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REGULI AND PSEUDO-REGULI IN $PG(3, s^2)$

0. Introduction

The purpose of this paper is to examine certain partial spreads of $PG(3, s^2)$ called pseudo-reguli. These arise from regular spreads of a 3-dimensional Baer subspace of $PG(3, s^2)$. It is well known that a translation plane given by a spread of lines of $PG(3, q)$ may be derivable with the derivation set being represented by a partial spread which is not a regulus [6]. Foulser [12] has given an explicit relationship between derivable nets of translation planes and partial spreads. This paper gives further information about derivation sets of translation planes which are not reguli by giving an explicit relationship (Theorem 3.1 and its Corollary) between a regulus over $GF(s^2)$ and a regular spread of $PG(3, s)$ via the regular switching sets of Bruck and Bose [4, p. 166].

In addition to Theorem 3.1 and its Corollary further results are the following. Theorems 2.1 and 2.2 show how i-dim. Baer subspaces of $PG(t, s^2) = \Sigma^*_t$, $1 \leqslant i \leqslant t$, are represented by certain i-reguli when a linear representation of Σ^*_t is given in $PG(2t + 1, s)$. This extends the work of Bose [2, 13].

In Section 4 using pseudo-reguli a construction for translation planes described by Bruen and Thas [7] is discussed. This point of view is used to examine the following question of Cofman [9]: 'Do there exist translation planes of order p^a containing more subplanes of order p^b for a divisor b of a than the Desarguesian affine plane of order p^a?' Theorem 4.1 gives an affirmative answer in the case $a = 4c$, $b = 2c$, for each $c \geqslant 1$, p an even or odd prime. Using the Klein quadric and a projection technique of Segre [15, p. 55] we show, Theorem 4.2, that a pseudo-regulus is contained in a spread. Its corollary provides a new class of complete partial spreads $\mathscr{S}$ of $PG(3, q)$ with

$$|\mathscr{S}| = q^2 - q + 2, \qquad q = 2^{2t}.$$

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1. Preliminary results and definitions

The theorems of this paper are based on the following results which are theorems 4.2, 4.7 and 5.3 of Bruck [3].

RESULT 1: Let Q be a doubly-ruled quadric of Σ_3 with reguli R, R'. Let $K = K(Q)$ be the subgroup of the projective linear group $PL(\Sigma_3)$ of Σ_3

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which maps Q to Q and let $K_0 = K_0(Q)$ be the subgroup of K which maps each of R, R' upon itself. Then K_0 is a subgroup of index 2 in K, and $K = K_1 \otimes K_2$ where $K_1(K_2)$ is the subgroup of K which maps $R(R')$ upon itself and fixes every line of $R'(R)$. In particular, each of K_1, K_2 is isomorphic to the projective linear group of the line $\mathrm{PG}(1, q)$.

A partial *t-spread* of $\Sigma_d = \mathrm{PG}(d, q)$ is a set S of t-dimensional subspaces of Σ_d such that a point of Σ_d is incident with at most one element of S. The set S is a *t-spread* if each point of Σ_d is incident with exactly one element of S. Following Dembowski [11, p. 220] a t-regulus in Σ_{2t+1} is a partial t-spread R having $q + 1$ elements with the property that if a line meets three distinct members of R in a point then the line meets each member of R. Each element of a t-regulus is called an *axis-space*. A line meeting three axis-spaces of a t-regulus in a point and hence all axis-spaces is called a *transversal*. For $t = 1$, a 1-regulus is one ruling of a nondegenerate hyperbolic quadric in Σ_3. Throughout this paper by a regulus we shall understand a 1-regulus. A t-spread S of Σ_{2t+1} is called *regular* if for any three distinct elements of S the t-regulus determined by the three is contained in S.

RESULT 2: Let S be a regular 1-spread of Σ_3 and A, B be two distinct lines of S. Then the $q^2 - 1$ lines of S distinct from A, B may be partitioned into $q - 1$ disjoint reguli R_i uniquely defined, apart from order, by the requirements that for each i, A, B are conjugate nonsecants of the quadric $Q_i = Q_i(R_i)$ ruled by R_i. The line-set, S', obtained from S by replacing each of the $q - 1$ reguli, R_i by its opposite regulus R_i', is a regular spread of Σ.

Call a subspace of order s in a projective space of order s^2 a *Baer subspace*.

RESULT 3: Let Σ_3 be embedded as a 3-dim. Baer subspace of Σ_3^*. Let τ be the unique involution of Σ_3^* which fixes every point of Σ_3. Let L be any line of Σ_3^* which contains no point of Σ_3. For each such line, L, let $\mathscr{S}(L)$ denote the set of all lines of Σ which meet L. Then

(i) $\mathscr{S}(L) = \mathscr{S}(L^\tau)$ is a regular spread of Σ_3. Every regular spread of Σ_3 can be represented in this manner for a unique pair of lines L, L^τ.

(ii) If P, Q, R are three distinct points of L, and if A, B, C are the unique lines of Σ_3 through P, Q, R, respectively, then the $q + 1$ lines of regulus $\mathscr{R}(A, B, C)$ of Σ_3 meet L in the points of the unique subline of order q of L which contains P, Q, R.

(iii) If two distinct lines A, B of $\mathscr{S}(L)$ meet L in points P, Q respectively, and if M is the line of Σ^* containing P and Q^τ, then M contains no points of Σ_3. The regular spreads $\mathscr{S}(L)$, $\mathscr{S}(M)$ of Σ_3 have the following properties: (a) A, B are the only common lines of $\mathscr{S}(L)$, $\mathscr{S}(M)$. (b) To each point X of Σ_3 which is not on A or B there corresponds a unique doubly-ruled quadric Q of Σ_3 containing X such that one regulus of Q is in $\mathscr{S}(L)$, the other regulus is in $\mathscr{S}(M)$, and A, B are conjugate nonsecants (in Σ_3) of Q.

Given a regular spread of Σ_3 let $\mathscr{F}$ be the partial spread of Σ_3^* obtained by extending the lines of the regular spread. Call $\mathscr{F}$ a *pseudo-regulus*. From above $\mathscr{F}$ has exactly two transversal lines. This partial spread is of interest because, for s odd, there is a collineation of Σ_3^* mapping $\mathscr{F}$ to D where D is the partial spread which represents the derivation set of the semi-field plane whose multiplication is constructed using the automorphism $\sigma: x \to x^s$ of $K = \mathrm{GF}(q^2) = \mathrm{GF}(s^4)$ as discussed by Bruen [6, p. 528].

2. A linear representation of PG (3, s^2) in Σ_7

The purpose of this section is to prove Theorem 2.1 which uses the idea of a linear representation of a projective plane as presented by Bruck and Bose [4] to obtain a linear representation in Σ_7 of PG(3, s^2). Building on the work of Bruck [3] and Bruen [6] we first discuss the embedding of Σ_7 in Σ_7^*.

Let $F = \mathrm{GF}(s) \subset \mathrm{GF}(s^2) = K$. Let $W_1 = V(K)$ be an eight-dimensional vector space over K and $W_2 = V_8(F)$ be a subset of vectors of W_1 forming an eight-dimensional vector space over F. A given basis $\{e_i\}$, $0 \leqslant i \leqslant 7$, for W_2 over F also forms a basis of W_1 over K. The elements of Σ_7 and Σ_7^* are the non-zero subspaces of W_2 and W_1, respectively. A point X of Σ_7^* is *real* or *imaginary* if X is a point of Σ_7 or not. A subspace of Σ_7^* is called imaginary if it has no real points. Let t be a primitive element of K. A point X of Σ_7^* will have homogeneous coordinates whose ith coordinate will be the form (w.r.t. the above basis)

$$tx_i + y_i, \qquad 0 \leqslant i \leqslant 7, \qquad x_i, y_i \in F.$$

It is immediate that a point X is real iff for some $0 \neq \lambda \in K$ all eight coordinates of λX belong to F. Further if X is imaginary the unique line p_x of Σ_7 passing through X is spanned by the two points (x_i) and (y_i), $0 \leqslant i \leqslant 7$.

Let $V_4(K)$ be the vector subspace over K spanned by v_0, v_1, v_2, v_3 where

$$(1) \qquad v_0 = e_0 + te_1, \qquad v_1 = e_2 + te_3, \qquad v_2 = e_4 + te_5, \\ v_3 = e_6 + te_7.$$

Let Σ_3^* be the associated three-dimensional projective space. It follows that Σ_3^* is imaginary.

Let $\mathscr{P}$ be the following collection of lines of Σ_7.

$$\mathscr{P} = \{p_x \mid p_x \text{ is the line of } \Sigma_7 \text{ through a point of } \Sigma_3^*\}$$

Since Σ_3^* is imaginary, for a point X of Σ_3^* the correspondence $X \to p_x$ is one-to-one and the collection $\mathscr{P}$ forms a 1-spread of Σ_7.

The multiplicative group of K determines a group of linear transformations of $V_4(K)$ given by scalar multiplication. This group induces a group of $G(\mathscr{P})$ of collineations of Σ_7 which leaves $\mathscr{P}$ elementwise invariant. Further, using the $s + 1$ collineations, $r(i)$, of Σ_7, $0 \leqslant i \leqslant s$, induced by the linear transformations

$$v \to t^i v \quad \text{for} \quad v \in V_4(K),$$

t a primitive element of K, it follows that $G(\mathscr{P})$ is transitive on the points of any line belonging to $\mathscr{P}$. The collineations $r(i)$ are the analogues of the rho-transformations of Bose [2], [13].

THEOREM 2.1. *Let Σ_3^* be the imaginary 3-space of Σ_7^* associated with $V_4(K)$ given in* (1). *Then Σ_3^* has a linear representation in Σ_7^* where* (i) *the points of Σ_3^* are given by the collection $\mathscr{P}$ of lines which forms a 1-spread of Σ_7*; (ii) *the lines of Σ_3^* are given by a collection $\mathscr{L}$ of 3-spaces of Σ_7 such that for any two elements, p_x, p_y of $\mathscr{P}$, the 3-space spanned by them belongs to $\mathscr{L}$*; (iii) *any two distinct elements $L(u^*)$, $L(v^*)$ of $\mathscr{L}$ either intersect in an element of $\mathscr{P}$ or are disjoint if, respectively, the lines u^*, v^* of Σ_3^* meet in a point or not*; (iv) *for each i, $1 \leqslant i \leqslant 3$, there is a one-to-one correspondence from the set of i-reguli whose transversals belong to $\mathscr{P}$ onto the set of the i-dimensional Baer subspaces of Σ_3^*.*

Proof. Part (i) has been established above. Consider a line u^* of Σ_3^* and two distinct points X, Y of u^*. The two lines p_x, p_y span a unique 3-space, $L(u^*)$, in Σ_7 whose extension to a 3-space over GF(s^2) contains u^*. The $s^2 + 1$ lines p_x for X belonging to u^* therefore lie in $L(u^*)$ and form a regular spread (Result 3) of $L(u^*)$. Let $\mathscr{L}$ be the collection of three-dimensional projective spaces of Σ_7 where

$$\mathscr{L} = \{L(u^*) \mid u^* \quad \text{is a line of} \quad \Sigma_3^*\}.$$

Using Result 3 and the fact that Σ_3^* is imaginary statements (ii) and (iii) follow. The correspondence $u^* \to L(u^*)$ for lines u^* of Σ_3^* and 3-spaces belonging to $\mathscr{L}$ is one-to-one. For a given line u^* of Σ_3^* we say that the associated 3-space $L(u^*)$ represents u^*. Each plane of Σ_3^* will be represented in a similar manner by a 5-space of Σ_7 having the expected intersection properties with $\mathscr{P}$ and $\mathscr{L}$.

Let v be a line of Σ_7 not in $\mathscr{P}$, and p_x, p_y be two of the $s + 1$ elements of $\mathscr{P}$ that meet v in a point. The 3-space $L(XY)$ spanned by p_x and p_y contains v and represents the line XY of Σ_3^*. This 3-space contains a subset of $\mathscr{P}$ which forms a regular spread of $L(XY)$ (Result 3). Therefore the $s + 1$ elements of $\mathscr{P}$ meeting v determine a unique regulus $R(v)$. The lines belonging to $R(v)$ represent points of a unique subline of $L(XY)$ (Result 3, part (ii)). Each line of the opposite regulus $R'(v)$ is a transversal of the 1-regulus $R(v)$. All axis lines of $R'(v)$ thus determine the same subline of $L(XY)$. Using part (ii) of Result 3 it follows that there is one-to-one correspondence between Baer sublines u in Σ_3^* and nondegenerate hyperbolic quadrics $Q(u)$ each of which has as one ruling a regulus $R(u)$ whose elements belong to $\mathscr{P}$.

Let π be a plane of Σ_7 not containing an element of $\mathscr{P}$. Using $G(\mathscr{P})$, the

$s^2 + s + 1$ elements of $\mathscr{P}$ meeting π in a point are the transversals of a 2-regulus with axis-planes $\{\pi^{r(i)}\}$, $0 \leqslant i \leqslant s$. This 2-regulus represents a projective plane in the following manner: the transversals of the 2-regulus being elements of $\mathscr{P}$ are the 'points'. The 'lines' are the 3-spaces belonging to $\mathscr{L}$ which meet π in a line. In particular as in the paragraph above each line of π determines $s + 1$ elements of $\mathscr{P}$ which form a 1-regulus contained in a unique element of $\mathscr{L}$. Therefore the lines of π determine $s^2 + s + 1$ elements of $\mathscr{L}$ each of which intersects the 2-regulus in a 1-regulus. By checking the axioms it follows that these 'points' and 'lines' of Σ_3^* form a projective plane of order s which is a Baer subplane of Σ_3^*.

Let Σ be a 3-space of Σ_7 not containing an element of $\mathscr{P}$. Using G($\mathscr{P}$) the elements of $\mathscr{P}$ meeting Σ in a point are the transversals of a 3-regulus with axis-spaces $\{\Sigma^{r(i)}\}$, $0 \leqslant i \leqslant s$. This 3-regulus represents a Baer 3-space of Σ_3^* in the following way. The transversals of the 3-regulus, being elements of $\mathscr{P}$, represent points of Σ_3^*. Further, each line (resp. plane) of Σ determines a 1-regulus (resp. 2-regulus) which represents a subline (subplane) of Σ_3^* as above. Therefore, there is a 1-to-1 correspondence between the points, lines, and planes of an axis-space which is itself a projective 3-space of order s and a subset of $\mathscr{P}$ and subsets of $\mathscr{L}$ which form points, lines, and planes of a 3-dimensional Baer subspace of Σ_3^*.

Conversely let θ be a 3-dimensional Baer subspace of Σ_3^*. A unique 3-regulus in Σ_7 which represents θ will be determined. First note that if u, v are lines of order s of Σ_3^* such that their extensions u^*, v^* are skew then there are exactly $s + 1$ Baer subspaces of dimension 3 that contain both u and v.

Let u, v be two skew lines of θ. Hence the lines u^*, v^* of Σ_3^* are skew. In Σ_7 the two 3-spaces $L(u^*)$, $L(v^*)$ are disjoint and each contains a quadric $Q(u)$, $Q(v)$ which represents the subline u, v respectively. Let $R(u)$, $R(v)$ be the reguli whose elements represent the points of u, v and $R'(u)$, $R'(v)$ be the opposite reguli. Let Σ be the space spanned by a line $m \in R'(u)$ and a line $n \in R'(v)$. Since $Q(u)$ and $Q(v)$ have no points in common Σ is a 3-space. By part (iii) of Theorem 2.1 already established Σ is not an element of $\mathscr{L}$. Hence Σ is not invariant under $G(\mathscr{P})$. Further Σ contains no element of $\mathscr{L}$. For if so, since the reguli $R(u)$, $R(v)$ are invariant under $G(\mathscr{P})$ the two 3-spaces, Σ, $\Sigma^{r(1)}$, would span at most a 5-space which would contain both $L(u^*)$ and $L(v^*)$. The two 3-spaces $L(u^*)$, $L(v^*)$ would then meet in at least a line, which is a contradiction. Therefore for a fixed line m of $R'(u)$ each of the $s + 1$ 3-spaces $\Sigma = \Sigma(m, n)$ as n varies over the lines of $R(v)$ is an axis space of a distinct 3-regulus which represents a Baer subspace of Σ_3^* containing the lines u, v. One of these 3-reguli therefore represents θ. Since each Baer subplane of Σ_3^* is in some Baer 3-space it follows that each such plane is represented by a unique 2-regulus in the above described manner. The theorem is now established.

As this investigation has shown, viewing PG(1, s^2), PG(2, s^2) and PG(3, s^2)

as imaginary lines, planes and 3-spaces of PG$(3, s^2)$, PG$(5, s^2)$, PG$(7, s^2)$ respectively, one may give a representation where the i-dim. Baer subspaces are given as certain i-reguli, $1 \leqslant i \leqslant 3$. The above proof may be extended to establish

THEOREM 2.2. *Embed* Σ_{2t+1} in $\Sigma^*{}_{2t+1}$ *as a Baer subspace. Let* Σ_t^* *be an imaginary t-dimensional projective space of* Σ_{2t+1}^*. *Then* Σ_t^* *has a representation in* Σ_{2t+1} *such that* (i) *the points of* Σ_t^* *are given by a collection* $\mathscr{P}$ *of lines which forms a* 1*-spread of* Σ_{2t+1}; (ii) *for each i*, $1 \leqslant i \leqslant t$, *each Baer subspace of dimension i of* Σ_t^* *is represented by an i-regulus whose transversals belong to* $\mathscr{P}$.

3. Reguli in PG$(3, s^2)$, regular spreads of PG$(3, s)$ and regular switching sets

The purpose of this section is to give an explicit connection between a regulus of PG$(3, s^2)$ and a regular spread of PG$(3, s)$ via regular switching sets in PG$(7, s)$. A *switching set* $\mathscr{K}$ is a proper partial t-spread of Σ_{2t+1} for which there is at least one conjugate partial t-spread, $\mathscr{K}'$, such that

(i) $\mathscr{K}$ and $\mathscr{K}'$ have no common members,
(ii) a point of Σ_{2t+1} is in a member of $\mathscr{K}$ iff the point is in a member of $\mathscr{K}'$.

A pair $(\mathscr{K}, \mathscr{K}')$ of conjugate switching sets is called a *regular* pair if dim $(J \cap J') = d - 1$ for a fixed integer d for $J \in K$, $J' \in \mathscr{K}'$.

As in Section 2 let Σ_7 be a Baer subspace of Σ_7^* with Σ_3^* a 3-space of order s^2 imaginary w.r.t. Σ_7.

Let R_1 be a regulus in Σ_3^* and R_2 its opposite regulus. It is immediate that a regular pair $(\mathscr{R}_1, \mathscr{R}_2)$ of switching sets of 3-spaces with $d = 2$, $t = 3$, is determined in Σ_7 when the lines of the reguli are viewed as 3-spaces in Σ_7 via the representation of Σ_3^* in Σ_7 described in Section 2.

The rest of this section is devoted to showing exactly how the regular pair $(\mathscr{R}_1, \mathscr{R}_2)$ may be obtained from a partial spread $\mathscr{F}$ in an imaginary 3-space $\Omega_3^* \neq \Sigma_3^*$ such that $\mathscr{F}$ is not a regulus in Ω_3^*. The reason allowing this fact is that having chosen two lines u, v of R_2 one will see that a line p_x in Σ_7 that represents a point of the regulus not on u or v will not represent a point of Ω_3^* but will determine a subline of Ω_3^*. Note that in this section u, v denote lines of order s^2 in the imaginary space Σ_3^* rather than lines in Σ_7 as in Section 2.

LEMMA 3.1. *Let* R_1 *be a regulus of* Σ_3^* *with* R_2 *its opposite regulus and* $u, v \in R_2$. *Then the lines of* $R_2 \backslash \{u, v\}$ *may be partitioned into* $s - 1$ *subsets,* $S_j(i)$, *each of size* $s + 1$, $1 \leqslant j \leqslant s - 1$, $0 \leqslant i \leqslant s$, *such that for each line d* *of* R_1 *and fixed* j, *the* $s + 1$ *points of intersection of* d *with elements of* $S_j(i)$ *form a subline* d_j *of* d.

Proof. Choose a line d of R_1. The $s^2 - 1$ points of d not on u or v may be uniquely partitioned into $s - 1$ sublines (Results 2, 3). Each such subline of d determines a subset of $R_2 \backslash \{u, v\}$ by saying a line of R_2 belong to a subset $S_j(i)$ if the line meets d in a point of the given subline. The statement now follows using Result 1.

LEMMA 3.2. *Let* $S_j(i)$ *be one of the sets of the partition of Lemma* 3.1. *For each line* d *of* R_1 *let* d_j *be the subline determined by* $S_j(i)$. *Then* (i) *the set of* 3-*spaces* $\{\theta_j(i)\}$ *in* Σ_7 *determined by* $S_j(i)$ *forms a* 3-*regulus* $\Gamma_j(i)$, (ii) *each transversal of* $\Gamma_j(i)$ *belongs to the unique regulus opposite to the regulus in* Σ_7 *representing the points of the subline* d_j.

Proof: The subline d_j is represented by a quadric, one regulus, $R_j(i)$, of which represents the points of the subline (Theorem 2.1). Each 3-space $\theta_j(i)$ (j fixed) meets the 3-space representing the line d of Σ_3^* in a line belonging to $R_j(i)$. Therefore a line of the opposite regulus $R_j'(i)$ meets each 3-space $\theta_j(i)$ in exactly a point and is therefore a transversal to the set $\{\theta_j(i)\}$, $0 \leqslant i \leqslant s$, which finishes the proof.

Let ξ be the unique involution of Σ_7^* leaving Σ_7 pointwise fixed. Let $\mathscr{P}$, $\mathscr{L}$, respectively, be the collection of lines and 3-spaces of Σ_7 determined by the representation (Theorem 2.1) of Σ_3^* in Σ_7. If X is an imaginary point then $X \neq X^\xi$ and the line XX^ξ meets Σ_7 in a line belonging to $\mathscr{P}$. Hence $p_x = p_y$ where $Y = X^\xi$. Any 3-space $L(u)$ of $\mathscr{L}$ extends to a 3-space $L^*(u)$ which is invariant under ξ. The restriction of ξ to $L^*(u)$ is therefore the unique involution of $L^*(u)$ leaving $L(u)$ pointwise fixed. This allows the use of Result 3 (iii).

LEMMA 3.3. *Let* u, v *be two lines of the regulus* R_2 *in* Σ_3^*. *Then the space spanned by* u, v^ξ *is an imaginary* 3-*space.*

Proof. Since Σ_3^* is imaginary its image under ξ is also imaginary and the two spaces have no point in common. Whence $\langle u, v^\xi \rangle$ is a 3-space, Ω_3^*. Let X be a point of Ω_3^* neither on u nor v^ξ. It is sufficient to show that X is imaginary. Let t be the line of Ω_3^* through X and meeting u, v^ξ in points, P, Q^ξ, for P on u, Q on v. The points P, Q are in Σ_3^* and the 3-space $L(PQ)$ in Σ_7 representing the line PQ extends to a unique 3-space $L^*(PQ)$ which is invariant under ξ. Therefore $L^*(PQ)$ contains Q^ξ as well as P, Q and contains the imaginary line PQ. The line $PQ^\xi = t$ containing X is therefore imaginary (Result 3, iii) which completes the proof.

Given a partial spread $\mathscr{U}$ of Σ^*, a subspace Γ having the property that each point of Γ is a point of an element of $\mathscr{U}$ and the intersection of each element of $\mathscr{U}$ with Γ is at least a point is called a *transversal space* of $\mathscr{U}$.

THEOREM 3.1. *In Σ_7^* let Σ_3^* be an imaginary 3-space w.r.t. Σ_7. For a regulus R_1 in Σ_3^* let u, v be two lines of the opposite regulus R_2. Let ξ be the involution of Σ_3^* leaving Σ_7 pointwise invariant. Let $\mathscr{F}$ be the collection of lines:*

$$\mathscr{F} = \{X(X')^\xi \mid X \in u,\ X' \text{ is the unique point of } v \text{ such that the line } XX' \text{ belongs to } R_1\}.$$

Then (i) *$\mathscr{F}$ is a partial spread of $\Omega_3^* = \langle u, v^\xi \rangle$ having two transversal lines, u, v^ξ.* (ii) *Each 3-regulus $\Gamma_j(i)$ in Σ_7 (j fixed) determined by the set of lines $S_j(i)$ in Σ_3^* as constructed in Lemma 3.1 represents a 3-dimensional Baer subspace of Ω_3^*.* (iii) *The intersection of $\mathscr{F}$ with the Baer subspace determined by $\Gamma_j(i)$ forms a regular spread of the Baer subspace: hence $\mathscr{F}$ is a pseudo-regulus.* (iv) *The regular switching set $(\mathscr{R}_1, \mathscr{R}_2)$ in Σ_7 obtained from the reguli R_1, R_2 in Σ_3^* is exactly the same switching set obtained from the pseudo-regulus $\mathscr{F}$ and its transversal spaces where the transversal spaces of $\mathscr{F}$ are the lines u, v^ξ, and the $s-1$ Baer subspaces $\Gamma_j(i)$, $1 \leqslant j \leqslant s-1$.*

Proof. Let the lines u, v and hence u^ξ, v^ξ be represented in Σ_7 by the 3-spaces Λ, M respectively. A line $XX' = d(k)$, $0 \leqslant k \leqslant s^2$, of the regulus R_1 determines a 3-space Δ_k in Σ_7 which contains a regular spread $\mathscr{S}_k$. Each element of $\mathscr{S}_k$ represents a point of Σ_3^*, where

$$\mathscr{S}_k = \{p_x, p_{x'}\} \bigcup_j R_j(i), \qquad 1 \leqslant j \leqslant s-1, \qquad i \text{ fixed}$$

with

$$p_x = \Delta_k \cap \Lambda, p_{x'} = \Delta_k \cap M.$$

A line $X(X')^\xi = d'(k)$ belongs to Ω_3^* and contains two points X, $(X')^\xi$ which by Result 3 are also represented in Σ_7 by the lines $p_x, p_{x'}$. Therefore in the representation of Ω_3^* in Σ_7 the line $d'(k)$ is represented by the same 3-space, Δ_k, as $d(k)$. The 3-spaces Δ_k, $0 \leqslant k \leqslant s^2$, are therefore pairwise disjoint and represent the lines belonging to $\mathscr{F}$. The collection $\mathscr{F}$ is then a partial spread of Ω_3^* having u, v^ξ as transversals. However (Result 3, iii) the regular spread in Δ_k representing the points of the line $d'(k)$ is

$$\mathscr{S}'_k = \{p_x, p_{x'}\} \bigcup_j R'_j(i), \qquad 1 \leqslant j \leqslant s-1,$$

where $R'_j(i)$ is the regulus opposite to $R_j(i)$. Therefore (Lemma 3.2) each transversal of the 3-regulus determined by $S_j(i)$ represents a point of Ω_3^*. The 3-regulus $\Gamma_j(i)$, j fixed, represents therefore a 3-dimensional Baer subspace of Ω_3^* (Theorem 2.1). Statements (i) and (ii) are now established. The Baer subspace $\Gamma_j(i)$ intersects each of the s^2+1 lines of $\mathscr{F}$ in a subline whose points are represented by the reguli $R'_j(i)$. Hence a 1-spread of the Baer subspace is determined by $\mathscr{F}$ which will now be shown to be regular. Consider an axis space $\theta_j(i)$ of the 3-regulus $\Gamma_j(i)$. Since $\theta_j(i)$ represents a line of Σ_3^* belonging to $S_j(i)$ the points of the represented line are given by a regular

1-spread $\mathscr{T}$ of $\theta_j(i)$. Even though each element of $\mathscr{T}$ represents a point of Σ_3^*, by Lemma 2 and Result 3, (iii), each element is an axis-line of a subline of Ω_3^* in the representation of Ω_3^* in Σ_7. Since $\mathscr{T}$ is regular it follows that the 1-spread of $\Gamma_j(i)$ determined by $\mathscr{F}$ is also regular. Finally, if there were a third transversal line to $\mathscr{F}$ in Ω_3^* then $\mathscr{F}$ would be a regulus. This is impossible by Result 3 (i). This establishes statement (iii).

Each point of the Baer subspace $\Gamma_j(i)$ is a point of a line belonging to $\mathscr{F}$ and each line belonging to $\mathscr{F}$ meets the Baer subspace in at least a point. Statement (iv) is now immediate and the proof is complete.

Conversely, consider a pseudo-regulus $\mathscr{F}$ in Σ_3^* with transversal lines u, v and Baer subspace θ. A collection of $s - 1$ transversal Baer 3-spaces of $\mathscr{F}$ including θ will now be obtained which, together with u, v will determine a set of points that will be the same point set as that determined by the elements of $\mathscr{F}$. First, Theorem 2.1 implies that θ is represented in Σ_7 by a 3-regulus Γ. From the proof of Theorem 3.1 using $\mathscr{F}$, a partial spread R is obtained in $\langle u, v^{\xi}\rangle = \Omega_3^*$. Further, each of the $s + 1$ axis spaces of Γ represents a unique line of Ω_3^* meeting each line of R in a point. Hence R is a regulus. Partitioning the regulus opposite to R using u, v^{ξ} as in Lemma 3.1 and reinterpreting the partition in Σ_3^* the sought after collection of Baer subspaces is obtained. This establishes

COROLLARY. *If $\mathscr{F}$ is a pseudo-regulus in Σ_3^* with transversal lines u, v and $s - 1$ transversal Baer subspaces $\theta(j)$, $1 \leqslant j \leqslant s - 1$, then in $\Omega_3^* = \langle u, v^{\xi}\rangle$, the collection of lines*

$$\{X(X')^{\xi} \mid X \in u,\ X' \in v,\ \&\ XX' \in \mathscr{F}\}$$

forms a regulus and the $s + 1$ axis-spaces in Σ_7 of each of the $s - 1$ 3-reguli representing $\theta(j)$ determine $s^2 - 1$ lines of Ω_3^* which together with u, v^{ξ} form the opposite regulus.

We exploit this point of view in Section 4.

4. Applications of pseudo-reguli

Let $\mathscr{S} = \mathscr{F} \dot{\bigcup} \mathscr{S}'$ (where $\dot{\bigcup}$ means disjoint union) be a 1-spread of PG(3, s^2) $= \Sigma_3^*$ which contains a pseudo-regulus $\mathscr{F}$ with transversal lines u, v. As in [4] let $\mathscr{A}T(\mathscr{S})$ be the associated affine translation plane. Let $\psi = \mathscr{S}' \bigcup \{u, v\} \bigcup \{\theta(j)\}$, where $\{\theta(j)\}$ is the set of $s - 1$ Baer 3-spaces which together with u, v are transversal spaces of $\mathscr{F}$. Embed Σ_3^* as a hyperplane in Σ_4^* and form an incidence structure $\mathscr{A}T(\psi)$ as follows: Points of $\mathscr{A}T(\psi)$ are the points of Σ_4^* not belonging to Σ_3^*. Blocks are of two types. Type I blocks are the planes of Σ_4^* that meet Σ_3^* in an element of $\mathscr{S}' \bigcup \{u, v\}$. Type II blocks are the 4-dimensional Baer subspaces of Σ_4^* that meet Σ_3^* in an element of $\{\theta(j)\}$. With incidence given by inclusion, it follows that $\mathscr{A}T(\psi)$ is an affine translation plane of order s^4. Further, it is easy to see that $\mathscr{AT}(\psi)$ is obtained from $\mathscr{AT}(\mathscr{S})$ by a derivation using $\mathscr{F}$ as a derivation set, see [11, p. 223].

The above partition ψ is an example of a general construction for translation planes mentioned by Bruen and Thas [7]. As in [7] let ψ be a partition of the points of PG$(2n-1, q^2)$, $n \geqslant 1$ where ψ consists of α_1 spaces $\Sigma(i)$, $1 \leqslant i \leqslant \alpha_1$, each isomorphic to PG$(n-1, q^2)$ and α_2 spaces $\Gamma(j)$, $1 \leqslant j \leqslant \alpha_2$, each isomorphic to PG$(2n-1, q)$. Embed PG$(2n-1, q^2)$ in PG$(2n, q^2)$ and define an incidence structure π by generalizing the example above. It follows that π is an affine translation plane of order q^{2n} with GF(q) contained in the kernel. As mentioned in [7] the case $n = 1$ gives an example of the now classical derivation procedure of Ostrom [11, p. 223]. The case of $\alpha_2 = 0$ yields the André construction emphasized by Bruck–Bose in their linear representation theory. For an extensive bibliography in the linear representation of projective planes see Barlotti [1].

By allowing the projective spaces used in the Bruck-Bose linear representation theory to include Baer subspaces like $\theta(j)$ it is easy to see how an affine triangle of $\mathscr{AT}(\mathscr{S})$ may be contained in more than one affine subplane. An affine triangle of $\mathscr{AT}(\mathscr{S})$ is represented by points A, B, C of Σ_4^* not in Σ_3^* that are not on a plane of Σ_4^* containing an element of $\mathscr{S}$. Hence they are not collinear in Σ_4^*. Let P, Q, R be the intersection points of Σ_3^* with the lines AB, AC, BC. Let λ be the Baer subplane of Σ_4^* spanned by A, B, C and the line $t = PQR$. Any affine triangle ABC of $\mathscr{AT}(\mathscr{S})$ with the line t contained in one of the transversal Baer subspaces $\theta(j)$ of $\mathscr{F}$ has the property that ABC is in at least two distinct Baer subplanes of $\mathscr{AT}(\mathscr{S})$. One will be represented by the plane over GF(s^2) determined by A, B, C. The second will be represented by the 4-dimensional Baer subspace Σ_4 spanned by $\theta(j)$ and λ. It is straightforward to check that the points of Σ_4 not in Σ_3^* together with the planes of Σ_4^* representing lines of $\mathscr{AT}(\mathscr{S})$ that meet Σ_4 in at least two affine points of Σ_4^* form an affine Baer subplane of $\mathscr{AT}(\mathscr{S})$. It is immediate that both are Desarguesian. Since each affine triangle in the Desarguesian affine plane is contained in exactly one affine Baer subplane [8] the following is established.

THEOREM 4.1. *Each affine translation plane of order p^{4c} given by a* 1-*spread containing a pseudo-regulus contains more subplanes of order p^{2c}, $c \geqslant 1$, than the Desarguesian affine plane of order p^{4c}.*

This answers in the affirmative one part of Cofman's question stated in the introduction. Also, the uniqueness condition $U(q)$ assumed by Bruck [3, p. 433] is therefore not true for $q = s^2$. However, as he mentions in [3], $U(q)$ not being true is not serious.

For a partition ψ of PG$(2n-1, q^2)$ as described by Bruen and Thas let $\mathscr{AT}(\psi)$ be the associated affine plane. If the partition has $\alpha_2 \neq 0$ it might be expected that $\mathscr{AT}(\psi)$ would not be Desarguesian. This is not true. For, as in

Section 3, let u, v be two lines of a regular spread of the imaginary space Σ_3^*. The remaining $s^4 - 1$ lines may be uniquely partitioned into $s^2 + 1$ reguli $F'(i)$, $0 \leqslant i \leqslant s^2$, each containing u, v with $\{F(i)\}$ the opposite reguli. Theorem 3.1 implies that each regulus $F(i)$ determines a pseudo-regulus $\mathscr{F}(i)$ in $\Omega_3^* = \langle u, v^\xi \rangle$, and in each regulus $F'(i)$, the $s^2 - 1$ lines not u, v determine $s - 1$ Baer 3-spaces in Ω_3^*. These $(s-1)(s^2+1) = \alpha_2$ Baer subspaces together with u, v^ξ form a partition ψ of the points of Ω_3^*. However it is immediate that the associated translation plane using ψ and Ω_3^* as the distinguished hyperplane is Desarguesian.

REMARK 4.1. Note that in Ω_3^* of the above paragraph each line meeting both u, v^ξ belongs to exactly one of the pseudo-reguli $\mathscr{F}(i)$, $0 \leqslant i \leqslant s^2$.

We now reinterpret this collection of pseudo-reguli on the Klein quadric to obtain a partition of the points of a ruled quadric in PG(3, s^2) by $s^2 + 1$ elliptic quadrics in Baer 3-spaces.

Let α be the Plucker correspondence ([10, [15]) from the lines of the 3-space Ω_3^* above to the points of the Klein quadric Q_5^*. The tangent 4-spaces at the points $\alpha(u)$, $\alpha(v^\xi)$ intersect in a 3-space Λ_3^* which sections Q_5^* in a non-degenerate ruled quadric H_3^*. Each point of H_3^* corresponds under α^{-1} to a line meeting both u, v^ξ. For each of the pseudo-reguli $\mathscr{F}(i)$ in Ω_3^*, $0 \leqslant i \leqslant s^2$, let $\Gamma(i)$ be one of its transversal Baer 3-spaces intersecting $\mathscr{F}(i)$ in a regular spread. Since the Plucker correspondence is independent of the defining points of any given line each $\Gamma(i)$ may be viewed, in turn, as a 'real' 3-space of Ω_3^*. Hence each pseudo-regulus $\mathscr{F}(i)$ corresponds under α to the points of an elliptic quadric in a Baer 3-space to yield

REMARK 4.2. The points of a ruled quadric of PG(3, s^2) are partitioned not only by two distinct reguli but also by a collection of $s^2 + 1$ elliptic quadrics of order s.

As indicated in [7] one motivation for obtaining the general partition ψ came from the examination of the partial spreads (maximal) given by Mesner [14]. A partial spread $\mathscr{S}$ is called *complete* if it is not a spread and not properly contained in a partial spread. Complete partial spreads are the maximal strictly partial spreads of Bruen [5]. For $q = 4$, q odd, or $q = 2^{2t+1}$, examples of complete partial spreads of PG(3, q) have been constructed with $\mathscr{S}| = q^2 - q + 2$, [14], [5], [7].

LEMMA 4.1. *Let* $\mathscr{S} = \mathscr{F} \bigcup \mathscr{S}'$ *be a spread of* PG(3, q), $q = s^2$ *containing a pseudo-regulus* $\mathscr{F}$ *with transversal lines* u, v *Then the partial spread* $\mathscr{S}' \bigcup \{u, v\}$ *is complete with*

$$|\mathscr{S}| = q^2 - q + 2.$$

Proof. Immediate.

For $q = s^2 = p^{2t}$, p an odd prime, the spreads arising from the indicator set of Theorem 3.2 of [6, p. 525], where $\sigma \in \mathrm{Aut}(\mathrm{GF}(q^2))$ is given by $x \to x^s$, contain a pseudo-regulus. Even though odd characteristic of GF(q) is critical in both [5], [6], note that it has played no significant role in this paper. The goal now is to show explicitly how a pseudo-regulus $\mathscr{F}$ may be contained in a full spread of PG(3, s^2), for any characteristic.

LEMMA 4.2. *Let* Γ *be a Baer subspace of* PG(3, s^2) *containing a regular spread and let* $\mathscr{F}$ *be the associated pseudo-regulus with transversal lines* u, v. *If a line of* PG(3, s^2) *is not in* $\mathscr{F}$ *and meets both* u, v, *then the line is imaginary w.r.t.* Γ.

Proof. Since u, v are disjoint each point X of PG(3, s^2) not on u or v is on a unique line meeting both u and v. If X is in Γ the unique line is in $\mathscr{F}$. The statement follows.

LEMMA 4.3. *Given a Baer* 3-*space* Γ *of* PG(3, s^2) *there is a hyperbolic quadric of* PG(3, s^2) *imaginary w.r.t.* Γ, *and conversely.*

Proof. We may identify PG(3, s^2) with Ω_3^* of Remark 4.1 and for a fixed i, $0 \leqslant i \leqslant s^2$, Γ may be taken as one of the transversal Baer subspaces $\Gamma(i)$ intersecting the pseudo-regulus $\mathscr{F}(i)$ with transversals u, v^ξ in a regular spread. Label this spread $\mathscr{S}$. In Γ let R be a regulus contained in $\mathscr{S}$, and R^* the regulus in Ω_3^* determined by R. The elements of R^* correspond under α to the points of a conic $\alpha(R^*)$ of Q_5^* on a plane π^*. The plane π^* contains a Baer subplane π containing the conic $\alpha(R)$, and π^* is contained in the 3-space Λ_3^*. Call a point P of the Klein quadric *real* if $\alpha^{-1}(P)$ is a line of Γ. In π^* choose a passing line m^* of the conic $\alpha(R^*)$. Necessarily m^* meets the subplane π in exactly one point. Of the $s^2 + 1$ planes of Λ_3^* through m^* one may choose, independent of the characteristic of GF(s^2), a plane intersecting the ruled quadric H_3^* in a conic C^* having no real points. Each line of the regulus in Ω_3^* corresponding to $\alpha^{-1}(C^*)$ meets u, v^ξ in a point each and is not an element of $\mathscr{F}$. The quadric determined by the regulus $\alpha^{-1}(C^*)$ therefore misses each point of Γ by Lemma 4.2. Since two ruled quadrics of PG(3, s^2) are equivalent under the collineation group of PG(3, s^2) [11, p. 46] the converse follows to finish the proof.

LEMMA 4.4. *Let* Ω *be a Baer* 3-*space of* PG(3, s^2) *containing a regulus* W, W^* *be the regulus in* PG(3, s^2) *determined by* W, *and* p *be a passing line of* W. *Then there is a Baer* 3-*space* Γ *containing* p *and having no point in common with the quadric determined by* W^*.

Proof. Let $Q_3(Q_3^*)$ be the ruled quadric in $\Omega(\Omega_3^*)$ determined by $W(W^*)$. The unique regular spread of Ω containing p and W determines a pseudo-regulus,

$\mathscr{F}(p, W)$, of Ω_3^*. The extension of p in Ω_3^*, p^*, is a secant to Q_3^* meeting it in points X, Y. Let $x^*(y^*)$ be the line of the regulus opposite to W^* containing $X(Y)$. The subline p is one of the $s - 1$ sublines that form a partition, Π, of the points of $p^*\backslash\{X, Y\}$. Thus Ω may be taken to be any one of the $s - 1$ Baer 3-spaces that are transversal spaces of $\mathscr{F}(p, W)$ (corollary to Theorem 3.1 above). From the proof of Lemma 4.3 let Γ and R^* be, respectively, the Baer 3-space and regulus imaginary w.r.t. Γ. Further, the regular spread of Γ, $\mathscr{S}$, determines a pseudo-regulus having u, v^ξ as transversals with u, v^ξ belonging to the regulus opposite to R^*. Let I_3^* be the quadric ruled by R^*. For a line $t \in \mathscr{S}$, its extension t^* is a secant to I_3^* meeting it in points A, B of u, v^ξ and t belongs to the subline partition of the points of $t^*\backslash\{A, B\}$. Again, there is a collineation δ of Ω_3^* with $(R^*)^\delta = W^*$, $u^\delta = x^*$, $(v^\xi)^\delta = y^*$. Using Result 1, the group of the quadric, $K(Q_3^*)$, is transitive on its secants. Hence there is a $\phi \in K(Q_3^*)$ with $(t^*)^{\delta\phi} = p^*$. The image of the subline t, $t^{\delta\phi}$, is in Π and $\Gamma^{\delta\phi}$ is one Baer 3-space having no point in common with Q_3^* to finish the proof.

A spread of PG(3, s^2) containing a given pseudo-regulus $\mathscr{F}$ is now described. Let Γ be one of the associated transversal Baer 3-spaces of $\mathscr{F}$. As above, a point P of Q_5^* is real or imaginary if $\alpha^{-1}(P)$ is a line of Γ or not. The real points lie on a quadric Q_a in a 5-dimensional Baer subspace. Let o, $x \in \mathscr{F}$ and $0 = \alpha(o)$, $X = \alpha(x)$ be the real points on Q_5^*. The tangent 4-spaces $T_4^*(0)$, $T_4^*(X)$ intersect in a 3-space Δ_3^* meeting Q_5^* in a ruled quadric I_3^*. When α is restricted to Γ let the distinguished Baer subspaces be, respectively, $T_4(0)$, $T_4(X)$, Δ_3.

Following Segre [15, p. 56] a *spread set* is a set of $q^2 + 1 = s^4 + 1$ mutually nonconjugate points of Q_5^*. Under the projection of the points of Q_5^* into $T_4^*(X)$ with center O, the points, $\alpha(\mathscr{F}\backslash\{o\})$, being on a real elliptic quadric, map 1-to-1 into the points of a Baer subplane π of $T_4^*(X)$ contained in $T_4(X)$. Also π intersects Δ_3 in a line p which is a passing line of the ruled quadric I_3 of Δ_3. By Lemma 4.4 let Γ be a Baer 3-space of Δ_3^* containing p and having no point in common with the quadric I_3^*. Let Σ_4 be the Baer 4-space of $T_4^*(X)$ spanned by the plane π and Γ. The set of points $\mathscr{A}_4$ of Σ_4 not in Γ is disjoint with Δ_3^*. Hence no point of $\mathscr{A}_4$ is in $T_4^*(0)$. The line OP for each point P of $\mathscr{A}_4$ must therefore intersect Q_5^* in a point distinct from 0. In this way there is a 1-to-1 correspondence between the elements of $\mathscr{A}_4$ and a set F of points on Q_5^*. Since the line joining any two points of $\mathscr{A}_4$ meets Δ_3^* in a point of Γ the line misses the ruled quadric I_3^*. Hence the set $F \bigcup \{0\}$ is a spread set containing the $s^2 + 1$ points of the real elliptic quadric $\alpha(\mathscr{F})$. We have therefore

THEOREM 4.2. *If* $\mathscr{F}$ *is a pseudo-regulus of* PG(3, s^2) *then* $\mathscr{F}$ *is contained in a spread of* PG(3, s^2).

COROLLARY. *There exist complete partial spreads* $\mathscr{S}$ *of* PG(3, q), $q = s^2 = 2^{2t}$, $t \geq 1$ *with* $|\mathscr{S}| = q^2 - q + 2$.

REMARK 4.3. In the above proof, by choosing a regular spread of Δ_3^* containing either regulus of the quadric I_3^* then $T_4^*(X)$ together with the spread is a linear representation of the indicator plane of Bruen [6] and the point set $\mathscr{A}_4$ is an indicator set.

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