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THE LAW OF SINES FOR TETRAHEDRA AND n-SIMPLICES

0. Introduction

Several trigonometric theorems—for example, the laws of sines and cosines—have natural analogues in higher dimensions. For tetrahedra, both these laws are more than 100 years old. It is a pity, and also surprising, that two such beautiful theorems have been all but completely forgotten.

This paper contains a new approach to the generalized law of sines, which easily applies also to simplices in n-dimensional Euclidean space for $n > 3$. In fact, I will define the '${}^n\sin$' of an n-dimensional corner in such a way that the law of sines becomes an almost trivial consequence. This n-dimensional sine will then also be expressed as a product of ordinary sines for certain angles. These results (Sections 1–6) have been published earlier in Swedish in [5].

The law of sines in n-dimensional non-Euclidean geometry was discovered in 1877 by the Italian mathematician Enrico d'Ovidio [11, p. 975]. This requires introduction of the 'n-dimensional polar sine', which is a kind of dual notion to the n-dimensional sine. A new proof for n-dimensional spherical geometry is given in Sections 8–9.

1. Some notation

In addition to the standard notation $|\mathbf{a}|$ for the length of the vector $\mathbf{a}$, we will use

$|[\mathbf{a}, \mathbf{b}]|$ = the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$,

$|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|$ = the volume of a parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]|$ = the content of an n-dimensional parallelotope with sides $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

(One may interpret $[\mathbf{a}, \mathbf{b}]$ as a bivector, $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ as a trivector and $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ as an n-vector, but this is not necessary for our arguments.)

2. Definition of n-dimensional sine

An ordinary angle with vertex O and sides along the vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, will be denoted by (O, AB). It is familiar that

$$\sin(O, AB) = \frac{|[\mathbf{a}, \mathbf{b}]|}{|\mathbf{a}||\mathbf{b}|},$$

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because $|[\mathbf{a}, \mathbf{b}]| = |\mathbf{a}|\,|\mathbf{b}| \sin(O, AB)$. By analogy, for a triangular corner in space formed by the vectors $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$, we introduce the notation (O, ABC) and

$$(1) \qquad {}^{3}\sin(O, ABC) = \frac{|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|^2}{|[\mathbf{b}, \mathbf{c}]|\,|[\mathbf{c}, \mathbf{a}]|\,|[\mathbf{a}, \mathbf{b}]|}.$$

In general, in a Euclidean space of dimension $\geqslant n$, we introduce the *n-dimensional sine* of a corner $(O, P_1 P_2 \cdots P_n)$ with n edges, determined by the vectors $\overrightarrow{OP_1} = \mathbf{v}_1$, $\overrightarrow{OP_2} = \mathbf{v}_2, \ldots, \overrightarrow{OP_n} = \mathbf{v}_n$, by the definition:

$$(2) \qquad {}^{n}\sin(O, P_1 P_2 \cdots P_n) = \frac{|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]|^{n-1}}{|[\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n]|\,|[\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n]| \cdots |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}]|}.$$

Note that ${}^{n}\sin(O, P_1 P_2 \cdots P_n)$ does not change if any vector $\mathbf{v}_k$ is replaced by $c\mathbf{v}_k$ for a number $c \neq 0$. Thus, it depends only on the directions of the vectors $\mathbf{v}_k$—not on the lengths. Therefore, it really belongs to the *corner*. For $c < 0$, we see that the n-dimensional sine is also unchanged if the direction is reversed for one or more of the vectors $\mathbf{v}_k$. For example, when three planes in 3-space meet at one point, they form eight corners, all of which have the same 3-dimensional sine.

From product formulas proved in Sections 5 and 6 below, it follows that the n-dimensional sine does not exceed 1 (cf. also Problem 2, p. 174, in the well-known textbook [14] by Shilov). The n-dimensional content of the simplex $OP_1 P_2 \cdots P_n$ is $m = |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]|/n!$. The $(n-1)$-dimensional facets of this simplex have the contents

$$m_1 = |[\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n]|/(n-1)!, \ldots,$$
$$m_n = |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}]|/(n-1)!.$$

We can therefore write (2) in the form

$$(3) \qquad m^{n-1} = \frac{(n-1)!}{n^{n-1}} m_1 m_2 \cdots m_n \, {}^{n}\sin(O, P_1 \cdots P_n).$$

Formulated in this way, our definition of the n-dimensional sine is an analogue in higher dimensions of the area theorem in ordinary trigonometry. In particular, for the volume m of the tetrahedron $OP_1 P_2 P_3$ we have

$$(4) \qquad m^2 = \tfrac{2}{9} m_1 m_2 m_3 \, {}^{3}\sin(O, P_1 P_2 P_3),$$

where m_1, m_2 and m_3 are the areas of the faces OP_2P_3, OP_3P_1 and OP_1P_2, respectively.

3. THE n-DIMENSIONAL LAW OF SINES

With the notation introduced above, and m_0 = the $(n-1)$-dimensional content of the simplex $P_1P_2\cdots P_n$, the n-dimensional law of sines can be written:

$$\frac{m_0}{{}^n\sin(O, P_1P_2\cdots P_n)} = \frac{m_1}{{}^n\sin(P_1, P_2P_3\cdots P_nO)}$$
$$= \frac{m_2}{{}^n\sin(P_2, P_3P_4\cdots OP_1)} = \cdots.$$

That is: *The contents of the $(n-1)$-dimensional facets of an n-simplex are proportional to the n-dimensional sines of the opposite corners.*

By our definition of the n-dimensional sine, the proof is almost trivial. According to (3)—and corresponding relations for the other corners—we have

$$\frac{(nm)^{n-1}}{(n-1)!m_0m_1\cdots m_n} = \frac{{}^n\sin(O, P_1P_2\cdots P_n)}{m_0}$$
$$= \frac{{}^n\sin(P_1, P_2P_3\cdots P_nO)}{m_1} = \cdots.$$

Thus, *the law of sines for a tetrahedron ABCD* can be written as

$$\text{(5)} \qquad \frac{T_A}{{}^3\sin A} = \frac{T_B}{{}^3\sin B} = \frac{T_C}{{}^3\sin C} = \frac{T_D}{{}^3\sin D},$$

if we use the simpler notation T_A = the area of the triangle BCD, ${}^3\sin A = {}^3\sin(A, BCD)$, etc.

4. A LEMMA ON ANGLES BETWEEN THE FACETS OF A SIMPLEX*

In Euclidean n-space we consider, for $k \leqslant n-1$, the angle α formed by two k-dimensional simplices $OP_1P_2\cdots P_{k-1}P_k$ and $OP_1P_2\cdots P_{k-1}P_{k+1}$, having the $(k-1)$-dimensional simplex $OP_1P_2\cdots P_{k-1}$ as intersection. By definition, α is the angle between two normals of the common subspace $OP_1P_2\cdots P_{k-1}$ in the respective k-dimensional spaces. The angle α can therefore be obtained as $(P, P'P_{k+1})$, where P and P' are the projections of P_{k+1} on the subspaces of the simplices $OP_1\cdots P_{k-1}$ and $OP_1\cdots P_{k-1}P_k$, respectively (cf. Figure 1, drawn for $k = 2$). In fact, PP' is orthogonal to $OP_1\cdots P_{k-1}$, because both PP_{k+1} and $P_{k+1}P'$ are. Thus we have (cf. Figure 1)

$$\text{(6)} \qquad \sin\alpha = h'/h,$$

where h and h' are the distances from P_{k+1} to P and P', respectively. But

* Discovered about 1850 by Schläfli [12, p. 232].

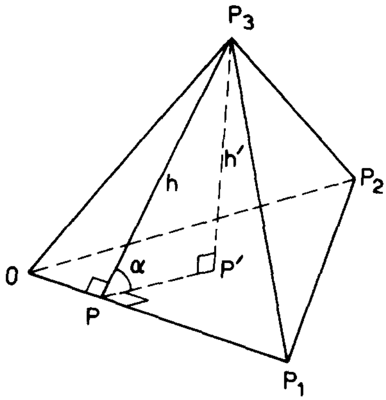

Fig. 1

these distances are also altitudes from P_{k+1} in the simplices $OP_1 \cdots P_{k-1}P_{k+1}$ and $OP_1 \cdots P_k P_{k+1}$, respectively, or in the corresponding parallelotopes. This yields

$$h = |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}]| / |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}]|$$

and

$$h' = |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}]| / |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]|.$$

Substituting these expressions in (6), we get:

LEMMA 1. $\sin \alpha = \dfrac{|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}]| \, |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}]|}{|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k]| \, |[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}]|}.$

5. The product formula for the 3-dimensional sine

The product formula is

$$^3\sin(O, P_1P_2P_3) = \sin \alpha_{13} \sin \alpha_{23} \sin(O, P_1P_2),$$

where α_{13} is the angle between the faces OP_2P_1 and OP_2P_3 at the edge OP_2, while α_{23} is the angle between the faces OP_1P_2 and OP_1P_3.

Proof. By definition (1)

$$^3\sin(O, P_1P_2P_3) = \frac{|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]|^2}{|[\mathbf{v}_2, \mathbf{v}_3]| \, |[\mathbf{v}_3, \mathbf{v}_1]| \, |[\mathbf{v}_1, \mathbf{v}_2]|}.$$

From Lemma 1 we have

$$\sin \alpha_{13} = \frac{|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]| \, |\mathbf{v}_2|}{|[\mathbf{v}_2, \mathbf{v}_1]| \, |[\mathbf{v}_2, \mathbf{v}_3]|}, \quad \sin \alpha_{23} = \frac{|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]| \, |\mathbf{v}_1|}{|[\mathbf{v}_1, \mathbf{v}_2]| \, |[\mathbf{v}_1, \mathbf{v}_3]|}.$$

These three formulas together yield

$$^3\sin(O, P_1P_2P_3) = \sin\alpha_{13}\sin\alpha_{23}\frac{|[\mathbf{v}_1, \mathbf{v}_2]|}{|\mathbf{v}_1|\,|\mathbf{v}_2|},$$

where the last factor of the right member is $\sin(O, P_1P_2)$.

Historical remark. The law of sines for tetrahedra, with this product expression for the 3-dimensional sine, was published in 1860 by the German mathematician Gustav Junghann [9]. Junghann [10] then developed an entire 'tetrahedrometry', that is, a large number of formulas for tetrahedra which are analogous to known trigonometric formulas. The law of sines for tetrahedra had already appeared by 1850 in a paper by Joachimsthal [8, p. 40], although there the 3-dimensional sine was expressed in terms of dihedral angles between faces:

$$\begin{aligned}^3\sin(O, P_1P_2P_3) = (1 &- \cos^2\alpha_{12} - \cos^2\alpha_{13} - \cos^2\alpha_{23} \\ &- 2\cos\alpha_{12}\cos\alpha_{13}\cos\alpha_{23})^{1/2}.\end{aligned}$$

The same law of sines was treated by Allendoerfer [1]. In 1969 W. Dörband [4, p. 304] gave a version of the n-dimensional law of sines, which can be stated as follows: *The contents of two facets, F_i and F_k, are proportional to the ordinary sines of the angles between the edge P_iP_k and those facets at the vertices P_i and P_k, respectively.*

6. THE PRODUCT FORMULA FOR THE n-DIMENSIONAL SINE

Similarly, the n-dimensional sine can be written as a product of ordinary sines of certain angles, also for $n > 3$. We now denote by α_{ik} the angle between the two $(k-1)$-dimensional simplices obtained from $OP_1P_2\cdots P_k$ by deleting the corner P_i and P_k, respectively. In shorthand notation, we will now write $\{k\}$ for $|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]|$ and $\{k; i\}$ for $|[\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k]|$; that is, the content of the $(k-1)$-dimensional parallelotope obtained by deleting $\mathbf{v}_i$. We then get, by definition (2) and Lemma 1,

$$\begin{aligned}\frac{^4\sin(O, P_1P_2P_3P_4)}{^3\sin(O, P_1P_2P_3)} &= \frac{\{4\}^3}{\{4;1\}\{4;2\}\{4;3\}\{3\}}\,\frac{\{3;1\}\{3;2\}\{3;3\}}{\{3\}^2} \\ &= \frac{\{4\}\{3;1\}}{\{4;1\}\{3\}}\,\frac{\{4\}\{3;2\}}{\{4;2\}\{3\}}\,\frac{\{4\}\{3;3\}}{\{4;3\}\{3\}} \\ &= \sin\alpha_{14}\sin\alpha_{24}\sin\alpha_{34}.\end{aligned}$$

Thus

$$^4\sin(O, P_1P_2P_3P_4) = {}^3\sin(O, P_1P_2P_3)\sin\alpha_{14}\sin\alpha_{24}\sin\alpha_{34}.$$

Analogously,

$$\begin{aligned}&^5\sin(O, P_1P_2P_3P_4P_5) \\ &\quad = {}^4\sin(O, P_1P_2P_3P_4)\sin\alpha_{15}\sin\alpha_{25}\sin\alpha_{35}\sin\alpha_{45},\end{aligned}$$

and, in general,

$$(7)\qquad {}^n\sin(O, P_1P_2\cdots P_n) = {}^{n-1}\sin(O, P_1P_2\cdots P_{n-1})\prod_{i=1}^{n-1}\sin\alpha_{in}.$$

7. THE n-DIMENSIONAL POLAR SINE

The n-dimensional sine has a twin, which we define as

$${}^n\text{polsin}(O, P_1P_2\cdots P_n) = \frac{|[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]|}{|\mathbf{v}_1|\,|\mathbf{v}_2|\cdots|\mathbf{v}_n|},$$

and call the *n-dimensional polar sine of the corner* $(O, P_1P_2\cdots P_n)$. If all the $\mathbf{v}_k = \overrightarrow{OP_k}$ are unit vectors, we will also write ${}^n\text{polsin}(P_1P_2\cdots P_n)$, and call it the n-dimensional polar sine of the *spherical simplex* with vertices P_k. It will play an important role in the next two sections. Here we shall prove the product formulas

$$
\begin{aligned}
&{}^3\text{polsin}(O, P_1P_2P_3)\\
&\qquad = \sin(O, P_1P_2)\sin(O, P_1P_3)\sin(OP_1, P_2P_3),\\
(8)\qquad &{}^4\text{polsin}(O, P_1P_2P_3P_4)\\
&\qquad = {}^3\text{polsin}(O, P_1P_2P_3)\sin(O, P_1P_4)\sin(OP_1, P_2P_4)\\
&\qquad\quad \times \sin(OP_1P_2, P_3P_4),\\
&\vdots\\
&{}^n\text{polsin}(O, P_1\cdots P_n)\\
&\qquad = {}^{n-1}\text{polsin}(O, P_1\cdots P_{n-1})\sin(O, P_1P_n)\\
&\qquad\quad \times \sin(OP_1, P_2P_n)\cdots\sin(OP_1\cdots P_{n-2}, P_{n-1}P_n).
\end{aligned}
$$

(OA, BC) denotes the angle formed along OA between the planes OAB and OAC, while (OAB, CD) denotes the angle formed along the plane OAB between the 3-spaces $OABC$ and $OABD$, and so on.

Proof of (8). The right member is, by the definition above and Lemma 1,

$$
\begin{aligned}
&\frac{|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]|}{|\mathbf{v}_1|\,|\mathbf{v}_2|\,|\mathbf{v}_3|}\,\frac{|[\mathbf{v}_1, \mathbf{v}_4]|}{|\mathbf{v}_4|\,|\mathbf{v}_1|}\,\frac{|\mathbf{v}_1|\,|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4]|}{|[\mathbf{v}_1, \mathbf{v}_4]|\,|[\mathbf{v}_1, \mathbf{v}_2]|}\\
&\qquad \times \frac{|[\mathbf{v}_1, \mathbf{v}_2]|\,|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]|}{|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4]|\,|[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]|}.
\end{aligned}
$$

By cancellations this is easily simplified to the definition of

$${}^4\text{polsin}(O, P_1P_2P_3P_4).$$

Obviously the same proof will also apply for general n.

Historical remark. Our definition of the 3-dimensional polar sine is equivalent to the formula for the volume of a tetrahedron

$$V = \tfrac{1}{6}|\mathbf{v}_1|\,|\mathbf{v}_2|\,|\mathbf{v}_3|\,{}^3\text{polsin}(O, P_1P_2P_3).$$

A similar formula has already been given by Euler [6], with the polar sine expressed in terms of the sides

$$s_1 = (O, P_2P_3), \qquad s_2 = (O, P_3P_1), \qquad s_3 = (O, P_1P_2)$$

of the spherical triangle determined by the corner:

$$\begin{aligned} &{}^3\mathrm{polsin}(O, P_1P_2P_3) \\ &\quad = \{1 - \cos^2 s_1 - \cos^2 s_2 - \cos^2 s_3 + 2\cos s_1 \cos s_2 \cos s_3\}^{1/2}. \end{aligned}$$

The corresponding expression for ${}^n\mathrm{polsin}^2 O$ as a determinant was used as definition by Grassmann [7, no. 195]. The formula for the content of an n-dimensional simplex

$$V = \frac{1}{n!} |\mathbf{v}_1|\,|\mathbf{v}_2| \cdots |\mathbf{v}_n|\ {}^n\mathrm{polsin}(O, P_1 \cdots P_n)$$

was (in different notation) given in 1882 by Study [16, p. 150]. This result can also be found in the books by Schoute [13] from 1902 and Sommerville [15, p. 124]. In [13, p. 274] Schoute also gives a formula (95), which in the present context can be written

$${}^4\mathrm{polsin}(O, P_1 \cdots P_4)/{}^3\mathrm{polsin}(O, P_1P_2P_3) = \sin\varphi.$$

φ is the angle between OP_4 and the space $OP_1P_2P_3$, which also appears in Dörband's law of sines [4, p. 304]. The formula follows from the fact that $|\mathbf{v}_4| \sin\varphi$ is the altitude from P_4 in the parallelotope $OP_1P_2P_3P_4$. In Study's paper [16, p. 152], we also find our (3).

8. Corners and n-dimensional sines in spherical geometry

We shall interpret n-dimensional spherical geometry as the geometry of the unit sphere S^n with center O in R^{n+1}, and consider points P_k on this sphere. By (P_1, P_2P_3) we now mean the angle between the great circles P_1P_2 and P_1P_3. This is also the angle between tangent directions P_1P_2' and P_1P_3', or between the planes OP_1P_2 and OP_1P_3 along OP_1; that is, (OP_1, P_2P_3) in our previous notation. Similarly, (P_1P_2, P_3P_4) denotes the angle formed by the spherical surfaces $P_1P_2P_3$ and $P_1P_2P_4$ along the circular arc P_1P_2, and this is also (OP_1P_2, P_3P_4). In general, we have

$$(9) \qquad (P_1 \cdots P_k, P_{k+1}P_{k+2}) = (OP_1 \cdots P_k, P_{k+1}P_{k+2}).$$

A corner $(P_1, P_2 \cdots P_{n+1})$ in the spherical geometry of S^m, where $m \geqslant n$, may be considered as the corresponding corner $(P_1, P_2' \cdots P_{n+1}')$ in R^{m+1} formed by tangent directions. Its n-dimensional sine can be obtained by our product formulas (Sections 5–6). We have also, in the shorthand notation of Section 6:

LEMMA 2. $^n\sin(P_1, P_2 \cdots P_{n+1}) = \dfrac{\{n+1\}^{n-1}}{\{n+1;2\}\{n+1;3\}\cdots\{n+1;n+1\}}.$

Proof by induction. For $n = 2$ we have

$$\sin(P_1, P_2P_3) = \sin(OP_1, P_2P_3) = \frac{\{3\}}{\{3;2\}\{3;3\}}$$

by Lemma 1, since $|\mathbf{v}_1| = 1$.

The product formula (7) yields

$$\text{(10)} \qquad {}^n\sin(P_1, P_2\cdots P_{n+1}) = {}^{n-1}\sin(P_1, P_2\cdots P_n)\prod_{i=2}^{n}\sin\beta_{i,n+1},$$

where $\beta_{i,n+1}$ is the angle formed by the two $(n-1)$-dimensional facets of the spherical simplex $P_1P_2\cdots P_{n+1}$, which are opposite to P_i and P_{n+1}, respectively. According to (9), $\beta_{i,n+1}$ is also the angle between the corresponding n-dimensional facets of the simplex $OP_1P_2\cdots P_{n+1}$. Thus, by Lemma 1

$$\sin\beta_{i,n+1} = \frac{\{n+1\}\{n;i\}}{\{n+1;i\}\{n\}}.$$

Using this in (10), and assuming our lemma true for $^{n-1}\sin(P_1, P_2\cdots P_n)$, we get

$$\begin{aligned} &{}^n\sin(P_1, P_2\cdots P_{n+1}) \\ &= \frac{\{n\}^{n-2}}{\{n;2\}\cdots\{n;n\}}\,\frac{\{n+1\}^{n-1}}{\{n\}^{n-1}}\,\frac{\{n;2\}}{\{n+1;2\}}\cdots\frac{\{n;n\}}{\{n+1;n\}} \\ &= \frac{\{n+1\}^{n-1}}{\{n+1;2\}\cdots\{n+1;n+1\}}, \end{aligned}$$

which completes the induction.

9. THE LAW OF SINES IN n-DIMENSIONAL SPHERICAL GEOMETRY

For the spherical simplex $P_1P_2\cdots P_{n+1}$, we now denote the corner $(P_k, P_{k+1}\cdots P_{n+1}P_1\cdots P_{k-1})$ simply by P_k and the opposite $(n-1)$-dimensional facet by F_k. Then:

$$\frac{^n\text{polsin } F_1}{^n\sin P_1} = \frac{^n\text{polsin } F_2}{^n\sin P_2} = \cdots = \frac{^n\text{polsin } F_{n+1}}{^n\sin P_{n+1}}.$$

That is: *The n-dimensional polar sines of the facets are proportional to the n-dimensional sines of the opposite corners.*

Proof. We have, according to the definition in Section 7,

$$\begin{aligned} {}^n\text{polsin } F_1 &= {}^n\text{polsin}(P_2P_3\cdots P_{n+1}) = {}^n\text{polsin}(O, P_2P_3\cdots P_{n+1}) \\ &= \{n+1;1\}. \end{aligned}$$

Combining this with Lemma 2, we get

$$\frac{{}^n\text{polsin}\, F_1}{{}^n\sin P_1} = \frac{\{n+1;1\}\{n+1;2\}\cdots\{n+1;n+1\}}{\{n+1\}^{n-1}}.$$

The right member is symmetric in the indices, and thus also equal to ${}^n\text{polsin}\, F_2/{}^n\sin P_2$, etc.

Historical remark. This law of sines reduces for $n = 2$ to the well-known law of sines for spherical triangles. As already mentioned, it is old also for general n, although d'Ovidio [11, p. 975] does not state it in the simple form used here. The 3-dimensional sine and polar sine have been much used in non-Euclidean trigonometry; cf., e.g., [3, pp. 236, 237], where they are denoted by $\sqrt{\Gamma}$ and $\sqrt{\gamma}$.

10. The Law of Cosines

It seems appropriate to mention here that the law of cosines has also been generalized to dimension $n \geqslant 3$ for Euclidean geometry. The law of cosines for a tetrahedron takes the following form:

$$\begin{aligned} m_0^2 = m_1^2 + m_2^2 + m_3^2 &- 2m_2m_3\cos\alpha_{23} \\ &- 2m_3m_1\cos\alpha_{31} - 2m_1m_2\cos\alpha_{12}, \end{aligned}$$

in the notation of section 3 and 5, with $\alpha_{12} = (OP_3, P_1P_2)$ etc. It was previously given in 1803 by Carnot [2, p. 310], together with the analogue statement for polyhedra. The generalization to dimension $n > 3$ is found in Dörband's paper [4, p. 303] from 1969.

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