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Freedom of choice and rational decisions

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Abstract. The paper formalizes a notion of preference-based freedom and examines to which extent such a notion is consistent with otherwise standard conditions of rational decision making. The central result is as follows. Suppose that a preference-based ranking of opportunity sets satisfies a very mild condition of "preference for freedom of choice". Then, either the ranking is degenerate in being discontinuous, or the underlying preference relation among the basic alternatives is incomplete. Hence, in any case preference-based rankings of freedom will violate at least some of the basic assumptions of traditional choice modelling. This conclusion is enhanced if the conditions on preference-based freedom are slightly strengthened.

1. Introduction

It has been argued that the concepts of freedom and preference are "very deeply interrelated, and that an affirmation of the importance of freedom must inter alia assign fundamental importance to preference" [16, p. 15]. The purpose of this paper is to formalize a certain notion of preference-based freedom and to examine to which extent such a notion is consistent with otherwise standard assumptions of rational decision making. Special attention is paid to the assumption that preferences among alternatives are *complete* and to the common requirement that "small" perturbations in the description of a decision problem should not lead to "large" changes in the outcome, i.e. to an appropriate *continuity* assumption. The central thesis of this paper is that there is a strong tension between the notion of

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preference-based freedom and these two basic assumptions of traditional decision theory.

The main result is as follows. Suppose that a preference-based ranking of opportunity sets satisfies a very mild condition of "preference for freedom of choice". Then, either the ranking is degenerate in being discontinuous, or the underlying preference relation among the alternatives is incomplete. One possible way to reconcile the continuity requirement with a "preference for freedom" is to sacrifice the assumption that preferences among alternatives are complete. Indeed, it is shown by means of examples that dropping the completeness assumption opens up some possibilities for modelling freedom of choice in a continuous framework. On the other hand, an appropriate modification of the above result shows that a slight strengthening of the conditions again produces an impossibility even when preferences are allowed to be incomplete. Thus, it seems that in the context of a preference-based notion of freedom the dividing line between possibility and impossibility is rather thin.

The plan of the paper is as follows. Section 2 introduces and discusses our basic conditions. Section 3 contains the central result showing that in a model of freedom the conditions of preference-basedness, continuity, and completeness of the underlying preference relation are together inconsistent. Section 4 provides some examples showing, in particular, that the conditions of preference-basedness and continuity are nevertheless compatible with underlying preferences which are in a sense *almost* complete. In Sect. 5, it is shown that stronger versions of the conditions of preference-basedness and continuity are together inconsistent even when the underlying preference relation is incomplete. Concluding remarks and some further comments on related literature are provided in Sect. 6.

2. Basic conditions

Let $X \subseteq \mathbf{R}^l$ be a set of basic alternatives. There are several possible interpretations of the set X which would fit our formal analysis. For the purpose of this paper, it will be convenient to think of the elements of X as either commodity bundles, or functioning l-tuples in the sense of Sen [14]. For a discussion of other interpretations and of the general question on which spaces of alternatives freedom of choice should be modelled we refer to [4, 15]. We just note here that with respect to the modelling of the universal set of alternatives the present approach is closer to Klemisch-Ahlert [4] than to Bossert et al. [1], Puppe [11], or Pattanaik and Xu [9, 10], who assume X to be finite. Throughout, we assume that X is endowed with the standard topology of $\mathbf{R}^l$ and that X has a non-empty interior.

Let $\mathscr{F}(X)$ be the set of all non-empty finite subsets of X. Elements of $\mathscr{F}(X)$ are referred to as *opportunity sets* or *menus*. The intended interpretation is that an element $A \in \mathscr{F}(X)$ represents a set of feasible alternatives from which, in a later stage of choice, exactly one element has to be chosen. Hence, the final outcomes of the decision process are always elements of X. Let $\succeq$ denote a binary relation on $\mathscr{F}(X)$. The interpretation of $\succeq$ is that $A \succeq B$ if and only if A offers *at least as much freedom as* B. Throughout, it is assumed that $\succeq$ is a *preorder*, i.e. transitive and reflexive. Thus, for all $A, B, C \in \mathscr{F}(X)$, ($A \succeq B$ and $B \succeq C$) implies $A \succeq C$, and for all $A \in \mathscr{F}(X)$, $A \succeq A$. The relations $\succ$ and $\sim$ are defined as the asymmetric and the symmetric part of $\succeq$, respectively, i.e. $A \succ B$ if ($A \succeq B$ and not $B \succeq$ A), $A \sim B$ if ($A \succeq B$ and $B \succeq A$). A preorder $\succeq$ is *complete* if for all $A, B \in \mathscr{F}(X)$, ($A \succeq B$ or $B \succeq A$). A complete preorder is also called a *weak order*.

Preferences among alternatives are modelled through a binary relation R on X. For any two alternatives $x, y \in X$, xRy is interpreted as "x is preferred or indifferent to y". Again, R is always assumed to be reflexive and transitive. In general, no assumptions are made with respect to completeness of R. As usual, the asymmetric and symmetric parts of R are denoted by P and I, respectively. Throughout, we will assume that preferences on X are continuous. There are two common formulations of continuity.

(C_R) *Continuity of* R: For all $x \in X$,

$$\{y \in X : xRy\} \text{ and } \{y \in X : yRx\} \text{ are closed in } X.$$

(C_P) *Continuity of* P: For all $x \in X$,

$$\{y \in X : xPy\} \text{ and } \{y \in X : yPx\} \text{ are open in } X.$$

Obviously, for a complete preorder R conditions (C_R) and (C_P) are equivalent. However, if R is not complete the equivalence no longer holds and the question arises which of the two conditions captures the "right" notion of continuity. Clearly, this will mainly depend on the specific interpretation of the alternatives and of the relation R. This question will be further discussed in the subsequent sections.

2.1. Conditions of "preference-basedness"

In this subsection some conditions specifying the relation between preferences R on X and a ranking $\succeq$ of menus in terms of freedom are suggested. Following the arguments of Sen [16] we will require that $\succeq$ is an *extension* of R in the sense that for all $x, y \in X$,

$$xRy \;\Rightarrow\; \{x\} \succeq \{y\} \tag{1}$$

and

$$xPy \;\Rightarrow\; \{x\} \succ \{y\}. \tag{2}$$

Condition (2) is exactly what has been suggested by Sen (cf. [16, p. 25]), and we will not repeat the underlying intuition here. Condition (1) is very much in the same spirit. Given (2), condition (1) just adds the requirement that if x and y are indifferent as alternatives then the singleton menus $\{x\}$ and $\{y\}$ should be indifferent in terms of freedom. It is noted that the condition $xIy \Rightarrow \{x\} \sim \{y\}$ is even satisfied in the non-preference-based approach of Pattanaik and Xu [9]. Indeed, if the relevant criterion is cardinality, as implied by the conditions used in [9], $\{x\} \sim \{y\}$ holds regardless of the preference between x and y. Hence, the conflict of the above conditions with the cardinality-based approach of [9] is entirely due to condition (2).

Given a preference relation R on X, conditions (1) and (2) specify a ranking $\succeq$ only among singleton sets implying no restrictions on the comparison of other sets. In order to capture the notion of preference-basedness stronger conditions are needed. For the formulation of the conditions below it will be convenient to introduce the following notation. For any $A \in \mathcal{F}$ (X) and every $x, y \in X$ such that

$y \in A$ and $x \notin A$, let A^x_{-y} denote the set $(A \setminus \{y\}) \cup \{x\}$. Thus, A^x_{-y} equals the set A with y replaced by x.

(PB) *Preference-Basedness*: For all $A \in \mathscr{F}(X)$, $x \in X \setminus A$, $y \in A$,

$$xRy \;\Rightarrow\; A^x_{-y} \succeq A.$$

Condition (PB) states that in a given situation the substitution of an alternative by a weakly preferred alternative should always be weakly preferred to the "status quo". Obviously, (PB) implies (1) by letting $A = \{y\}$. There are several slightly different aspects of condition (PB). First, condition (PB) may be viewed as a weak *dominance* condition. Indeed, (PB) is equivalent to the following condition which is part of the condition of *Weak Preference Dominance* suggested in [16] in the context of freedom of choice (see also [14, 17]).

> *Weak Preference Dominance*: Let $A, B \in \mathscr{F}(X)$ with $\# A = \# B$, where $\# A$ and $\# B$ denote the cardinality of A and B, respectively. Suppose that there exists a one-to-one correspondence $f: A \to B$ such that for all $x \in A$, $xRf(x)$. Then, $A \succeq B$.

Clearly, Weak Preference Dominance implies (PB). To see the converse implication suppose that $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ are ordered in such a way that $a_i R b_i$ for all i. Assume for a moment that A and B are disjoint. Repeated application of (PB) yields

$$\begin{aligned}
\{a_1, \ldots, a_{n-1}, a_n\} &\succeq \{a_1, \ldots, a_{n-1}, b_n\},\\
\{a_1, \ldots, a_{n-1}, b_n\} &\succeq \{a_1, \ldots, b_{n-1}, b_n\},\\
&\ \vdots\\
\{a_1, b_2, \ldots, b_n\} &\succeq \{b_1, b_2, \ldots, b_n\}.
\end{aligned}$$

Hence by transitivity, $A \succeq B$ as required by Weak Preference Dominance. The general case, when A and B are not necessarily disjoint, follows along the same lines by applying the argument to those elements by which A and B differ (cf. [7, Lemma 1]).

An alternative interpretation of condition (PB) is as an *independence* condition. From this point of view, (PB) states that a preference for x over y should be independent from the context, i.e. the specific menu, in which these alternatives occur. In this context, it is worth noting that for an extension $\succeq$ of R condition (PB) is implied by a stronger independence condition which has been widely used in the literature on ranking of sets[2]. To further illustrate the independence character of condition (PB), suppose that a decision maker decides that $\{x\} \succeq \{y\}$, thereby "revealing" that his preferences are xRy. Now suppose that there is a third option, say z, and the same decision maker is asked to rank $\{x, z\}$ *vis-à-vis* $\{y, z\}$. If his ranking is, even in a weak sense, "preference-based" he should indeed find that

[2] See Kannai and Peleg [3] and the various comments on their well-known impossibility theorem in the *J Econ Theory*, Vols. 32 and 33. In the context of freedom of choice, variants of the stronger independence condition, referred to as condition (IND) there, have been used by Bossert et al. [1], Pattanaik and Xu [9, 10], and Suppes [18]. For the proof that (IND) implies (PB) if $\succeq$ is an extension of R see [7].

$\{x, z\} \succeq \{y, z\}$. The reason is that in comparing *menus* the decision maker knows that the final outcome will be exactly one alternative from one of these menus. Indeed, the comparison in our specific example is between the options "*x or z*" and "*y or z*". Hence, it seems that the common alternative z is "irrelevant" for the ranking of $\{x, z\}$ *vis-à-vis* $\{y, z\}$. This line of thought finally suggests a third inerpretation of condition (PB), namely as a condition of *consistency*.

It is noted that – just as other independence conditions used in the literature on freedom of choice – condition (PB) does not take into account the aspect of "variety" discussed in Pattanaik and Xu [9]. The following example is taken from [9, pp. 389–390]. Suppose that a person weakly prefers travelling by a blue car to travelling by train. Furthermore, suppose that there is a third option, travelling by a red car. Would not one plausibly feel that – contrary to what condition (PB) requires – the set $\{$*train*, *red car*$\}$ offers more freedom than the set $\{$*blue car*, *red car*$\}$, because it entails a greater variety of opportunities? Our intuition is that the plausibility of this example hinges to a large extent on the possibility of *uncertain future preferences*. Indeed, in the presence of uncertainty greater variety is likely to increase the probability that some desirable alternative will be included in the set of future opportunities. This connection between freedom and uncertainty is clearly an important one[3]. However, there is no uncertainty modelled in our present approach, and our aim is to examine a notion of freedom which does not *depend* on the presence of uncertainty. In this case, it is no longer clear what role the notion of variety could play in the assessment of freedom. Certainly, this is not to say that "variety" is not an issue in the context of freedom. It may eventually turn out that concepts different from preference, such as e.g. the desire for greater variety in the presence of uncertainty, have to explicitly enter the analysis of freedom. And indeed the results to be presented in the subsequent sections may suggest this. But in the absence of uncertainty the notion of variety might not be *intrinsically* relevant for the assessment of freedom. In this case, condition (PB) seems to be very attractive.

By condition (PB), the replacement of an alternative y by an alternative x should be valued according to the preference between those two alternatives. However, since R may be incomplete the alternatives x and y may be incomparable. In this case, condition (PB) has no impact on the ranking of menus. Consider therefore the following variant of the condition of preference-basedness.

(PB′) For all $A \in \mathscr{F}(X)$, $x \in X \setminus A$, $y \in A$,

$$A \succ A^{x}_{-y} \;\Rightarrow\; yPx.$$

At first, it might seem that (PB′) is just the contraposition of (PB). This is, however, only true if both, R and $\succeq$, are complete relations. In this case one clearly has (PB) $\Leftrightarrow$ (PB′). If only R is complete one still has (PB) $\Rightarrow$ (PB′). But in general the two conditions are independent. Although formally similar, the spirit of condition (PB′) is different. Indeed, (PB) states that *if there is* preference between x and y then it should be decisive in the comparison of A^{x}_{-y} and A. On the other hand, (PB′) states that a strict preference for one of the sets, A^{x}_{-y} or A, must be *due* to a strict preference for one of the two alternatives by which these sets differ. However, although (PB′) is implied by (PB) for a complete preorder R it might be a too strong condition if R is not complete. Suppose for example that x and y are incomparable

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[3] For an analysis of this connection in terms of Kreps' [5] concept of "preference for flexibility" see [11].

with respect to R but that there exists a third alternative $a \in A$ such that aPx but not aPy. Thus, although x and y are incomparable with respect to R the alternative $a \in A$ allows to discriminate between x and y with respect to preference, favouring y over x. In this case, it seems plausible to conclude $A \succ A^x_{-y}$ in violation of condition (PB′). In the subsequent sections we will therefore only use the following weaker version of (PB′).

(PB″) For all $A \in \mathscr{F}(X)$, $x \in X \setminus A$, $y \in A$,

$$A \succ A^x_{-y} \;\Rightarrow\; \exists a \in A \text{ such that } aPx \text{ and not } aPy.$$

Obviously, (PB′) $\Rightarrow$ (PB″), with equivalence if R is complete (for an illustration of the difference between (PB′) and (PB″) see Example 5 in Sect. 4).

2.2. Conditions of "preference for freedom"

We turn now to the problem of formulating explicit conditions of "preference for freedom of choice". The conditions suggested in this subsection correspond to different notions of monotonicity with respect to set inclusion. The most uncontroversial condition in this context is the following.

(M) *Weak Monotonicity w.r.t. Set Inclusion*: For all $A, B \in \mathscr{F}(X)$,

$$B \subseteq A \;\Rightarrow\; A \succeq B.$$

However, this condition is obviously not sufficient to capture even a very weak notion of freedom. Indeed, condition (M) seems to be satisfied in *any* model of ranking sets of alternatives in which the decision maker himself (and not "nature", or an opponent) determines the final outcome. At least, this will be true if one neglects "decision costs", or other restrictions on the decision maker's ability to choose from his opportunities.

Next, consider the following condition which has been introduced in [11].

(F) *Preference for freedom of choice*: For all $A \in \mathscr{F}(X)$ with $\# A \geq 2$,

$$\exists x \in A \text{ such that } A \succ A \setminus \{x\}.$$

In [11] it has been argued that on a finite domain X condition (F) is a *necessary* condition for a ranking of opportunity sets to be regarded as a ranking in terms of freedom of choice. The intuition behind condition (F) is best explained by reference to the notion of *availability*. Indeed, the most important factor in the assessment of freedom is the availability of certain alternatives (and not *only* their utility). Note that there is an essential conceptual difference between the notions of utility and availability. Indeed, unlike utility, availability cannot be subject to substitution. That is, availability of one alternative is always another entity than availability of another alternative, whereas the utility of two different alternatives might just be the *same*. What condition (F) requires is that in every opportunity set A with $\# A \geq 2$ there is *at least one* alternative the availability of which constitutes an essential contribution to the freedom offered by A. In the light of our former issue of preference-basedness, a clear candidate for such an alternative would be any alternative $x \in A$ such that x is maximal in A with respect to the underlying preference relation R. Note that, given a preorder R on X and a non-empty finite subset A of X, such maximal alternatives always exist. However, it would be misleading if one would concentrate on R-maximal alternatives in order to verify condition (F). Indeed, $A \succ A \setminus \{x\}$ will in general hold for other alternatives as well. For instance, suppose that travelling by car is strictly preferred to travelling by train. Then, while the freedom is certainly reduced by removing the option "travelling by car", it may as well be reduced by removing the option "travelling by train".

Clearly, condition (F) is much weaker than the requirement that $A \succ A \setminus \{x\}$ holds for *every* $x \in A$ as implied e.g. in the approach of Pattanaik and Xu [9]. However, although condition (F) may be a necessary condition for ranking freedoms on a *finite* domain X it might still be too strong if considered on infinite domains. Imagine a set A of alternatives between all of which a decision maker is indifferent. If the cardinality of A is very large, it is in fact not clear that removing *one single* alternative would appreciably reduce the extent of freedom. It seems that on infinite domains condition (F) would be justified only by the assumption that the decision maker has *perfect* discrimination power with respect to the number of available alternatives. On the other hand, even when the decision maker has only imperfect discrimination power it seems clear that if alternatives *continue* to be removed, then eventually freedom is reduced. This suggests the following weakening of condition (F).

(MF) *Minimal preference for freedom*: For all $A \in \mathscr{F}(X)$ with $\# A \geq 2$,

$\exists B \subseteq A$ such that $A \succ B$.

Note that since $\succeq$ is only defined on $\mathscr{F}(X)$ the set B in condition (MF) is necessarily non-empty. Also, since $\succeq$ is reflexive, one must have $B \neq A$. Hence, what condition (MF) requires is that for any $A \in \mathscr{F}(X)$ with $\# A \geq 2$ there should exist a proper non-empty subset B of A such that A offers more freedom than B. Obviously, (F) implies (MF). Indeed, suppose that A has n elements with $n \geq 2$. Then, condition (F) requires that A should offer more freedom than some proper subset of A containing exactly $n-1$ alternatives.

Certainly, we do not want to argue that condition (MF) characterizes an interesting notion of freedom. Condition (MF) is presumably too weak to achieve this. Rather, as our labelling indicates, we consider (MF) to be a *minimal* requirement for a ranking of opportunity sets to display a "preference for freedom of choice". The aim of this paper is to demonstrate that in a continuous framework there is a strong tension between the conditions of preference-basedness and even such a weak condition as (MF). Obviously, this would hold *a forteriori* if condition (MF) would be replaced by a stronger condition of "preference for freedom of choice".

Comparing conditions (MF) and (M) it seems natural to combine them into one single condition.

(MF′) For all $A \in \mathscr{F}(X)$ with $\# A \geq 2$,

$(\forall B \subseteq A\colon A \succeq B)$ and $(\exists B \subseteq A\colon A \succ B)$.

In fact, condition (MF′) seems to be a very natural way of formalizing the notion of *strict* monotonicity with respect to set inclusion.

2.3. Continuity conditions

In this subsection, we will use the additional structure on X given by the topology of $\mathbf{R}^l$. The basic idea is that a small perturbation of the alternatives in a given set of opportunities should only have a small impact on the ranking of this set *vis-à-vis* all other sets. To be specific, let x^i, $i = 1, \dots, n$, be (not necessarily distinct) elements of X. Furthermore, for each $i = 1, \dots, n$, let $(x^i_j)_{j \in \mathbf{N}}$ be a sequence converging to x^i. We will consider the following two conditions.

($C_{\succeq}$) *Continuity of* $\succeq$: For all $A \in \mathscr{F}(X)$,

$$[\forall j \in \mathbf{N}\colon \{x^1_j, \dots, x^n_j\} \succeq A] \;\Rightarrow\; \{x^1, \dots, x^n\} \succeq A$$

and

$$[\forall j \in \mathbf{N}\colon A \succeq \{x^1_j, \dots, x^n_j\}] \;\Rightarrow\; A \succeq \{x^1, \dots, x^n\}.$$

($C_{\succ}$) *Continuity of* $\succ$: For all $A \in \mathscr{F}(X)$,

$$\{x^1, \dots, x^n\} \succ A \;\Rightarrow\; [\exists j_0 \forall j \geq j_0\colon \{x^1_j, \dots, x^n_j\} \succ A]$$

and

$$A \succ \{x^1, \dots, x^n\} \;\Rightarrow\; [\exists j_0 \forall j \geq j_0\colon A \succ \{x^1_j, \dots, x^n_j\}].$$

Either of the two continuity conditions captures the following general idea. Let $\{x^1, \dots, x^n\}$ be a given set of alternatives. Suppose that for each $i = 1, \dots, n$, the alternative x^i is replaced by an alternative $\hat{x}^i$. Then, the ranking of $\{\hat{x}^1, \dots, \hat{x}^n\}$ *vis-à-vis* any other set should not suddenly change if for each $i = 1, \dots, n$, $\hat{x}^i$ gets arbitrarily close to x^i. Indeed, suppose that for each $i = 1, \dots n$, $\hat{x}^i$ is arbitrarily close to x^i. Then, if a decision maker's set of opportunities $\{x^1, \dots, x^n\}$ is replaced by $\{\hat{x}^1, \dots, \hat{x}^n\}$ he can still guarantee himself for any x^i an alternative arbitrarily close to x^i. Note that this applies not only to the alternative actually chosen but to any *counterfactual* choice as well. In terms of possible outcomes the two situations represented by $\{x^1, \dots, x^n\}$ and $\{\hat{x}^1, \dots, \hat{x}^n\}$ are arbitrarily close to each other. Hence, given that preferences over alternatives are continuous and that opportunity sets are valued by their possible outcomes, a replacement of $\{x^1, \dots, x^n\}$ by $\{\hat{x}^1, \dots, \hat{x}^n\}$ cannot suddenly change the ranking *vis-à-vis* any other set.

Note that some of the sequences $(x^i_j)_{j \in \mathbf{N}}$ in conditions ($C_{\succeq}$) and ($C_{\succ}$) may have the *same* limit. Thus, suppose for example that the sequences (x_j) and (y_j) both converge to x_0, and that for all $j \in \mathbf{N}$, $x_j \neq y_j$. Condition ($C_{\succeq}$) requires that if for all $j \in \mathbf{N}$, $\{x_j, y_j\} \succeq A$ (resp. $A \succeq \{x_j, y_j\}$) for some $A \in \mathscr{F}(X)$, then one should also have $\{x_0\} \succeq A$ (resp. $A \succeq \{x_0\}$). Similarly, condition ($C_{\succ}$) requires that if $\{x_0\} \succ A$ (resp. $A \succ \{x_0\}$) then one should have $\{x_j, y_j\} \succ A$ (resp. $A \succ \{x_j, y_j\}$) for large j, i.e. if x_j and y_j are sufficiently close to x_0. In the context of freedom, it may seem as if both conditions would run counter the intuition that the cardinality of opportunity sets could play some role in the assessment of freedom. It is, however, important to realize that both conditions *do* allow the cardinality to play an important role. For example, let $x_j = x_0$ for all j, and assume that the y_j's are pairwise different alternatives, each indifferent to x_0. Conditions ($C_{\succeq}$) and ($C_{\succ}$) are perfectly consistent with the assumption that for all $j \in \mathbf{N}$, $\{x_0, y_j\} \succ \{x_0\}$. All what the continuity conditions require is that, intuitively, the "advantage" of having an additional indifferent alternative tends to zero as y_j converges to x_0.

As with conditions (C_R) and (C_P), conditions ($C_{\succeq}$) and ($C_{\succ}$) are equivalent if $\succeq$ is complete, but mutually independent in the general case. Also, it is easily verified that if R is a complete preference order on X and $\succeq$ is a (not necessarily complete) extension of R, then either condition, ($C_{\succeq}$) or ($C_{\succ}$), implies both (C_R) and (C_P). However, if R is not complete it is not necessarily true that ($C_{\succeq}$) implies (C_R)

or that $(C_{\succ})$ implies (C_P). The reason is that our formal definition of an extension (see (1) and (2)) does not rule out the possibility that x and y are incomparable with respect to R but at the same time e.g. $\{x\} \sim \{y\}$.

Although conditions $(C_{\succeq})$ and $(C_{\succ})$ are similar in spirit, we do not claim that they are equally plausible in our present context. Indeed, there is a subtle but important difference with respect to the logical form of the two conditions. Formally, condition $(C_{\succeq})$ deduces *one single* preference judgement from a given *set* of preference judgements. Conversely, condition $(C_{\succ})$ deduces a *set* of preference judgements from *one single* given preference judgement. Hence, $(C_{\succ})$, but not $(C_{\succeq})$, implies a "local completeness" property at strict preference judgements. Therefore, if completeness is an issue, and we shall see in the next section that it might be, then $(C_{\succ})$ is possibly a too strong condition. Note that the same observation applies to conditions (C_R) and (C_P) on the level of alternatives.

To further illustrate the difference between the two continuity conditions, assume as in [10] that what is relevant for the assessment of freedom is not the agent's individual preference ordering on X but the whole *set* of individual preference orderings $\{R_1, \ldots, R_n\}$ of the members of the society. In this case, it could be argued that $\succeq$ should be an extension of the *intersection* R of the R_i's, i.e. of the relation R defined by

$$xRy \;:\Leftrightarrow\; \text{for all } i, xR_iy.$$

It is easily verified that if all individual orderings are complete and continuous in the usual sense, the intersection R satisfies (C_R), but in general not (C_P). Consequently, under this specific interpretation the appropriate condition is $(C_{\succeq})$ and not $(C_{\succ})$.

3. An impossibility result

In this section we will prove the following theorem.

Theorem 1. *Let $X \subseteq \mathbf{R}^l$ with $l \geq 2$, and let R be a complete preorder on X. There does not exist an extension $\succeq$ of* R *to $\mathscr{F}(X)$ such that $\succeq$ satisfies (PB), (MF) and one of the conditions, $(C_{\succeq})$ or $(C_{\succ})$.*

It is emphasized that Theorem 1 does not require completeness of the extension $\succeq$ but only completeness of R. The proof of Theorem 1 is based upon the following two lemmata.

Lemma 1. *Let $X \subseteq \mathbf{R}^l$ with $l \geq 2$ and let R be a complete preorder on X satisfying (C_R) or, equivalently, (C_P). Then, there exist $x_0 \in X$ and a sequence $(x_j)_{j \in \mathbf{N}}$ in X such that $x_j \to x_0$ and for all $j \in \mathbf{N}$, x_jIx_0 and $x_j \neq x_0$.*

Proof. Let R be a continuous and complete preorder on X. By Debreu [2], there exists a continuous utility function $u: X \to \mathbf{R}$ such that $xRy \Leftrightarrow u(x) \geq u(y)$. Let w be a point in the interior of X. Choose $\varepsilon > 0$ such that the closure of the open ball $B_\varepsilon(w)$ around w is contained in the interior of X. If u is constant on $B_\varepsilon(w)$ the assertion of the lemma is obviously true. If u is not constant choose $y, z \in B_\varepsilon(w)$ such that $u(y) > u(z)$. Furthermore, choose $x \in B_\varepsilon(w)$ such that $u(y) > u(x) > u(z)$. There are an infinite number of continuous paths in $B_\varepsilon(w)$ connecting y and z such that any pair of these paths has no points in common other than y and z. By continuity of u, on every path there exists a point which has the same image under

u as the point x. Thus, there is an infinite set $\mathscr{I}$ of points in $B_\varepsilon(w)$ which are indifferent (with respect to R) to x. By the Bolzano–Weierstrass theorem there exists an accumulation point x_0 of the set $\mathscr{I}$. Obviously, this x_0 together with a suitable sequence in $\mathscr{I}$ converging to x_0 satisfy all the conditions required in the lemma.

Lemma 2. *Let $\succeq$ be a (not necessarily complete) preorder on $\mathscr{F}(X)$ such that $\succeq$ is an extension of some preorder R on X. Furthermore, let $\mathscr{I} \subseteq X$ be such that xIy for any $x, y \in \mathscr{I}$. Then (PB) and (MF) imply that for all $x, y, z, w \in \mathscr{I}$ with $x \neq y$, $z \neq w$,*

$$\{x, y\} \sim \{z, w\}, \quad \{x, y\} \succ \{z\}.$$

Proof. Without loss of generality assume that x, y, z, w are pairwise different. By condition (PB), yIw implies $\{x, y\} \sim \{x, w\}$ and xIz implies $\{x, w\} \sim \{z, w\}$. Hence, by transitivity, $\{x, y\} \sim \{z, w\}$. This shows the first part of Lemma 2.

To prove the second part, observe that by condition (MF) one must have either $\{x, y\} \succ \{x\}$, or $\{x, y\} \succ \{y\}$. Since both, zIx and zIy, one obtains $\{z\} \sim \{x\}$ and $\{z\} \sim \{y\}$, hence $\{x, y\} \succ \{z\}$.

Remark. Lemma 2 may be viewed as a very rudimentary version of the result of Pattanaik and Xu [9], which characterizes the ranking of opportunity sets given by their cardinalities. Indeed, by Lemma 2 conditions (PB) and (MF) together imply that an extension $\succeq$ of R ranks opportunity sets according to their cardinalities, provided that (i) the sets under consideration contain at most two elements, and (ii) the domain is restricted to a set $\mathscr{I}$ of mutually indifferent alternatives. It can be shown that in this result the restriction (i) can be dropped if condition (MF) is strengthened to condition (F). That is, if an extension $\succeq$ of R satisfies (PB) and (F), then for all $A, B \subseteq \mathscr{I}$, $A \succeq B \Leftrightarrow \#A \geq \#B$.

Despite the similarity with the conclusion of Pattanaik and Xu [9], the conditions in Lemma 2 are different from those considered in [9]. In particular, the condition, $\{x\} \sim \{y\}$ for all x, y ("Indifference between no-choice situations"), used in [9] is not *assumed* here. By Lemma 1, the conditions of continuity and completeness of R, together with the assumption that $\succeq$ is an extension of R, *imply* that there always exists a non-trivial domain $\mathscr{I} \subseteq X$ where this condition is satisfied. Lemma 2 applies only to such a domain.

Proof of Theorem 1. Assume, contrary to what Theorem 1 claims, that $\succeq$ is an extension to $\mathscr{F}(X)$ of a given weak order R on X such that $\succeq$ satisfies conditions (PB), (MF) and one of the conditions, ($C_\succeq$) or ($C_\succ$). First, we show that either condition, ($C_\succeq$) and ($C_\succ$), implies continuity of R in the usual sense. Let R' denote the preorder on X defined by

$$xR'y :\Leftrightarrow \{x\} \succeq \{y\}. \tag{3}$$

Clearly, continuity of $\succeq$ in the sense of condition ($C_\succeq$) and ($C_\succ$) imply continuity of R' in the sense of condition (C_R) and (C_P), respectively. However, since $\succeq$ is an extension of R and R is complete, the induced preorder R' coincides with R, i.e. for all $x, y \in X$, xRy if and only if $xR'y$. In particular, R' is complete. Consequently, continuity in the sense of (C_R) is equivalent to continuity in the sense of (C_P). Thus, R is continuous in the usual sense.

Next, choose x_0 and $(x_j)_{j \in \mathbf{N}}$ according to Lemma 1. By Lemma 2 one has for all $j \in \mathbf{N}$,

$$\{x_0, x_j\} \sim \{x_0, x_1\} \succ \{x_0\}.$$

However, this contradicts either continuity condition, $(C_{\succeq})$ and $(C_{\succ})$. Indeed, by $(C_{\succeq})$, $\{x_0, x_j\} \sim \{x_0, x_1\}$ for all $j \in \mathbf{N}$ implies $\{x_0\} \sim \{x_0, x_1\}$; and by $(C_{\succ})$, $\{x_0, x_1\} \succ \{x_0\}$ implies $\{x_0, x_1\} \succ \{x_0, x_j\}$ for sufficiently large $j \in \mathbf{N}$. This completes the proof of Theorem 1.

Remark. The proof of Theorem 1 actually proves a slightly stronger result. Indeed, the assumption that $\succeq$ is an extension of R to $\mathscr{F}(X)$ can be replaced by the weaker assumption that $\succeq$ is only an extension of R to the set of all non-empty subsets of X containing at most two elements. On the set of all two-element subsets of X, condition (MF) can be reformulated as follows. For all $x \neq y$, $(\{x, y\} \succ \{x\}$ *or* $\{x, y\} \succ \{y\})$. We note that this is a weakening of the condition of "Strict Monotonicity" suggested in [9].

Theorem 1 might be a rather disturbing result. After all, completeness of preferences is one of the basic assumptions in standard models of rational choice. However, we believe that Theorem 1 has a quite intuitive interpretation. Suppose that a decision maker has complete preferences among the basic alternatives and is asked to rank opportunity sets based on these preferences. Since there is no uncertainty involved in our setting the decision maker knows his preferences which will prevail when the final choice of the alternative has to be made. In such a deterministic framework it is indeed hard to see why this decision maker – knowing in every respect exactly what he wants – should exhibit a "preference for freedom of choice". On the other hand, incompleteness of preferences, i.e. an inability or unwillingness to completely rank the basic alternatives, can be regarded as an indication of a general *vagueness* of the person's preference relation as a whole. In such a situation of vagueness it could be advantageous to delay the decision while trying to retain the freedom to choose in a later stage of the decision process.

An important consequence of Theorem 1 is that, except for the case where X is one-dimensional, any preference-based ranking of freedoms satisfying one of the conditions, $(C_{\succeq})$ or $(C_{\succ})$, is by *itself* necessarily incomplete.

Corollary 1. *Let $X \subseteq \mathbf{R}^l$ with $l \geq 2$. There does not exist a complete preorder $\succeq$ on $\mathscr{F}(X)$ such that $\succeq$ satisfies (MF), one of the conditions, $(C_{\succeq})$ or $(C_{\succ})$, and the following condition. For all $A \in \mathscr{F}(X)$, $x \in X \backslash A$, $y \in A$,*

$$\{x\} \succeq \{y\} \;\Rightarrow\; A^{x}_{-y} \succeq A. \tag{4}$$

Remark. Condition (4) is of course closely related to (PB), highlighting the consistency character of condition (PB). On a purely formal level, however, the two conditions are not interchangeable. Indeed, Corollary 1 would be false if (4) would be replaced by (PB) together with the requirement that $\succeq$ be an extension of some preorder R. The reason is again that our definition of an extension is rather weak. For example, one could choose R to be the trivial preorder declaring any two distinct alternatives incomparable. Then *every* reflexive ranking of opportunity sets would trivially satisfy condition (PB).

Corollary 1 immediately follows from Theorem 1 once it is observed that $\succeq$ is an extension of the relation R' defined by (3), and that condition (4) is just condition (PB) applied to R'.

It is noted that the conclusion of Corollary 1 supports the view expressed in Sen [17, p. 529] who writes: "… comparisons of freedom must frequently take the form of incomplete orderings. While some set comparisons would be obvious enough, others would remain undecidable". Clearly, Theorem 1 goes even beyond this, revealing a conflict between a preference-based notion of freedom and the completeness assumption not only on the level of sets, but already on the level of alternatives.

4. Examples and independence of the conditions

This section provides some examples which, in particular, demonstrate the independence of the conditions stated in Theorem 1.

Example 1. *The Maximax Extension.*
Let R be a complete preorder on X with continuous utility representation $u: X \to \mathbf{R}$. For any $A \in \mathscr{F}(X)$, let $U(A) := \max_{x \in A} u(x)$. The weak order $A \succeq_{\max} B :\Leftrightarrow U(A) \geq U(B)$ clearly satisfies (PB) and every derivative form of this condition considered in Sect. 2. Also, $\succeq_{\max}$ satisfies both conditions, ($C_{\succeq}$) and ($C_{\succ}$). However, $\succeq_{\max}$ is obviously not concerned with freedom of choice, and consequently it violates (MF) if X is at least two-dimensional. Indeed, if $u(x) = u(y)$ and $x \neq y$, one has $\{x, y\} \sim_{\max} \{x\} \sim_{\max} \{y\}$.

Example 2. *The Lexicographic Maximax Extension.*
Let R and $u: X \to \mathbf{R}$ be as in the previous example. For any $A = \{x_1, \ldots, x_m\} \in \mathscr{F}(X)$ such that $u(x_1) \geq u(x_2) \geq \cdots \geq u(x_m)$, and for any $n > m$ we denote

$$v_n(A) := (u(x_1), \ldots, u(x_m), \overbrace{-\infty, \ldots, -\infty}^{n-m}) .$$

For $A, B \in \mathscr{F}(X)$ and $n > \max\{\#A, \#B\}$, define the lexicographic maximax extension by $A \succeq_{\text{lex}} B :\Leftrightarrow v_n(A) \succeq_L v_n(B)$, where $\succeq_L$ is the lexicographic ordering on $\mathbf{R}^n$. Note that due to the different motivation our definition of the lexicographic maximax extension differs from the definition of Pattanaik and Peleg [8]. Clearly, the definition does not depend on the choice of $n \in \mathbf{N}$. It is easily verified that $\succeq_{\text{lex}}$ is a complete preorder on $\mathscr{F}(X)$ satisfying conditions (PB), (F) and (M). In particular, it satisfies any derivative form of (PB), as well as (MF) and (MF′). However, $\succeq_{\text{lex}}$ neither satisfies ($C_{\succeq}$), nor ($C_{\succ}$). This fact is indeed part of the reason why we feel that $\succeq_{\text{lex}}$ cannot plausibly be interpreted as a ranking of freedoms. Consider the following example. Let $x, y \in X$ be such that $x \neq y$, $u(x) = u(y)$, and let $(x_j)_{j \in \mathbf{N}}$ be a sequence converging to x with $u(x_j) = u(x) + 1/j$. Then, for every j, $\{x_j\} \succ \{x, y\}$ but in the limit $\{x, y\} \succ \{x\}$ in violation of both, ($C_{\succeq}$) and ($C_{\succ}$). We note that although lexicographically structured rankings of freedom seem to be problematic in our present context, they may have some appeal on *finite* domains (see e.g. [1, 4]). Indeed, conditions ($C_{\succeq}$) and ($C_{\succ}$) are always trivially satisfied if X is endowed with the discrete topology.

Example 3. *Non-Preference-Based Extensions.*
Let R and $u: X \to \mathbf{R}$ be as above. For simplicity we assume $X \subseteq \mathbf{R}^2$ in the following example. For $A \in \mathscr{F}(X)$ denote by $\Delta(A)$ the *diameter* of A, i.e.

$$\Delta(A) := \max_{x, y \in A} d(x, y),$$

where $d(\cdot, \cdot)$ denotes the euclidean distance in $\mathbf{R}^2$. Furthermore, let $\text{Conv}(A)$ denote the convex hull of A, and let $\Phi_{\text{Conv}}(A)$ denote the area of $\text{Conv}(A)$ in $\mathbf{R}^2$. For any $A \in \mathscr{F}(X)$, let

$$V(A) := \Delta(A) + \Phi_{\text{Conv}}(A). \tag{5}$$

Define a complete preorder $\succsim_{u,\mathrm{Conv}}$ on $\mathscr{F}(X)$ by

$$A \succsim_{u,\mathrm{Conv}} B :\Leftrightarrow U(A) + V(A) \geq U(B) + V(B),$$

where $U(A)$ is defined as in Example 1. It is easily verified that $\succsim_{u,\mathrm{Conv}}$ is an extension of R and satisfies conditions (M), (F), ($C_{\succsim}$), and ($C_{\succ}$). In a sense, the ordering $\succsim_{u,\mathrm{Conv}}$ may be considered to be a ranking of freedoms[4]. However, $\succsim_{u,\mathrm{Conv}}$ violates any of the conditions of preference-basedness. This can be seen as follows. Let $x, y, z \in X$ be such that $u(x) \geq u(y) \geq u(z)$ and $d(x,z) > d(x, y)$. In this case one obtains

$$u(x) + d(x, z) > u(x) + d(x, y),$$

which implies $(\{x, z\} \succ_{u,\mathrm{Conv}} \{x, y\})$ but $(\{y\} \succsim_{u,\mathrm{Conv}} \{z\})$, in violation of any version of condition (PB).

Example 4. *Extensions based on incomplete preferences.*

The examples considered so far demonstrate that (PB), (MF), and either of the continuity conditions form an independent set of conditions, in the sense that no subset of two of the conditions implies the third. We will now show that the conditions can be simultaneously satisfied if the underlying preference relation is incomplete. It has already been noted that this can be done in a rather trivial way. Indeed, let R_0 denote the trivial preorder on X declaring any two distinct elements of X as incomparable. Then, if V is defined as in (5), the ordering $A \succsim B :\Leftrightarrow V(A) \geq V(B)$ is a complete extension of R_0 which satisfies (F), both continuity conditions, and – in a trivial sense – condition (PB). Note, however, that this ordering satisfies none of the conditions (PB′), (PB″), or (4).

A different example is the very simple ranking given by set inclusion: $A \succsim_{\subseteq} B :\Leftrightarrow B \subseteq A$. It is easily verified that $\succsim_{\subseteq}$ is an extension of R_0 satisfying conditions (F) (and hence (MF)), ($C_{\succsim}$), and – again in a trivial way – (PB), as well as any derivative form of condition (PB) with respect to R_0. The fact that $\succsim_{\subseteq}$ violates condition ($C_{\succ}$) does not seem to be too disturbing. As we have argued, ($C_{\succ}$) might be too strong in the context of incomplete orderings.

Although the two examples demonstrate the consistency of our basic conditions in the case of incomplete preferences, they are rather unsatisfactory as candidates for rankings of freedoms. Indeed, both examples apply only to the case where the underlying preference relation is *maximally* incomplete. We already know that under continuity and conditions (PB) and (MF) one cannot have completeness. But the question arises whether there exist preorderings on $\mathscr{F}(X)$ which would satisfy our conditions and which at the same time would extend a preference relation on X displaying fewer instances of noncomparability. Fortunately, the answer is affirmative.

Example 5. *Extensions based on "almost" complete preferences.*

Let $u: X \to \mathbf{R}$ be a continuous function. For any $\varepsilon > 0$ define a preorder R_ε on X by

$$xR_\varepsilon y \;:\Leftrightarrow\; x = y \text{ or } u(y) \leq u(x) - \varepsilon.$$

It will be convenient to think of $\varepsilon > 0$ as a *just noticeable difference*. Thus, x is weakly preferred to y if either $x = y$, or if $u(x)$ is greater than $u(y)$ by at least the just

[4] For a (distantly) related ranking and an interpretation of the convex hull in the context of freedom of choice, see [4, pp. 196–198].

noticeable difference. Note that, according to our definition, two different alternatives, x and y, such that $u(x)$ and $u(y)$ differ from each other by less than ε are treated as *incomparable* rather than *indifferent*. Also note that since ε may be arbitrarily small (but positive), the relation R_ε may be "almost" complete in the following sense. Let K be any compact subset of X, and assume that u satisfies some regularity conditions (e.g. strict monotonicity). Then for every fixed $x \in X$, the set of all $y \in K$ such that x and y are incomparable with respect to R_ε has arbitrarily small measure as ε tends to 0.

For the asymmetric part of this relation one obtains

$$xP_\varepsilon y \Leftrightarrow u(x) \geq u(y) + \varepsilon. \tag{6}$$

It is easily shown that P_ε satisfies the following two conditions which characterize a *semiorder* (see [6, 13]). The first condition is called *semitransitivity*. The second is the condition for an *interval order*. For all $x, y, z, a, b \in X$,

$$xPy, yPz \;\Rightarrow\; xPa \text{ or } aPz$$

and

$$xPy, aPb \;\Rightarrow\; xPb \text{ or } aPy.$$

Next, let for every $A \in \mathscr{F}(X)$, $\max_{R_\varepsilon}(A)$ denote the set of maximal elements in A with respect to R_ε, i.e.

$$\max_{R_\varepsilon}(A) := \{x \in A\text{: for no } y \in A, yP_\varepsilon x\}.$$

Thus, $\max_{R_\varepsilon}(A)$ is the set of alternatives $x \in A$ such that $u(x)$ differs from $U(A)$ by less than the just noticeable difference, where U is derived from u as in Example 1. Now define a preorder $\succeq_\varepsilon$ on $\mathscr{F}(X)$ by

$$A \succeq_\varepsilon B \;:\Leftrightarrow\; \max_{R_\varepsilon}(A \cup B) \subseteq A.$$

It is easily verified that for every $\varepsilon > 0$, $\succeq_\varepsilon$ is an extension of R_ε satisfying conditions (PB), (F) (and hence (MF)), and ($C_\succeq$). Furthermore, $\succeq_\varepsilon$ satisfies conditions (4) and (PB″). On the other hand, $\succeq_\varepsilon$ does not satisfy (PB′). For example, let $a, x, y \in X$ be such that $u(a) = u(y) + \varepsilon/2$ and $u(y) = u(x) + 2\varepsilon/3$. Then, one has $\{a, y\} \succ_\varepsilon \{a, x\}$ but not $yP_\varepsilon x$. As the ranking $\succeq_\subseteq$ considered in the previous example, the ranking $\succeq_\varepsilon$ does not satisfy ($C_\succ$). Indeed, in the next section it is shown that an extension of a (possibly incomplete) preorder satisfying conditions (PB″) and (MF′) must *necessarily* violate at least one of the two continuity conditions.

Finally, we note that the weak inequality on the right hand side of (6) is not standard. Usually, a semiorder is represented by the right hand side of (6) with strict inequality. However, this deviation is not essential for the interpretation of a semiorder. It is necessary here to insure continuity of the preorder $\succeq_\varepsilon$ in the sense of condition ($C_\succeq$).

5. A Further impossibility result

In this section it is shown that the conclusion of Theorem 1 can be strengthened if the condition of preference-basedness is slightly modified and if both continuity conditions, $(C_{\succeq})$ and $(C_{\succ})$, are simultaneously imposed. Specifically, we will prove the following theorem.

Theorem 2. *Let $X \subseteq \mathbf{R}^l$ with $l \geq 2$, and let R be a (not necessarily complete) preorder on X. There does not exist an extension $\succeq$ of R to $\mathscr{F}(X)$ such that $\succeq$ satisfies conditions (PB″), (MF′), $(C_{\succeq})$, and $(C_{\succ})$.*

Note that the conclusion of Theorem 2 is indeed much stronger than the conclusion of Theorem 1 since R is not assumed to be complete.

Proof. The proof consists in deriving a contradiction from the assumption that there exists a preorder $\succeq$ satisfying the stated conditions. Let y_0 be a point of the interior of X, and let $Y \subseteq X$ be a connected neighbourhood of y_0. Let R' be defined as in (3), i.e. $xR'y \Leftrightarrow \{x\} \succeq \{y\}$. Since $\succeq$ satisfies $(C_{\succeq})$ and $(C_{\succ})$, the preorder R' satisfies (C_R) and (C_P). We distinguish two cases.

Case 1. There exist $x, y \in Y$ such that $xP'y$. By a theorem of Schmeidler [12], the relation R' must be complete on Y, since Y is connected and R' satisfies (C_R) and (C_P). Apply Lemma 1 to R' in order to obtain a point $x_0 \in Y$ and a sequence $(x_j)_{j \in \mathbf{N}}$ converging to x_0 such that for all $j \in \mathbf{N}$, $x_j I' x_0$ and $x_j \neq x_0$. By condition (MF′) one has

$$\{x_0, x_1\} \succ \{x_0\}. \tag{7}$$

Indeed, since (MF′) subsumes condition (M), one has $\{x_0, x_1\} \succeq \{x_0\}$. If $\{x_0, x_1\} \sim \{x_0\}$ would hold, one would obtain $\{x_0, x_1\} \succ \{x_1\}$ by (MF), and hence $x_0 P' x_1$ which is false by assumption. By condition $(C_{\succ})$, (7) implies $\{x_0, x_1\} \succ \{x_0, x_j\}$ for sufficiently large $j \in \mathbf{N}$. This implies by condition (PB″), either $x_1 P x_j$, or $x_0 P x_j$. Since $\succeq$ is an extension of R, one finally obtains either $\{x_1\} \succ \{x_j\}$, or $\{x_0\} \succ \{x_j\}$ by applying (2). Hence, either $x_1 P' x_j$, or $x_0 P' x_j$ which again contradicts the assumptions.

Case 2: For no pair $x, y \in Y$, $xP'y$. Consequently, in this case one must have that any two distinct elements of Y are either indifferent or incomparable with respect to R'. Choose a sequence $(y_j)_{j \in \mathbf{N}}$ in Y converging to y_0. As before, (MF′) implies $\{y_0, y_1\} \succ \{y_0\}$, which in turn implies by $(C_{\succ})$, $\{y_0, y_1\} \succ \{y_0, y_j\}$ for sufficiently large $j \in \mathbf{N}$. However, this contradicts condition (PB″) as in Case 1 since y_0, y_1 and y_j are pairwise either indifferent or incomparable with respect to R'. This completes the proof of Theorem 2.

The intended interpretation of Theorem 2 is not primarily as a proper impossibility result. Indeed, our intuition is that the conditions used in Theorem 2 are not as compelling as their weaker counterparts considered in Section 3. The replacement of (MF) by (MF′) does not seem to be problematic. After all, the only difference is that (MF′) entails condition (M). The replacement of (PB) by (PB″) is less straightforward. Intuitively, condition (PB″) more strongly emphasizes the role of preference in the assessment of freedom. However, the most problematic part of the conditions in Theorem 2 are the continuity conditions. It is not only that condition $(C_{\succ})$ is somewhat less plausible than $(C_{\succeq})$ because of its "local completeness" character. Presumably, the driving force behind Theorem 2 is the *simultaneous* imposition of the two continuity conditions. Indeed, as Schmeidler's theorem shows, the two continuity conditions *together* imply completeness under relatively mild additional assumptions.

Despite these qualifications, Theorem 2 has been included here for two reasons. First, it shows that even if the completeness assumption is dropped the possibilities of modelling notions of preference-based freedom in a continuous framework may be rather limited. Secondly, it demonstrates that in the case of incomplete rankings of sets the appropriate continuity conditions have to be chosen very carefully.

6. Conclusion

The central result of this paper, Theorem 1, is an impossibility result in the context of ranking opportunity sets in terms of freedom. Although Kannai and Peleg [3] also provide an impossibility result in the context of ranking sets, Theorem 1 is more closely related to Nehring and Puppe [7, Corollary 1], who characterize the maximax extension rule under different structural assumptions but using conditions similar to some of the conditions considered here. Theorem 1 states that the following conditions are together inconsistent: preference-basedness, "minimal preference for freedom", continuity, and completeness of the underlying preference relation among alternatives. As we have argued, the condition of preference-basedness, which is closely related to Sen's "Weak Preference Dominance", is very attractive in the absence of uncertainty. On the other hand, there might be the following objection to condition (MF)[5]. Suppose that in a certain opportunity set there are only terrible and dreadful alternatives, between all of which the decision maker is indifferent. It could be argued that in such an extreme situation there does not exist a set of alternatives such that their exclusion would be a disadvantage. Indeed, in this situation one might feel that the fewer sorts the better. Note that this objection does not only apply to condition (MF), but to any notion of monotonicity with respect to set inclusion as well. However, the example does not disturb the results of this paper. Indeed, suppose that there exists some alternative w in the interior of X such that w is at least as good as the "status quo". (In fact, it would probably suffice to assume that w is *not* a terrible or dreadful alternative.) Then the results of this paper would apply without change if condition (MF) is assumed to hold only in a neighbourhood of w.

If conditions (PB) and (MF) are not easily dismissed, one has to scarifice either both continuity conditions, or the assumption that preferences among alternatives are complete. The possibility of modelling freedom of choice based on incomplete preferences has been demonstrated in Example 5 of Sect. 4. Clearly, one might also consider sacrificing the continuity conditions. But as we have tried to argue, at least ($C_{\succeq}$) seems to be a very reasonable condition in the context of ranking opportunity sets on infinite domains. Moreover, sacrificing the continuity conditions would mark a quite radical departure from traditional ways of modelling choice behaviour.

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[5] I am grateful to an anonymous referee for the following example.

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