∂K by

$$(3.1) \qquad \Delta A = \int_{\Delta S} \sigma(u)\, \mathrm{d}S_u,$$

where $\sigma(u)$ is the supporting function of $T \in \mathcal{T}^n$. An analogue of the Gaussian curvature of ∂K at p may be introduced as follows: Let H be the tangent hyperplane to K at an elliptic point p. Let J_h be the small convex body cut off from K be a hyperplane H_h parallel to H at distance h. Let ΔA be the relative surface area of that portion ΔS of ∂K which forms the cap of J_h. Then we define the relative curvature by

$$(3.2) \qquad C(K, p; T) = \lim_{\Delta A \to 0} \frac{(n+1)^{n-1} \pi_{n-1}^2 \, |J_h|^{n-1}}{(\Delta A)^{n+1}}.$$

To justify this definition, we will show that, under the usual regularity assumptions on ∂K, $C(K, p; B_n)$ is the ordinary Gaussian curvature. We remark that an advantage of this definition over the one commonly used in relative differential geometry (see [3, p. 64]) is that the existence of (3.2) depends only on the regularity of ∂K and not on T.

Let ∂K be of class C^2 with positive Gaussian curvature. Let p be the origin z and let H have the equation $x_n = 0$ with K contained in the halfspace $x_n \geqslant 0$. Let the remaining coordinate axes be in the principal directions of ∂K at z and let $\{R_i\}$ be the principal radii of curvature of ∂K at z. Then locally, the surface ∂K is given by

$$(3.3) \qquad x_n = \sum_{i=1}^{n-1} (2R_i)^{-1} x_i^2 + r(x_1, \ldots, x_{n-1}),$$

where $\varepsilon = r(x_1, \ldots, x_{n-1}) \left(\sum_{i=1}^{n-1} x^2\right)^{-1} \to 0$ as $\sum_{i=1}^{n-1} x_i^2 \to 0$. Let $J_h = \{x \in K \,|\, 0 \leqslant x_n \leqslant h\}$ and let A_h be the projection of J_h on H. Then $\Delta A / |A_h|_{n-1} \to 1$ as $h \to 0$. Let A_h' be the image of A_h under the dilatation $y_i = h^{-1/2} x_i$, $i = 1, \ldots, n-1$. Then $|A_h|_{n-1} = h^{(n-1)/2} |A_h'|_{n-1}$. As $h \to 0$, A_h' converges to the $(n-1)$-dimensional ellipsoid (essentially Dupin's indicatrix)

$$E_{n-1} = \{(y_1, \ldots, y_{n-1}, 0) \,|\, \sum_{i=1}^{n-1} (2R_i)^{-1} y_i^2 \leqslant 1\},$$

and $|E|_{n-1} = \pi_{n-1} (2^{n-1} R_1 \cdots R_{n-1})^{1/2}$. Consequently

$$(3.4) \qquad \lim_{h \to 0} \frac{2^{n-1} \pi_{n-1}^2 \, h^{n-1}}{(\Delta A)^2} = (R_1 \ldots R_{n-1})^{-1}.$$

A calculation (see [21, p. 1542]) shows that

$$\int_{E_{n-1}} y_i^2 \, \mathrm{d}y_1 \ldots \mathrm{d}y_{n-1} = \frac{2\pi_{n-1} R_i}{(n+1)} (2^{n-1} R_1 \ldots R_{n-1})^{1/2}.$$

Also

$$|J_h| = \int_{A_h} (h - x_n)\, dx_1 \dots dx_{n-1}$$
$$= h^{(n+1)/2}\left[|A'_h|_{n-1} - \sum_{i=1}^{n-1} \int_{A'_h} [(2R_i)^{-1} + \varepsilon]\, y_i^2\, dy_1 \dots dy_{n-1}\right].$$

Consequently

$$(3.5) \qquad \lim_{h\to 0} \frac{2^{n+1}\pi_{n-1}^2 h^{n+1}}{(n+1)^2 |J_h|^2} = (R_1 \dots R_{n-1})^{-1}.$$

Combining (3.4) and (3.5), we obtain $C(K, z; B_n) = (R_1 \dots R_{n-1})^{-1}$.

If u is the outer normal to K at p, then from (3.1) and (3.2) we have

$$(3.6) \qquad C(K, p; T) = \sigma^{-n-1}(u)\, C(K, p; B_n).$$

In E^n the Gaussian curvature of ∂K at p may also be defined as the limit of the ratio of the area of the spherical image of ΔS to the area of ΔS as ΔS shrinks to the point p. This definition does not presuppose the regularity of the hypersurface ∂K and it may exist when the ordinary Gaussian curvature does not (see [5, Chapter 1]). The reciprocal Gaussian curvature in this sense, viewed as a function on Ω, will be called the *curvature function* of K. A necessary and sufficient condition for a given positive continuous function $f(u)$ defined on Ω to be a curvature function of a convex body K is that

$$(3.7) \qquad \int_{\Omega} u f(u)\, d\omega = 0.$$

See [3, Chapter 13] and [5, pp. 60–67].

From (3.6), a necessary condition for ∂K to have constant relative curvature k^{-1} is that its curvature function $f(u) = k\sigma^{-n-1}(u)$. But, by (1.4), for each $T \in \mathscr{T}^n$, the function $f(u)$ defined by $f(u) = k\sigma^{-n-1}(u)$ satisfies (3.7) and is therefore the curvature function of a convex body K, uniquely determined up to a translation. Without presupposing the regularity of ∂K we state the following definition:

(3.8) DEFINITION. For $T \in \mathscr{T}^n$, the convex surface ∂K is said to have constant relative curvature $k^{-1} > 0$ if K possesses the curvature function $f(u) = k\sigma^{-n-1}(u)$ for $u \in \Omega$.

Now let $f(u)$ be any positive continuous curvature function on Ω and set $H(u) = |u| f^{-1/n+1}(u/|u|)$ for $u \neq z$ and $H(z) = 0$. Let $\mathscr{V}^n$ denote the set of all convex bodies in E^n with curvature function $f(u)$ such that $H(u)$ is a convex function on E^n. The affine surface area of a convex body K with a positive continuous curvature function $f(u)$ is defined by

$$(3.9) \qquad A_a(K) = \int_{\Omega} f(u)^{n/n+1}\, d\omega .$$

Each $K \in \mathscr{V}^n$ uniquely determines $T \in \mathscr{T}^n$ such that ∂K has constant relative curvature. By (1.5) and (3.9), the supporting function $\sigma(u)$ of T is given by

$$(3.10) \qquad \sigma(u) = [(n\pi_n)^{-1} A_a(K)]^{1/n} f(u)^{-1/n+1}, \qquad u \in \Omega .$$

We remark that there is a one-to-one correspondence between $\mathscr{T}^n$ and the equivalence classes of $\mathscr{V}^n$ consisting of homothetic members. Thus, $\mathscr{V}^n$ is a fairly wide class. On the other hand, the results (3.11), (3.12), (3.18), (3.21) which follow, show that $\mathscr{V}^n$ is a rather special subclass of $\mathscr{K}^n$.

A geometrical connection between $K \in \mathscr{V}^n$ and the corresponding $T \in \mathscr{T}^n$ is given by the following result.

(3.11) THEOREM. *Let $K \in \mathscr{V}^n$ have constant relative curvature with respect to $T \in \mathscr{T}^n$. If Z is the polar reciprocal of T with respect to $s(T) = z$, then the projection body of K is homothetic to the centroid body of Z.*

Proof. The geometric interpretation and properties of centroid bodies are given in [21]. The supporting function $H(u)$ of the centroid body of Z is given by the volume integral

$$H(u) = \frac{1}{\pi_n} \int_{Z} |u \cdot x|\, dV_x .$$

Since $r = \sigma^{-1}(u)$ is the polar equation of the boundary of Z, we have

$$H(u) = \frac{1}{(n+1)\pi_n} \int_{\Omega} |u \cdot \tau|\, \sigma^{-n-1}(\tau)\, d\omega_\tau .$$

Now, from (3.8), K possesses the curvature function $f(u) = k\sigma^{-n-1}(u)$ for some $k > 0$. Since the mixed volume integral representations are valid for curvature functions, the supporting function of the projection body P of K, by (2.16), is given by

$$p(u) = \tfrac{1}{2} \int_{\Omega} |u \cdot \tau|\, k\sigma^{-n-1}(\tau)\, d\omega_\tau .$$

Consequently, P is homothetic to the centroid body of Z.

A fundamental link with affine differential geometry is the following result.

(3.12) THEOREM. *Let K be a convex body with a continuous positive curvature function $f(u)$. Then*

$$n\pi_n G^n(K) \geqslant [A_a(K)]^{n+1}$$

with equality if and only if $K \in \mathscr{V}^n$.

Proof. For any $T \in \mathscr{T}^n$,

$$A(K, T) = \int_{\Omega} \sigma(u) f(u) \,\mathrm{d}\omega .$$

A form of Hölder's inequality for integrals [15, p. 140] states that for positive continuous functions g, h

$$\int gh \geqslant \left(\int g^k\right)^{1/k} \left(\int h^{k'}\right)^{1/k'},$$

where $k<0$, $k'=k/k-1$, and the equality holds if and only if g^k is proportional to $h^{k'}$. Setting $k=-1/n$, $g=\sigma$, $h=f$ and using (1.5), we obtain

$$A(K, T) \geqslant (n\pi_n)^{-1/n} [A_a(K)]^{(n+1)/n}.$$

with equality if and only if $f(u)\,\sigma^{n+1}(u)$ is a constant on Ω. The inequality in (3.12) follows directly. If $K \in \mathscr{V}^n$, then it is easily seen that equality must hold in (3.12). On the other hand, if equality holds in (3.12), then, since $\mathscr{T}^n$ is distinguished, there exists a unique $T \in \mathscr{T}^n$ such that $A(K, T) = G(K)$ and consequently $K \in \mathscr{V}^n$. A corollary of this proof is the following:

(3.13) COROLLARY. *If* $K \in \mathscr{V}^n$ *and* $T \in \mathscr{T}^n$, *then* ∂K *has constant relative curvature with respect to T if and only if* $T = T(K, \mathscr{T}^n)$. *If* $T = T(K, \mathscr{T}^n)$, *then* $A^n(K, T) = G^n(K) = (n\pi_n)^{-1} [A_a(K)]^{n+1}$.

If $T \in \mathscr{S}^n$, we define

$$\text{(3.14)} \qquad \mathscr{K}(T, \mathscr{S}^n) = \{K \in \mathscr{K}^n \mid A(K, T) = G(K, \mathscr{S}^n)\}.$$

(3.15) COROLLARY.

a. *If* $T \in \mathscr{S}^n$, *then* $\mathscr{K}(T, \mathscr{S}^n)$ *is nonempty.*

b. (*Converse of* (2.5)). *If for each* $K \in \mathscr{K}^n$ *there exists* $T \in \mathscr{S}^n$ *such that* $G(K, \mathscr{S}^n) = A(K, T)$, *then* $\mathscr{S}^n$ *is closed.*

Proof. (a) Clearly $\mathscr{K}(T, \mathscr{S}^n) \supset \mathscr{K}(T, \mathscr{T}^n)$ and $\mathscr{K}(T, \mathscr{T}^n)$ always contains the convex bodies whose surfaces have constant relative curvature. These may, however, be the only bodies in $\mathscr{K}(T, \mathscr{T}^n)$, see (3.25b).

(b) Let $T \in \mathrm{cl}(\mathscr{S}^n)$ and let $K \in \mathscr{K}(T, \mathscr{T}^n)$. Then, by hypothesis there exists $T_0 \in \mathscr{S}^n$ such that $A(K, T_0) = G(K, \mathscr{S}^n)$. But $G(K, \mathscr{S}^n) = G(K, \mathrm{cl}(\mathscr{S}^n)) = A(K, T) = G(K)$. Since $T = T(K, \mathscr{T}^n)$ is unique, $T = T_0 \in \mathscr{S}^n$.

(3.16) *Remark.* From (1.3), (2.14) and (3.12), we obtain

$$(n\pi_n)^2 (n|K|)^{n-1} \geqslant n^{n+1} |K|^n I_m(K) \geqslant n\pi_n G^n(K) \geqslant [A_a(K)]^{n+1}.$$

The inequality between the first and last members is called the affine isoperimetric inequality. It is due to Blaschke for $n=2,3$ and generalized by Santaló [26] to all $n \geqslant 2$.

(3.17) THEOREM. *For $T \in \mathscr{S}^n$, the set $\mathscr{K}(T, \mathscr{S}^n)$ is closed under Blaschke addition.*

Proof. For the definition and properties of Blaschke addition, see [9], [10] and [12, pp. 331–340]. From the integral representation of $V_1(K, T)$ (see [5, p. 62]), and the additivity with respect to the area functions, we obtain $A(K_1 \# K_2, T) = A(K_1, T) + A(K_2, T)$. Hence, $G(K_1 \# K_2, \mathscr{S}^n) \geqslant G(K_1, \mathscr{S}^n) + G(K_2, \mathscr{S}^n)$. Suppose $K_1, K_2 \in \mathscr{K}(T, \mathscr{S}^n)$. Then $G(K_1, \mathscr{S}^n) + G(K_2, \mathscr{S}^n) = A(K_1, T) + A(K_2, T) = A(K_1 \# K_2, T) \geqslant G(K_1 \# K_2, \mathscr{S}^n)$. Consequently, $K_1 \# K_2 \in \mathscr{K}(T, \mathscr{S}^n)$.

(3.18) LEMMA. *The set $\mathscr{V}^n$ is an affine invariant subset of $\mathscr{K}^n$ and is closed relative to $\mathscr{K}^n$.*

Proof. A proof of the affine invariant property of $\mathscr{V}^n$ is contained in [24, p. 38]. Now let $\{K_i\} \subset \mathscr{V}^n$ and let $K_i \to K \in \mathscr{K}^n$. From (3.10) and (3.12), we obtain $f_i(u) = [(n\pi_n)^{-1} G(K_i)]\, \sigma_i^{-n-1}(u)$ where $\sigma_i(u)$ is the supporting function of $T(K_i, \mathscr{T}^n)$. By Lemma (2.11c), $T(K_i, \mathscr{T}^n) \to T(K, \mathscr{T}^n)$ and, consequently, $\sigma_i(u)$ converges uniformly on Ω to the supporting function $\sigma(u)$ of $T(K, \mathscr{T}^n)$. Thus, using (2.6), $f_i(u)$ converges uniformly to $f(u) = [(n\pi_n)^{-1} G(K)]\, \sigma^{-n-1}(u)$ on Ω. By the continuity of mixed volumes,

$$nV(H, K, \ldots, K) = \int_{\Omega} L(u) f(u)\, d\omega$$

for an arbitrary compact convex set H with supporting function $L(u)$. Consequently, by the uniqueness theorem [3, p. 115], $f(u)$ is the curvature function of K and $K \in \mathscr{V}^n$.

(3.19) THEOREM. *For $T \in \mathscr{T}^n$, let ∂V be a surface of unit constant relative curvature. Then $A(V, T) = G(V) = n\pi_n$ and $|V| \geqslant \pi_n$ with equality if and only if T (or V) is an ellipsoid.*

Proof. The equality $A(V, T) = n\pi_n$ follows directly from (3.8), (1.7) and (1.5). The inequality $|V| \geqslant \pi_n$ now follows from (2.15), with equality if and only if V is an ellipsoid. If T is an ellipsoid, then, from (1.8) and (1.6), $|V| \leqslant \pi_n$ and therefore V is a translate of T. On the other hand, if V is an ellipsoid, then (2.12b) implies that T is an ellipsoid.

(3.20) *Remark.* The max $|V|$ is unknown. However, by (3.18) and (2.22), $\max|V|^{n-1}=(n\pi_n)^n a^{-1}(\mathscr{V}^n,\mathscr{T}^n)$ and the maximum is attained when $V\in\mathscr{A}(\mathscr{V}^n,\mathscr{T}^n)$.

Winternitz ([29], also [2, p. 200]) proved that if a convex body K with affine surface area $A_a(K)$ is properly contained in an ellipsoid E, then $A_a(K)<A_a(E)$. Since ellipsoids are members of $\mathscr{V}^n$, the following theorem generalizes this result.

(3.21) THEOREM. *Let K_1 be a convex body with affine surface area $A_a(K_1)$ and let $K_2\in\mathscr{V}^n$. If $K_1\subset K_2$, then $A_a(K_1)\leqslant A_a(K_2)$ with equality if and only if $K_1=K_2$.*

Proof. By (3.12) and (2.1b), we have

$$[A_a(K_1)]^{n+1}\leqslant n\pi_n G^n(K_1)\leqslant n\pi_n G^n(K_2)=[A_a(K_2)]^{n+1}.$$

If the equality holds throughout, then $K_1\in\mathscr{V}^n$ and $G(K_1)=G(K_2)$. Consequently, from (2.11b), $T(K_1,\mathscr{T}^n)=T(K_2,\mathscr{T}^n)$. By (3.13), K_1 and K_2 are homothetic and, by (2.1a), $K_1=K_2$. This completes the proof.

A proof of the following Lemma is given in [24, p. 39].

(3.22) LEMMA. *Let Q be a convex body with projection body P_Q such that $T_Q=\pi_{n-1}^{-1}P_Q\in\mathscr{H}^n$. Then*

$$A(K,T_Q)=\pi_{n-1}^{-1}\int_{\partial Q}p(u)\,\mathrm{d}S_u,$$

where $p(u)$ is the supporting function of the projection body of K. Moreover, $\pi_n\geqslant|Q|$ with equality if and only if Q is an ellipsoid.

(3.23) THEOREM. *Let K be a convex body with projection body P. Then*

$$|P|\geqslant n^{-n}\pi_{n-1}^n\pi_n^{1-n}G^n(K,\mathscr{H}^n)$$

with equality if and only if K is an affine transform of an euclidean body of constant brightness.

Proof. In Lemma (3.22), choose Q homothetic to the projection body P of K. Then $G^n(K,\mathscr{H}^n)\leqslant A^n(K,T_Q)=n^n\pi_{n-1}^{-n}V^n(P,Q,\ldots,Q)=n^n\pi_{n-1}^{-n}|Q|^{n-1}\times$ $\times|P|\leqslant n^n\pi_{n-1}^{-n}\pi_n^{n-1}|P|$, which establishes the inequality. Affine transforms of euclidean bodies of constant brightness are characterized by the fact that their projection bodies are ellipsoids. Now if the equality holds then Q, and hence P, is an ellipsoid. On the other hand suppose P is an ellipsoid. Let E be an ellipsoid with projection body P. By (2.17b), $G(K,\mathscr{H}^n)=$ $=G(E,\mathscr{H}^n)$. Applying the above proof to E in place of K, we obtain equality throughout since T_Q is homothetic to E and E is selfminimal.

(3.24) *Remark.* As a companion to (3.17), given $K\in\mathscr{K}(T_Q,\mathscr{H}^n)$, Lemma (3.22) permits the construction of other members of $\mathscr{K}(T_Q,\mathscr{H}^n)$. Suppose Q is not differentiable, e.g. Q is a polytope. Let R be the intersection of all closed supporting halfspaces to the projection body P of K whose outer normals correspond to tangent hyperplanes to Q. Then $V_1(Q,P)=$ $=V_1(Q,R)$. Suppose P_1 is the projection body of K_1 such that $P\subset P_1\subset R$. Then $G(K,\mathscr{H}^n)=A(K,T_Q)=A(K_1,T_Q)\geqslant G(K_1,\mathscr{H}^n)$. But (2.17a) implies $G(K_1,\mathscr{H}^n)\geqslant G(K,\mathscr{H}^n)$ and therefore $K_1\in\mathscr{K}(T_Q,\mathscr{H}^n)$. Now let ∂K have constant relative curvature with respect to T_Q. Then $K\in\mathscr{K}(T_Q,\mathscr{T}^n)$. Let W be the intersection of all infinite cylinders $K+L$ where L is a line through z parallel to a normal to a tangent hyperplane to Q. If K_1 satisfies $K\subset K_1\subset W$, then its projection body P_1 satisfies $P\subset P_1\subset R$. Therefore $G(K,\mathscr{T}^n)=$ $=G(K,\mathscr{H}^n)=G(K_1,\mathscr{H}^n)\geqslant G(K_1,\mathscr{T}^n)$. But (2.1b) implies $G(K_1,\mathscr{T}^n)\geqslant$ $\geqslant G(K,\mathscr{T}^n)$ and therefore $K_1\in\mathscr{K}(T_Q,\mathscr{T}^n)$. For $n=2$ and Q a polygon, one may verify the following: The set $\mathscr{K}(T_Q,\mathscr{T}^2)$ contains an infinite subset, no two of which are homothetic. There exist polygons in $\mathscr{K}(T_Q,\mathscr{T}^2)$ with an arbitrarily large number of sides and, in contrast to (2.13a), $\mathscr{K}(T_Q,\mathscr{T}^2)$ contains members which are not centrally symmetric.

When $T\in\mathscr{T}^n_0$, we define the relative brightness of a convex body K in the direction u. Let L be the line through z parallel to u. The relative brightness of K in the direction u is defined as the minimal relative cross sectional area of the infinite cylinder $K+L$ where the relative area of a section is obtained from (3.1). Since $\sigma(u)=\sigma(-u)$, there is no ambiguity due to the orientation of the section. In [24, Theorem 2] it is proved that K has constant relative brightness if and only if its projection body is homothetic to the polar reciprocal Z of T. When $E\in\mathscr{T}^n_0$ is an ellipsoid, this definition of constant relative brightness coincides with the definition given in [6] applied to E.

(3.25) COROLLARY. *Let $E\in\mathscr{T}^n$ be an ellipsoid.*

a. *If $\mathscr{H}^n\subset\mathscr{S}^n\subset\mathscr{T}^n_0$, then $\mathscr{K}(E,\mathscr{S}^n)$ is the set of all convex bodies of constant relative brightness.*

b. *$\mathscr{K}(E,\mathscr{T}^n)$ is the set of all ellipsoids homothetic to E.*

Proof. Let P be the projection body of $K\in\mathscr{K}^n$. Now K has constant relative brightness if and only if P is homothetic to the polar reciprocal Z of E. In (3.22), we may choose $Q=Z$ which gives $T_Q=E$. Since $|Z|=\pi_n$. Minkowski's inequality for mixed volumes applied to (3.22) gives

$$(3.26)\qquad A^n(K,E)\geqslant n^n\pi_{n-1}^{-n}\pi_n^{n-1}\,|P|,$$

with equality if and only if K has constant relative brightness. Consequently, by (3.23), $\mathscr{K}(E,\mathscr{H}^n)$ is the set of all convex bodies of constant relative brightness. But, by (2.17b), $\mathscr{K}(E,\mathscr{T}_0)$ must also contain these

bodies and, since $\mathscr{K}(E, \mathscr{H}^n) \supset \mathscr{K}(E, \mathscr{S}^n) \supset \mathscr{K}(E, \mathscr{T}_0^n)$, the result (a) follows.
The result (b) follows directly from (1.8), (1.6) and (2.15).

(3.27) *Remark.* Let K have euclidean surface area S. Then from (3.26), (3.23) and (3.12) we obtain

$$S^n \geqslant n^n \pi_{n-1}^{-n} \pi_n^{n-1} |P| \geqslant G^n(K, \mathscr{H}^n) \geqslant$$
$$\geqslant G^n(K) \geqslant (n\pi_n)^{-1} [A_a(K)]^{n+1}.$$

The inequality between the first and last members is Berwald's generalization of an inequality of Winternitz [2, p. 206]. The inequality between the second and last members was obtained by the author in [23, p. 240].

There is a close connection between affine distance and relative curvature when these concepts are applied to $T \in \mathscr{T}^n$. For $z \in \operatorname{int} K$ and $p \in \partial K$, the affine distance from z to p may be defined by $a(K, p) = H(u)\, C^{-1/n+1}(u)$, where $H(u)$ is the supporting function of K and $C(u)$ is the Gaussian curvature of ∂K at the point p with outer normal u. From (3.2), we obtain

$$[a(K, p)]^{n+1} = \lim_{h \to 0} \frac{n^{n+1} |J_1|^{n+1}}{(n+1)^{n-1} \pi_{n-1}^2 |J_h|^{n-1}},$$

where J_1 is the cone with vertex z and base $H_h \cap K$. This is a generalization of Blaschke's geometric interpretation [2, p. 128]. If $a(K, p)$ is constant for $p \in \partial K$ (the affine normals will pass through z), then ∂K is an affine sphere. From (3.6) we obtain $C(T, p; T) = [a(T, p)]^{-n-1}$. We may now give the Blaschke-Deicke theorem a new interpretation.

(3.29) THEOREM. *For $n \geqslant 2$, let $T \in \mathscr{T}^n$ and for $n \geqslant 3$, suppose that the distance function $F(x)$ of T with respect to z is of class C^4 (except at z). Then, if ∂T has constant relative curvature, T is an ellipsoid.*

Proof. Let $n \geqslant 3$. Set $g_{ij} = \partial^2(\frac{1}{2}F^2)/\partial x_i \partial x_j$. Then $\det[g_{ij}] = C(T, x; T) =$ $=$ constant. See Deicke [7]. Simplifying Deicke's proof, Brickell [4] shows that if F is of class C^4 (except at z) and $\det[g_{ij}]$ is constant, then $[g_{ij}]$ is constant. Consequently T is an ellipsoid.

The case $n = 2$ admits an elementary and explicit solution. Reviewing briefly the planar situation [3, pp. 65–66], if $h(\phi)$ is any function of class C^2 on the unit circle Ω, then the corresponding positively homogeneous function $H(u)$ is the supporting function of a convex body if and only if $h'' + h \geqslant 0$ on Ω. This follows from

$$a^2 \frac{\partial^2 H}{\partial u_1^2} + 2ab \frac{\partial^2 H}{\partial u_1 \partial u_2} + b^2 \frac{\partial^2 H}{\partial u_2^2} = (h'' + h)\, |u|^{-1} \times$$
$$\times (a \sin\phi - b \cos\phi)^2.$$

If $f(\phi)$ is any non-negative continuous function on Ω satisfying (3.7), then

$$h(\phi) = c_1 \cos\phi + c_2 \sin\phi + \int_0^\phi \sin(\phi - t) f(t)\,\mathrm{d}t$$

is defined on Ω and $h''+h=f$. Consequently, if T satisfies (3.8), then $\sigma(\phi)$ must satisfy the differential Equation (1) $\sigma''+\sigma=k\sigma^{-3}$. Choose the reference axis $\phi=0$, through a point of ∂T at maximal distance from z. Then $\sigma'(0)=0$, $\sigma''(0)\leqslant 0$. Set $a=\sigma(0)$ and $e^2=|\sigma''(0)|\,a^{-1}$. From (1), we obtain (2)

$$(\sigma')^2 + k\sigma^{-2} + \sigma^2 = a^2(2-e).$$

From (1) and (2), we obtain

$$(\sigma^2)'' + 4\sigma^2 = 2a^2(2-e).$$

Consequently

$$\sigma^2 = a^2(1-e^2\sin^2\phi),$$

and σ is the supporting function (on Ω) of an ellipse with semi-major axis a and eccentricity e.

(3.30) *Remark.* It appears likely that a complete solution to the Minkowski problem would eliminate any a priori differentiability assumptions in (3.29) and one need only assume that T satisfies (3.8). A survey of this problem for $n=3$ is given in [5, pp. 33–40] and ([25], Chapter 7]. A partial solution for $n\geqslant 4$ has been obtained by Pogorelov [26].

Let U be the unit disk (center z) of a Minkowski plane. A line through z cuts U into pieces of equal area and the centroids of all such pieces constitute the centroid curve of U. The curves homothetic to this centroid curve are the curves of constant (Minkowskian) curvature. The following result was stated without proof in [20, p. 279].

(3.31) THEOREM. *If a Minkowski circle has constant curvature, then it is an ellipse* (i.e. *the Minkowski plane is euclidean*).

Proof. The normalized solution T to the Minkowski isoperimetric problem is obtained from the unit disk U of area π by rotating U about z through 90: and taking the polar reciprocal. Consequently $T\in\mathscr{T}_0^2$ and Minkowskian curvature and arclength are the same as relative curvature and arclength with respect to T. Now $|U|\,I_m(U)=\pi|T|$ and, from the hypothesis and

(3.13), $G(U)$ is the perimeter of U. Thus, by (2.14) and (2.11a), U is homothetic to T. Therefore the result follows from (3.29).

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