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Partially efficient voting by committees

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Abstract. On the separable preference domain, voting by committees is the only class of voting rules that satisfy strategy-proofness and unanimity, and dictatorial rules are the only ones that are strategy-proof and Pareto efficient. To fill the gap, we define a sequence of efficiency conditions. We prove that for strategy-proof rules on the separable preference domain, the various notions of efficiency reduce to three: unanimity, partial efficiency, and Pareto efficiency. We also show that on the domain, strategy-proofness and partial efficiency characterize the class of voting rules represented as *simple games* which are independent of objects, *proper* and *strong*. We call such rules *voting by stable committee*.

1. Introduction

The purpose of this paper is to study the implications in a simple public good model of "strategy-proofness" and a newly-defined sequence of efficiency conditions. We introduce a measure of similarity of agents' preferences to define notions of efficiency, corresponding to degrees of their similarity, with "unanimity" being the weakest, and "Pareto efficiency" the strongest. We study social choice rules satisfying strategy-proofness and these efficiency conditions. It is known that on a large enough domain of preference relations, all strategy-proof and unanimous rules are dictatorial. However, our domain restriction enables us to analyze a non-empty class of strategy-proof, unanimous, and nondictatorial rules.

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We consider the following problem: there is a finite set of *objects*, and each object can be chosen, or not chosen. The *feasible set* is the set of all combinations of objects (including the empty set). These objects may be interpreted as indivisible public goods: bills considered for adoption, members of a club, and so on. Further, there is no combination of "contradicting" objects. There is a finite set of *voters*. Each voter has a strict preference relation on the feasible set. A *coalition* is a nonempty subset of the whole set of voters.

A *voting rule* is a function which associates with each preference profile a set of objects. The empty set may be interpreted as the *status quo*. A voting rule is *strategy-proof* if there is no preference profile such that a voter can obtain an outcome which he prefers by misrepresenting his preference relation. A voting rule satisfies *voter-sovereignty* if no subset of objects is *a priori* barred from emerging as the outcome. The Gibbard–Satterthwaite theorem implies that if there are at least three alternatives, on the domain of strict preference relations only dictatorial rules satisfy strategy-proofness and voter-sovereignty ([4, 7]; see also [1, 8]). We investigate preference relations called *separable*, for which there is no substitutability or complementarity between goods. Barberà et al. [3] prove that on the domain of separable preferences, the class of voting rules that they call *voting by committees*, to which dictatorial, and many nondictatorial, rules belong, are the only rules that satisfy strategy-proofness and voter sovereignty. Further, they show that if a rule of voting by committee satisfies strategy-proofness and certain regularity conditions implying nondictatorship, then the domain is a subset of the separable preference domain. In voting by committees, each object is considered separately. For each object, there is a rule according to which the object is chosen or not, and the rule is represented as what is known in game theory as a *simple game*: if a coalition is decisive, so is any coalition including it (see [5, 6]).

They also examine the existence of strategy-proof rules satisfying *Pareto efficiency*, a condition which implies voter sovereignty. They demonstrate that on the separable preference domain, dictatorial rules are the only ones that are strategy-proof and Pareto efficient. That is, even if preferences are separable, there does not exist a voting rule satisfying strategy-proofness, nondictatorship and Pareto efficiency.

Even a voting rule satisfying strategy-proofness and nondictatorship has a nonempty subset of the domain on which it selects Pareto efficient outcomes. Consider voting by committees on the separable preference domain. On the set of all separable preference profiles such that the most preferred set is the same for all agents, the outcome is the top set. This property of selecting the top set when it is at the top of every voter is called *unanimity*. Such an outcome is of course Pareto efficient. On the whole domain of separable preferences, however, only dictatorial rules are Pareto efficient. Notice that the former set of profiles is an extremely restricted subset of the latter. Then it will be interesting to study on what kinds of subsets of the domain, which class of voting by committees give rise to Pareto efficient outcomes.

In this paper, we define a measure of similarity of voters' preference relations to formulate a sequence of efficiency conditions of increasing restrictiveness, unanimity being the weakest and Pareto efficiency the strongest. When attention is limited to the separable preference domain and to strategy-proof rules, the conditions reduce to three: unanimity, Pareto efficiency, and an intermediate efficiency condition. We call the condition *partial efficiency*: if a set is the most preferred or the second most preferred by every voter, then the chosen outcome should be Pareto efficient.

We also prove that, on the separable preference domain, strategy-proofness and partial efficiency characterize voting rules represented as simple games which are *proper* and *strong*: for any decomposition of the grand coalition into two disjoint coalitions, one is decisive and the other is not. We call such voting rules *voting by stable committee.* It is known that if there is a veto player in a proper and strong simple game, then he is the dictator (see [6, p. 35]). Then we conclude that on the separable preference domain, rules of voting by stable committee with no veto player are the only ones that satisfy strategy-proofness, partial efficiency, and nondictatorship. In addition, we have that on the same domain, majority rule is the unique rule that satisfies strategy-proofness, partial efficiency, nondictatorship, and *anonymity* if the number of voters is odd.

The results presented here suggest that on a restricted domain, there may exist an efficiency condition which is stronger than unanimity but does not imply dictatorship, and reinforce the appeal of majority rule.

2. The model and voting by committees

We introduce the model discussed in this paper, and state the basic conditions to characterize voting by committees.

The set of *objects* is the finite set K with $|K| \geq 2$, and the *feasible set* is 2^K, the power set of K. The set of *voters* is the finite set N with $|K| \geq 2$. Let $\mathscr{L}_0$ be the set of admissible preference relations. If is the set of all *linear orders* on 2^K: $R_0 \in \mathscr{L}_0$ if for all $A, B, C \in 2^K$, we have

$$A\,R_0\,B \text{ or } B\,R_0\,A \text{ (completeness),}$$

$$A\,R_0\,B\,\&\,B\,R_0\,A \;\Rightarrow\; A = B \text{ (antisymmetry),}$$

$$A\,R_0\,B\&B\,R_0\,C \;\Rightarrow\; A\,R_0\,C \text{ (transitivity).}$$

For every $i \in N$, let $\mathscr{R}_i \subset \mathscr{L}_0$ be the set of linear orders that voter i may have as his preference relations, and preference relations to announce. Then $\mathfrak{R} = \prod_{i \in N} \mathfrak{R}_i$ represents the set of possible preference profiles, and possible announcements for the voters.

For all $R_i \in \mathfrak{R}_i$ and $\mathscr{A} \subset 2^K$, let

$$B_1(R_i) = \text{argmax}(R_i, 2^K),$$

$$B_t(R_i) = \text{argmax}(R_i, 2^K \backslash \{B_1(R_i), \ldots, B_{t-1}(R_i)\})$$

$$\text{for each } t = 2, \ldots, 2^{|K|},$$

where $\text{argmax}(R_i, \mathscr{A}) = \{A \in \mathscr{A} \mid \forall B \in A, A\,R_i\,B\}$. Then $B_t(\cdot)$ gives the tth best subset of K for each preference relation. For notational simplicity, we sometimes denote by $B(\cdot)$ the correspondence $B_1(\cdot)$, which selects the most preferred set.

A *voting rule* is a function from $\mathfrak{R}$ to 2^K. A *coalition* is a nonempty subset of N. The following class of coalitions and voting rules will be central to the analysis to follow.

Definition. *A simple game* is a nonempty class $\mathscr{W}$ of coalitions such that

$$\forall M \in \mathscr{W}, \forall M' \subset N: \quad M \subset M' \;\Rightarrow\; M' \in \mathscr{W}.$$

Each element of $\mathscr{W}$ is a *winning coalition.*

Definition. A voting rule $f: \mathfrak{R} \to 2^K$ is *voting by committees* if each $x \in K$, there is a simple game $\mathscr{W}_x$ such that for all $R \in \mathfrak{R}$,

$$x \in f(R) \;\Leftrightarrow\; \{i \in N \mid x \in B(R_i)\} \in \mathscr{W}_x.$$

Given $x \in K$, we can construct $\mathscr{W}_x$ so that x is chosen by unanimity rule, simple majority rule, a weighted majority rule, and a dictatorial rule, respectively (we will give formal definitions in Sects. 3 and 4). That is, these rules are voting by committees. We see two neat features of voting by committees. One is "tops only": such a rule depends only on the subset of K ranked highest by each voter. The other is "object by object": a decision is made for each object separately.

We are interested in domains on which voting by committees has some desirable properties. A domain $\mathfrak{R}$ is *rich* if for all $A \in 2^{|K|}$ and all $i \in N$, there is $R_i \in \mathfrak{R}_i$ such that $A = B(R_i)$. Among rich domains, the following is simple and well-behaved.

Definition. A preference relation $R_0 \in \mathscr{L}_0$ is *separable* if

$$\forall x \in K,\ \forall A \in 2^{K \setminus \{x\}}: (A \cup \{x\})\, R_0\, A \;\Leftrightarrow\; \{x\}\, R_0\, \emptyset.$$

Let $\mathscr{S}_0$ be the class of all separable preference relations. The class $\mathscr{S} = \prod_{i \in N} \mathscr{S}_0$ is the *separable preference domain.*

For each $R_0 \in \mathscr{L}_0$, $x \in K$ is a "good" for R_0 if $\{x\}\, R_0\, \emptyset$. Otherwise, we call it a "bad" for R_0. Then $R_0 \in \mathscr{S}_0$ means that each object is a good or a bad for R_0, regardless of which objects are combined together with it. In particular, for a separable preference relation, the top set of a voter is simply the set of all its goods.

A voting rule is *strategy-proof* if there is no preference profile such that a voter can obtain an outcome which he prefers by misrepresenting his preference relation. A rule satisfies *unanimity* if for a preference profile where voters' top sets are identical, the set emerges as the outcome. (Then, unanimity implies the condition of *voter-sovereignty.*) These properties are formally defined as follows.

Strategy-proofness: $\forall R \in \mathfrak{R},\ \forall i \in N,\ \forall R_i' \in \mathfrak{R}_i, f(R)\, R_i\, f(R_i', R_{\cdot i})$.

Unanimity: $\forall R \in \mathfrak{R},\ \forall A \in 2^K,\ [\forall i \in N, B(R_i) = A] \;\Rightarrow\; f(R) = A$,

where $(R_i', R_{\cdot i})$ represents the profile $R^* \in \mathfrak{R}$ such that $R_i^* = R_i'$, and $R_j^* = R_j'$ for all $j \in N \setminus \{i\}$.

The Gibbard–Satterthwaite theorem says that if there are at least three social states, on the domain of linear orders $\mathscr{L}_0$ only dictatorial rules satisfy strategy-proofness and unanimity. Barberà et al. [3] show that on the separable preference domain the class of voting by committees, to which many dictatorial, and nondictatorial, rules belong, is the only class of rules satisfying strategy-proofness and unanimity.

Possibility Theorem [3, Theorem 1]: A *voting rule $f: \mathscr{S} \to 2^K$ satisfies strategy-proofness and unanimity if and only if it is voting by committees.*

3. Partial efficiency on a restricted domain

The reason why the Gibbard–Satterthwaite theorem does not apply in our model is that the separability assumption reduces the problem to $|K|$ social choices, each with two alternatives. Then it may not be very surprising that there exist nondictatorial voting rules satisfying strategy-proofness and unanimity.

Barberà et al. [3, Theorem 3] state that if $\mathfrak{R}$ is a rich domain on which there is a rule $f: \mathfrak{R} \to 2^K$ of voting by committees satisfying strategy-proofness and certain regularity conditions, then $\mathfrak{R}$ must be a subclass of $\mathscr{S}$. That is, strategy-proofness and fairly natural requirements of a voting rule imply that the domain consists of separable preferences. We thus mainly limit our attention to the domain of separable preferences, and to the class of voting by committees.

If we additionally impose an efficiency condition, however, it is not obvious if there exists a nondictatorial rule satisfying them all. We will formulate notions of efficiency and investigate their implications under the strategy-proofness and nondictatorship constraints.

For each $R_i \in \mathfrak{R}_i$ and $A \in 2^K$, the *strong upper-contour set of* R_i *at* A is $U^0(R_i, A) = \{B \in 2^K \setminus \{A\} \mid B\, R_i\, A\}$, which is the open segment above A for the linear order R_i. For each $R \in \mathfrak{R}$, the set of *Pareto efficient outcomes for* R *is* $\mathscr{P}(R) = \{A \in 2^K \mid \bigcap_{i \in N} U^0(R_i, A) = \emptyset\}$. Then, *Pareto efficiency*, and *dictatorship*, of a voting rule f are expressed as follows:

Pareto efficiency: $\forall R \in \mathfrak{R},\ \ f(R) \in \mathscr{P}(R)$.

Dictatorship: $\exists i \in N, \forall R \in \mathfrak{R}, B(R_i) = f(R)$.

Barberà et al. [3] also demonstrate that dictatorial rules are the only voting rules that are strategy-proof and Pareto efficient on $\mathscr{S}$.

Impossibility Theorem [3, Theorem 4]. *Suppose* $|K| \geq 3$. *Then a voting rule* $f: \mathscr{S} \to 2^K$ *satisfies strategy-proofness and Pareto efficiency if and only if it satisfies dictatorship.*

For the cases where $|N| = 2, 3$, and 4, the numbers of possible dictatorial rules are respectively 2, 3, and 4. On the other hand, the corresponding numbers of voting by committees for one object are 4, 18, and 166. Then the total number of possible voting by committees are respectively $4^{|K|}$, $18^{|K|}$, and $166^{|K|}$ since the number of objects is $|K|$. Notice that there is a considerable gap between the two classes of rules. The Possibility, and Impossibility, Theorems establish characterizations of a very big class, and a very small class, of rules, respectively. This gap is caused by the difference between unanimity and Pareto efficiency. These theorems tell us that many nondictatorial rules satisfy strategy-proofness and unanimity but are not Pareto efficient.

On the set of preference profiles such that all voter's top sets are identical, however, unanimity requires that the chosen outcome should be the top set, which is the unique Pareto efficient outcome. Further, we may make the conjecture: "The more similar voters' top sets are, the greater the number of the rules of voting by committees that produce Pareto efficient outcomes is". Hence, every strategy-proof and unanimous rule has a subclass of the domain on which it selects Pareto optimal outcomes, so that we may describe notions of efficiency of voting rules in terms of similarity of preference relations in such subclass. It will be meaningful to find nondictatorial strategy-proof rules which associate Pareto efficient outcomes with as many types of preference profiles (including identical ones) as possible.

Now, we consider the following problem: for what types of preference profiles, which class of strategy-proof rules give rise to Pareto efficient outcomes? For this purpose, we introduce several concepts of efficiency of voting rules on which our main results are based. These concepts are defined with respect to "similarity" of voters' top sets. We measure similarity of the top sets by separating all possible configurations of preference profiles into the following cases:

Case 1: There is a (unique) set ranked first by every voter.

Case 2: There is a set ranked first or second by every voter.

....

Case T: There is a set ranked first, second, ... , or Tth by every voter.

Table 1 exposes the above configurations. These cases describe decreasing similarity of voters' top sets. For all $R \in \mathfrak{R}$, let

$$v(R) = \min_{A \in 2^K} \max_{i \in N} \{t \mid B_t(R_i) = A\}.$$

This is the measure of similarity of the top sets we employ. We see that $v(R) \leq T$ if and only if Case T holds for R. Then $v(R)$ may be interpreted as a sort of "variance" of the top sets of R. The following condition requires Pareto efficiency of outcomes when diversity of voters' top sets is of order T (i.e. when $v(R) \leq T$):

T-Partial efficiency: $\forall R \in \mathscr{R}$, $[\bigcap_{i \in N}\{B_1(R_i), \ldots, B_T(R_i)\} \neq \emptyset \Rightarrow f(R) \in \mathscr{P}(R)]$.

Note that this condition implies unanimity. Similarly, we have the following alternative expression of unanimity:

1-partial efficiency: $\forall R \in \mathfrak{R}$, $[\bigcap_{i \in N}\{B_1(R_i)\} \neq \emptyset \Rightarrow f(R) \in \mathscr{P}(R)]$.

Pareto efficiency is equivalent to $2^{|K|}$-partial efficiency. Note that $\{B_t(R_i) \mid t = 1, \ldots, 2^{|K|}\} = 2^K$. Suppose that we check the top half, $2^{|K|-1}$, of $B_t(R_i)$'s for all $i \in N$ to find $\bigcap_{i \in N}\{B_1(R_i), \ldots, B_{2^{|K|-1}}(R_i)\} = \emptyset$. Then for all $T > 2^{|K|-1}$, and for all $j \in N$, $B_T(R_i) \in \bigcap_{i \in N \setminus \{j\}}\{B_1(R_i), \ldots, B_{2^{|K|-1}}(R_i)\}$, so that $\bigcap_{i \in N}\{B_1(R_i), \ldots, B_T(R_i)\} \neq \emptyset$. It thus follows that $\bigcap_{i \in N}\{B_1(R_i), \ldots, B_T(R_i)\} \neq \emptyset$ for all $T > 2^{|K|-1}$ no matter how different preferences are. Then the conditions of $(2^{|K|-1} + 1)$-, ... , $2^{|K|}$-partial efficiency are all equivalent to Pareto efficiency. When $|K| = 2$, 3-partial efficiency is Pareto efficiency. We also have that, for all T, T' with $T > T'$, T-partial efficiency implies T'-partial efficiency. The following theorem tells us that for strategy-proof rules on the domain of separable preferences, 2-partial efficiency is equivalent to 3-partial efficiency (a proof is in Sect. 5).

Theorem 1. *Let $f: \mathscr{S} \to 2^K$ satisfy strategy-proofness. Then f satisfies 2-partial efficiency if and only if it satisfies 3-partial efficiency.*

We present Remarks 1 and 2 to check the independence of assumptions in Theorem 1 (proofs are in Sect. 5). The assumptions are the following two: one is that the set of admissible preference relations is the whole class $\mathscr{S}_0$ of separable preference relations, which is so large that for every $R_0 \in \mathscr{S}_0$, there is R_0^* in $\mathscr{S}_0$ with

Table 1. Configurations of preference profiles

	Case 1	Case 2	...	Case T	...
1st best sets	$AAA \ldots A$	$ACB \ldots C$	...	$ACB \ldots C$	...
2nd best sets	...	$BAA \ldots A$	...	$BAC \ldots B$	...
...	...	...	...	...	...
Tth best sets	...	...	...	$CDA \ldots A$	...
...	...	...	...	...	...

$B_1(R_0^*) = B_1(R_0)$ and $B_2(R_0^*) = B_3(R_0)$, i.e. second best sets and third best sets are exchangeable in $\mathscr{S}$. The other is that f is strategy-proof.

We consider strategy-proof rules on proper subsets of $\mathscr{S}$ which violate the exchangeability of second best subsets and third best subsets. We show the existence of a rule on a domain which does not satisfy exchangeability but satisfies richness. Rules of voting by committees are strategy-proof on a subset of $\mathscr{S}$, so that it will be appealing to examine voting by committees.

Remark 1. (1) *Let* $|K| = 2$, $\mathfrak{R} \subset \mathscr{S}$ *and* $f: \mathfrak{R} \to 2^K$ *be voting by committees. Then* f *satisfies* 2-*partial efficiency if and only if it satisfies Pareto efficiency.* (2) *There is a proper subset* $\mathfrak{R}$ *of* $\mathscr{S}$ *on which at least one rule* $f: \mathfrak{R} \to 2^K$, *where* $|K| \geq 3$, *of voting by committees satisfies strategy-proofness and* 2-*partial efficiency but does not satisfy* 3-*partial efficiency.*

Next we consider voting rules on $\mathscr{S}$ which are not strategy-proof. The following shows that for such rules, 2-partial efficiency does not generally imply 3-partial efficiency.

Remark 2. *There is a voting rule* $f: \mathscr{S} \to 2^K$ *which satisfies* 2-*partial efficiency but does not satisfy strategy-proofness or* 3-*partial efficiency.*

We focus on 2-partial efficiency in the following argument. For convenience, we simply refer to 2-partial efficiency as *partial efficiency.*

Partial efficiency: $\forall R \in \mathfrak{R}, [\bigcap_{i \in N}\{B_1(R_i), B_2(R_i)\} \neq \emptyset \Rightarrow f(R) \in \mathscr{P}(R)]$.

Fact 1. *Pareto efficiency* implies *partial efficiency. Partial efficiency* implies *unanimity.*

We can prove that for strategy-proof rules on the domain of separable preferences, 4-partial efficiency is equivalent to dictatorship, and hence it is also equivalent to Pareto efficiency (proof is in Sect. 5).

Theorem 2. *Let* $f: \mathscr{S} \to 2^K$ *satisfy strategy-proofness. Then* f *satisfies* 4-*partial efficiency if and only if it satisfies Pareto efficiency. Further, if* $|K| \geq 3$, *then* f *satisfies* 4-*partial efficiency if and only if it satisfies dictatorship.*

The two assumptions in Theorem 2 are exactly the same as in Theorem 1. If $|K| = 2$, then $|2^K| = 4$, so that 4-partial efficiency is equivalent to Pareto efficiency. If $|K| \geq 3$, however, we can show that dropping any one of them leads to the failure of the conclusion (proofs are in Sect. 5).

Remark 3. *There is a proper subset* $\mathfrak{R}$ *of* $\mathscr{S}$ *on which at least one rule* $f: \mathfrak{R} \to 2^K$, *where* $|K| \geq 3$, *of voting by committees satisfies strategy-proofness and* 4-*partial efficiency but does not satisfy Pareto efficiency.*

Remark 4. *There is a voting rule* $f: \mathscr{S} \to 2^K$, *where* $|K| \geq 3$, *which satisfies* 4-*partial efficiency but does not satisfy strategy-proofness or Pareto efficiency.*

In Theorem 2, we have the equivalence of 4-partial efficiency, ... , and $2^{|K|}$ partial efficiency. Hence, all of the efficiency conditions of strategy-proof rules on $\mathscr{S}$ (i.e. voting by committees on $\mathscr{S}$) reduce to only three conditions:

unanimity (1-partial efficiency),

partial efficiency (2-partial efficiency, 3-partial efficiency), and

Pareto efficiency (4-partial efficiency , ... , $2^{|K|}$-partial efficiency).

By definition, partial efficiency is close to unanimity and rather far from Pareto efficiency. However, on the separable preference domain, partial efficiency is the only real intermediate efficiency condition for our criterion.

4. Voting by stable committee

We give the class of voting rules characterized by strategy-proofness and partial efficiency on the separable preference domain.

Consider the following class of rules of voting by committees: each of them has a unique simple game (that is independent of objects) to decide which subset of K is chosen. Then they satisfy the condition of "object by object" in a special form.

Definition. A voting rule $f:\mathscr{S} \to 2^K$ is *voting by neutral committee* if there is a simple game $\mathscr{W}$ such that for all $x \in K$, and for all $R \in \mathfrak{R}$,

$$x \in f(R) \Leftrightarrow \{i \in N | x \in B(R_i)\} \in \mathscr{W}.$$

The following are the standard properties of a simple game (see e.g. [6, p 34]). We will study simple games satisfying them.

Properness: $\forall M \in \mathscr{W}, \quad N \backslash M \notin \mathscr{W},$

Strength: $\forall M \notin \mathscr{W}, \quad N \backslash M \in \mathscr{W}.$

If the number of voters is odd, *majority rule* (i.e. $\mathscr{W} = \{M \subset N \,|\, |M| > |N|/2\}$) is proper and strong. If the number of voters is even, it is proper but is not strong. *Unanimity rule* (i.e. $\mathscr{W} = \{N\}$) is also a typical example which is proper but is not strong. *Voting by quota* 1 (i.e. $\mathscr{W} = \{M \subset N \,|\, |M| \geq 1\}$, which is the rule of "Ask, and it shall be given you") is an extreme case which is strong but is not proper. If a rule is represented as a simple game satisfying properness and strength, then it always gives an answer as a decision by a winning coalition (strength), and the decision is never objected by any other winning coalition (properness). What kind of rules of voting by neutral committee satisfy such "stability"? We analyze this class of rules called *voting by stable committee.*

Definition. A voting rule $f:\mathfrak{R} \to 2^K$ is *voting by stable committee* if f is voting by neutral committee with a simple game which satisfies properness and strength.

Dictatorial rules are voting by stable committee (if $|N| = 2$, then rules of voting by stable committee are dictatorial). So is majority rule if the number of voters is odd. In general cases, dictatorship is described by a simple game satisfying the above two properties and the following.

No veto power: $\forall i \in N, \quad N \backslash \{i\} \in \mathscr{W}.$

Fact 2. *A voting rule $f:\mathfrak{R} \to 2^K$ satisfies dictatorship if and only if f is voting by stable committee with a simple game which does not satisfy no veto power.*

Fact 3. *Dictatorial rules $\subset$ voting by stable committee $\subset$ voting by neutral committee $\subset$ voting by committees.*

We now claim the main theorem in this paper: on the separable preference domain, voting by stable committee is the only class of voting rules that satisfy strategy-proofness and partial efficiency.

Theorem 3. *A voting rule $f: \mathscr{S} \to 2^K$ satisfies strategy-proofness and partial efficiency if and only if it is voting by stable committee.*

By combining this with Fact 2, we also have a characterization of nondictatorial voting by stable committee on $\mathscr{S}$.

Corollary. *A voting rule $f: \mathscr{S} \to 2^K$ satisfies strategy-proofness, partial efficiency, and does not satisfy dictatorship if and only if it is voting by stable committee with no veto power.*

We show Theorem 3 in two steps. First, we prove that the class of all voting rules on $\mathscr{S}$ satisfying strategy-proofness and partial efficiency is a subclass of voting by stable committee on the same domain. We apply techniques employed in the main part of the proof of Theorem 4 in [3].

Lemma 1. *If a voting rule $f: \mathscr{S} \to 2^K$ satisfies strategy-proofness and partial efficiency, then it is voting by stable committee.*

Proof. Suppose that $f: \mathscr{S} \to 2^K$ satisfies strategy-proofness sand partial efficiency. By Fact 1, partial efficiency implies unanimity. Then f satisfies strategy-proofness and unanimity. By the Possibility Theorem, it must be voting by committees. Let $x \in K$, and $\{N_{\mathrm{I}}, N_{\mathrm{II}}\}$ be a partition of N such that $R_i = R_{\mathrm{I}}$ for all $i \in N_{\mathrm{I}}$, and $R_i = R_{\mathrm{II}}$ for all $i \in N_{\mathrm{II}}$.

Claim 1. *$N_{\mathrm{I}} \in \mathscr{W}_x \Rightarrow N_{\mathrm{I}} \in \mathscr{W}_y$ for all $y \in K \backslash \{x\}$.*

Let $y \in K \backslash \{x\}$. Define $R_{\mathrm{I}}, R_{\mathrm{II}} \in \mathscr{S}_0$ as follows:

$$\{x, y\}\, R_{\mathrm{I}}\, \{y\}\, R_{\mathrm{I}}\, \{x\}\, R_{\mathrm{I}}\, \emptyset\, R_{\mathrm{I}} \ldots,$$

$$\emptyset\, R_{\mathrm{II}}\, \{y\}\, R_{\mathrm{II}}\, \{x\}\, R_{\mathrm{II}}\, \{x, y\}\, R_{\mathrm{II}} \ldots$$

Since f is voting by committees, we have $f(R) \subset \bigcup_{i \in N} B(R_i) = \{x, y\}$. Since $N_{\mathrm{I}} \in \mathscr{W}_x$, we have $x \in f(R)$. If $N_I \notin \mathscr{W}_y$, we have $y \notin f(R)$, so that $f(R) = \{x\}$. Since $\{y\} R_{\mathrm{I}} \{x\}$ and $\{y\} R_{\mathrm{II}} \{x\}$, this contradicts partial efficiency. Hence $N_{\mathrm{I}} \in \mathscr{W}_y$.

Note that Claim 1 is equivalent to the following:

$$N_{\mathrm{I}} \notin \mathscr{W}_x \Rightarrow N_{\mathrm{I}} \notin \mathscr{W}_y \quad \text{for all } y \in K \backslash \{x\}.$$

Claim 2. *$N_{\mathrm{I}} \in \mathscr{W}_x \Rightarrow N_{\mathrm{II}} \notin \mathscr{W}_y$ for all $y \in K \backslash \{x\}$.*
Let $y \in K \backslash \{x\}$. Define $R_{\mathrm{I}}, R_{\mathrm{II}} \in \mathscr{S}_0$ as follows:

$$\{x\}\, R_{\mathrm{I}}\, \emptyset\, R_{\mathrm{I}}\, \{x, y\}\, R_{\mathrm{I}}\, \{y\}\, R_{\mathrm{I}} \ldots,$$

$$\{y\}\, R_{\mathrm{II}}\, \emptyset\, R_{\mathrm{II}}\, \{x, y\}\, R_{\mathrm{II}}\, \{x\}\, R_{\mathrm{II}} \ldots$$

Since f is voting by committees, we have $f(R) \subset \bigcup_{i \in N} B(R_i) = \{x, y\}$. Since $N_{\mathrm{I}} \in \mathscr{W}_x$, we have $x \in f(R)$. If $N_{\mathrm{II}} \in \mathscr{W}_y$, we have $y \in f(R)$, so that $f(R) = \{x, y\}$. Since $\emptyset R_{\mathrm{I}} \{x, y\}$ and $\emptyset R_{\mathrm{II}} \{x, y\}$, this contradicts partial efficiency. Hence $N_{\mathrm{II}} \notin \mathscr{W}_y$.

Claim 3. *$N_{\mathrm{I}} \notin \mathscr{W}_x \Rightarrow N_{\mathrm{II}} \in \mathscr{W}_y$ for all $y \in K \backslash \{x\}$.*

Let $y \in K \backslash \{x\}$. Define $R_{\mathrm{I}}, R_{\mathrm{II}} \in \mathscr{S}_0$ as follows:

$$\{x\}\, R_{\mathrm{I}}\, \{x, y\}\, R_{\mathrm{I}}\, \emptyset\, R_{\mathrm{I}}\, \{y\}\, R_{\mathrm{I}} \ldots,$$

$$\{y\}\, R_{\mathrm{II}}\, \{x, y\}\, R_{\mathrm{II}}\, \emptyset\, R_{\mathrm{II}}\, \{x\}\, R_{\mathrm{II}} \ldots$$

Since f is voting by committees, we have $f(R) \subset \bigcup_{i \in N} B(R_i) = \{x, y\}$. Since $N_{\mathrm{I}} \notin \mathscr{W}_x$, we have $x \notin f(R)$. If $N_{\mathrm{II}} \notin \mathscr{W}_y$, we have $y \notin f(R)$, so that $f(R) = \emptyset$. Since $\{x, y\}\, R_{\mathrm{I}}\, \emptyset$ and $\{x, y\}\, R_{\mathrm{II}}\, \emptyset$, this contradicts partial efficiency. Hence $N_{\mathrm{II}} \in \mathscr{W}_y$.

Let $M \in \mathscr{W}_x$. By Claim 1, we have $M \in \mathscr{W}_a$ for all $a \in K$. Then there is a simple game $\mathscr{W}$ such that $\mathscr{W} = \mathscr{W}_a$ for all $a \in K$. Thus f is represented as the simple game $\mathscr{W}$. By Claim 2, $\mathscr{W}$ satisfies properness. By Claim 3, $\mathscr{W}$ satisfies strength. Hence f is voting by stable committee. □

We can check that the assumptions in Lemma 1 are independent. Remark 1 tells us that there is a voting rule on a proper subset of $\mathscr{S}$ which is strategy-proof and partially efficient but is not voting by stable committee. Remark 2 also tells us that there is a voting rule on $\mathscr{S}$ which is partially efficient but is not strategy-proof or voting by stable committee. In addition, we can see that the unanimity rule on $\mathscr{S}$ is a voting rule which is strategy-proof but is not partially efficient or voting by stable committee.

Next, we prove that the class of voting by stable committee on $\mathscr{S}$ is included in the class of voting rules on the same domain satisfying strategy-proofness and partial efficiency. Since voting by stable committee is voting by committees, the Possibility Theorem tells us that it is strategy-proof. To check partial efficiency, it is sufficient to prove the following:

Lemma 2. *If $\mathfrak{R} \subset \mathscr{S}$, and a voting rule $f: \mathfrak{R} \to 2^K$ is voting by stable committee, then it satisfies partial efficiency.*

Proof. Suppose that f satisfies dictatorship. Then it satisfies Pareto efficiency. By Fact 1, it satisfies partial efficiency.

Suppose that f does not satisfy dictatorship. Then $\mathscr{W}$ satisfies no veto power, so that $\min\{|M| \mid M \in \mathscr{W}\} \geq 2$. Choose $R \in \mathscr{S}_0$, and $A \in 2^K$ so that for all $i \in N, A = B(R_i)$ or $A = B_2(R_i)$. For every $i \in N$, R_i is separable, then for each $x \in K$.

(1) $[x \notin A \,\&\, x \in B(R_i)] \Rightarrow B(R_i) = A \cup \{x\}$,

(2) $[x \in A \,\&\, x \notin B(R_i)] \Rightarrow B(R_i) = A \setminus \{x\}$.

We show that there is $i \in N$ such that $f(R)\, R_i\, A$.

Case 1. There is $i \in N$ with $f(R) = B(R_i)$: then $f(R)\, R_i\, A$.

Case 2. There is no $i \in N$ with $f(R) = B(R_i)$: we show $f(R)$: we show $f(R) \subset A$. Let $x \in f(R)$, then

$$\exists M \in \mathscr{W}, \ \forall i \in M, \quad x \in B(R_i).$$

Suppose $x \notin A$. Let $M \in \mathscr{W}$ satisfy the above condition, the $B(R_i) \neq A$ for all $i \in M$. Since each R_i is separable and $x \in B(R_i)$ for all $i \in M$, it follows by (1) that for all $i \in M, B(R_i) = A \cup \{x\}$. Then $f(R) = A \cup \{x\} = B(R_i)$ for all $i \in M$. This contradicts that $f(R) \neq B(R_i)$ for all $i \in N$. Then $x \in A$. Hence $f(R) \subset A$.

Next, we show $A \subset f(R)$. Let $x \notin f(R)$, then we have

$$\forall M \in \mathscr{W}, \exists i \in M, x \notin B(R_i).$$

Suppose $x \in A$. Let $M^* = \{i \in N \mid x \notin B(R_i)\}$, then $N \setminus M^* = \{i \in N \mid x \in B(R_i)\}$. Since $x \notin f(R)$, we have $N \setminus M^* \notin \mathscr{W}$. Since $\mathscr{W}$ satisfies strength, we have $M^* \in \mathscr{W}$. Since each R_i is separable and $x \notin B(R_i)$ for all $i \in M^*$, it follows by (2) that for all

Table 2. Number of voting rules

Rules\\|N|	2	3	4
Voting by committees	$4^{\|K\|}$	$18^{\|K\|}$	$166^{\|K\|}$
Dictatorial rules	2	3	4
Voting by stable committee	2	4	8

$i \in M^*, B(R_i) = A \setminus \{x\}$. Then $f(R) = A \setminus \{x\} = B(R_i)$ for all $i \in M^*$. This contradicts that $f(R) \neq B(R_i)$ for all $i \in N$. Then $x \notin A$. Hence $A \subset f(R)$.

We thus have $A = f(R)$, so that $f(R) R_i A$ for all $i \in N$. □

If $|K| = 2$, then it can be shown that voting by stable committee satisfies Pareto efficiency regardless of the domain [2, Proposition 11]. If $|N| = 2$, then voting by stable committee satisfies dictatorship, which gives rise to Pareto efficient outcomes. However, we can show that voting by stable committee does not satisfy partial efficiency if $|K| \geq 3$, $|N| \geq 3$, and the domain of f contains nonseparable preferences (proof is in Section 5):

Remark 5. *There is a proper superset* $\mathfrak{R}$ *of* $\mathcal{S}$ *on which at least one rule* $f: \mathfrak{R} \to 2^K$, *where* $|K| \geq 3$ *and* $|N| \geq 3$, *of voting by stable committee does not satisfy partial efficiency.*

From Lemmas 1, 2, and the Possibility Theorem, we have a full characterization of voting by stable committee on $\mathcal{S}$. Table 2 summarizes the numbers of rules of voting by committees, dictatorial rules, and rules of voting by committee when $|N| = 2, 3, 4$. Since voting by stable committee is voting by neutral committee, partial efficiency makes the power $|K|$ vanish. The table shows how partial efficiency refines the class of voting by committees to characterize voting by stable committee.

Finally, we mention some results when we impose *anonymity*: for every permutation π of N and every $R \in \mathfrak{R}$ satisfying $(R_{\pi(i)})_{i \in N} \in \mathfrak{R}$, $f(R) = f(\{R_{\pi(i)}\}_{i \in N})$. In anonymous voting by committees, if a coalition has the same size as that of a winning coalition for an object, then it is also winning. It can be shown that there is no anonymous voting by committees on $\mathcal{S}$ when the number of voters is even. Also majority rule is the only anonymous rule of voting by stable committee on $\mathcal{S}$ when the number of voters is odd. Then, on the separable preference domain with an odd number of voters, majority rule is the unique voting rule that satisfies strategy-proofness, partial efficiency, and anonymity.

5. Proofs

Proof of Theorem 1. By definition, 3-partial efficiency implies 2-partial efficiency. Let $f: \mathcal{S} \to 2^K$ be a voting rule which satisfies strategy-proofness and 2-partial efficiency, then f satisfies unanimity. By the Possibility Theorem, f is voting by committees. Let $R \in \mathcal{S}$ be such that $A \in \bigcap_{i \in N} \{B(R_i), B_2(R_i), B_3(R_i)\}$. We show $f(R) \in \mathcal{P}(R)$.

Suppose $\bigcap_{i \in N} \{B(R_i), B_2(R_i)\} \neq \emptyset$. By 2-partial efficiency, $f(R) \in \mathcal{P}(R)$. Suppose $A \in \bigcap_{i \in N} \{B(R_i), B_2(R_i), B_3(R_i)\}$.Then $A \in \mathcal{P}(R)$, and $M = \{i \in N \mid A = B_3(R_i)\}$ is nonempty. Let $i \in M$, then $B(R_i) \neq A$. Hence $[A \setminus B(R_i)] \cup [B(R_i) \setminus A]$ is nonempty. That is, there is $x \in K$ such that $x \notin B(R_i) \cap A$. We then prove two claims corresponding to two exclusive cases:

Claim 1. $[x \notin A \,\&\, x \in B(R_i)] \Rightarrow B(R_i) = A \cup \{x\}$.

Since $x \notin A$ and $x \in B(R_i)$, we have $(A \cup \{x\}) R_i A$. We first show $B(R_i) \subset A \cup \{x\}$. Let $y \in B(R_i) \backslash \{x\}$, then $y \neq x$. If $y \notin A$, then $(A \cup \{y\}) R_i A$. Since $x, y \in B(R_i)$, we have $(A \cup \{x, y\}) R_i A$. Hence according to R_i, $A \cup \{x\}$, $A \cup \{y\}$, and $A \cup \{x, y\}$ are preferred to A. This contradicts $A = B_3(R_i)$, so that $y \in A$. Hence $B(R_i) \subset A \cup \{x\}$. We next show $A \cup \{x\} \subset B(R_i)$. Let $z \in A$, then $z \neq x$. If $z \notin B(R_i)$, then $(A \backslash \{z\}) R_i A$. Since $x \in B(R_i)$ we have $(A \cup \{x\} \backslash \{z\}) R_i A$. Hence according to R_i, the sets $A \cup \{x\}$, $A \backslash \{z\}$, and $A \cup \{x\} \backslash \{z\}$ are preferred to A. This contradicts $A = B_3(R_i)$, so that $z \in B(R_i)$. Hence $A \subset B(R_i)$. Recall $x \in B(R_i)$, then $A \cup \{x\} \subset B(R_i)$. We thus have $B(R_i) = A \cup \{x\}$.

We can show the following in the same way as above.

Claim 2. $[x \in A \,\&\, x \notin B(R_i)] \Rightarrow B(R_i) = A \backslash \{x\}$.

By Claims 1 and 2, it follows that either $B(R_i) = A \cup \{x\}$ or $B(R_i) = A \backslash \{x\}$. There exist separable preference relations R_0 such that $B(R_0) = B(R_i)$ and $B_2(R_0) = A$ (if $B(R_i) = A \cup \{x\}$ (resp., $B(R_i) = A \backslash \{x\}$), assume that x is a good (resp., bad) for R_0). Hence, we may choose a preference profile $R^* \in \mathscr{S}$ so that

$$\forall i \in M, \quad B(R_i^*) = B(R_i) \,\&\, B_2(R_i^*) = A,$$

$$\forall i \notin M, \quad R_i^* = R_i.$$

Then for all $i \in N$, $B(R_i^*) = B(R_i)$. Since f is voting by committees, it follows by "tops only" that $f(R^*) = f(R)$. By construction, $A \in \bigcap_{i \in N} \{B(R_i^*), B_2(R_i^*)\}$. By 2-partial efficiency, $f(R^*) \in \mathscr{P}(R^*)$. Hence $f(R) \in \mathscr{P}(R^*)$, so that there is $j \in N$ with $f(R) R_j^* A$. If $f(R) = A$, recall $A \in \mathscr{P}(R)$. Thus $f(R) \in \mathscr{P}(R)$. If $f(R) \neq A$, recall $A \in \{B(R_i^*) B_2(R_i^*)\}$. Thus $f(R) R_j^* A$ means that $f(R) = B(R_j^*)$, and $A = B_2(R_j^*)$. Since $f(R) = B(R_j^*) = B(R_j)$, we have $f(R) \in \mathscr{P}(R)$. □

Proof of Remark 1. (1) By definition, Pareto efficiency implies 2-partial efficiency. Let $f: \mathfrak{R} \to 2^K$ be voting by committees which satisfies 2-partial efficiency, and $R \in \mathscr{S}$. We show $f(R) \in \mathscr{P}(R)$.

Suppose that $f(R) = \{x\}$. If there is $i \in N$ with $B(R_i) = \{x\}$, then $f(R) = B(R_i) \in \mathscr{P}(R)$. If there is no $i \in N$ with $B(R_i) = \{x\}$, by the definition of voting by committees, there exist coalitions N_{I}, and N_{II} with $B(R_i) = \{x, y\}$ for all $i \in N_{\mathrm{I}}$, and $B(R_i) = \emptyset$ for all $i \in N_{\mathrm{II}}$. If $N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$ is nonempty, we must have $B(R_i) = \{y\}$ for all $i \in N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$. Since these preferences are separable, we have $\{x\} R_i \emptyset$ for all $i \in N_{\mathrm{I}}$, $\{x\} R_i \{x, y\}$ for all $i \in N_{\mathrm{II}}$. Hence $\{y\}$ is the only possible outcome that Pareto-dominates $\{x\}$, i.e. $\{y\} R_i \{x\}$ for all $i \in N$. If $\{y\}$ dominates $\{x\}$, it follows that

$$\{x, y\} R_i \{y\} R_i \{x\} \emptyset \quad \text{for all } i \in N_{\mathrm{I}},$$

$$\emptyset R_i \{y\} R_i \{x\} R_i \{x, y\} \quad \text{for al } i \in N_{\mathrm{II}}.$$

Whether or not there exists $N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$, we thus have $\{y\} \in \bigcap_{i \in N} \{B_1(R_i), B_2(R_i)\}$. By 2-partial efficiency, no outcome Pareto-dominates $f(R) = \{x\}$, a contradiction. Hence $\{x\} \in \mathscr{P}(R)$.

In the same way as above, we can prove that if $f(R) = \{y\}$, then $\{y\} \in \mathscr{P}(R)$.

Suppose that $f(R) = \{x, y\}$. If there is $i \in N$ with $B(R_i) = \{x, y\}$, then $f(R) = B(R_i) \in \mathscr{P}(R)$. If there is no $i \in N$ with $B(R_i) = \{x, y\}$, by the definition of voting by committees, there exist coalitions N_{I}, N_{II} with $B(R_i) = \{x\}$ for all $i \in N_{\mathrm{I}}$, $B(R_i) = \{y\}$ for all $i \in N_{\mathrm{II}}$. If $N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$ is nonempty, we must have $B(R_i) = \emptyset$ for all $i \in N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$. Since these preferences are separable, we have $\{x, y\} R_i \{y\}$ for all $i \in N_{\mathrm{I}}$, and $\{x, y\} R_i \{x\}$ for all $i \in N_{\mathrm{II}}$. Hence $\emptyset$ is the only possible outcome that Pareto-dominates $\{x, y\}$, i.e. $\emptyset R_i \{x, y\}$ for all $i \in N$. If $\emptyset$ dominates $\{x, y\}$, it follows that

$$\{x\} R_i \emptyset R_i \{x, y\} R_i \{y\} \quad \text{for all } i \in N_{\mathrm{I}},$$

$$\{y\} R_i \emptyset R_i \{x, y\} R_i (x\} \quad \text{for all } i \in N_{\mathrm{II}}.$$

Then, whether or not there exists $N \backslash (N_{\mathrm{I}} \cup N_{\mathrm{II}})$, we have $\emptyset \in \bigcap_{i \in N} \{B_1(R_i), B_2(R_i)\}$. By 2-partial efficiency, no outcome Pareto-dominates $f(R) = \{x, y\}$, a contradiction. Hence $\{x, y\} \in \mathscr{P}(R)$.

In the same way as above, we can prove that if $f(R) = \emptyset$, then $\emptyset \in \mathscr{P}(R)$.

(2) We show this by way of an example. Let $K = \{x, y, z\}$, $N = \{1, 2\}$, and $\mathscr{W}_a = \{\{1, 2\}\}$ for all $a \in K$. Define the proper subset $\mathfrak{R}_0$ of $\mathscr{S}_0$ by

$$\mathfrak{R}_0 = \{R_0 \in \mathscr{S}_0 \mid B_1(R_0) \neq \emptyset \Rightarrow |B_1(R_0)| > |B_2(R_0)|\}.$$

Let $\mathfrak{R} = (\mathfrak{R}_0)^2$ and $f: \mathfrak{R} \to 2^K$ be voting by committees with simple games $\mathscr{W}_x$, $\mathscr{W}_y$, $\mathscr{W}_z$(i.e. unanimity rule). It can be verified that f satisfies 2-partial efficiency. Let $R_1, R_2 \in \mathfrak{R}$ be such that

$$\{x, y\} R_1 \{y\} R_1 \{x, y, z\} R_1 \{y, z\} R_1 \{x\} R_1 \emptyset R_1 \{x, z\} R_1 \{z\},$$

$$\{x, z\} R_2 \{z\} R_2 \{x, y, z\} R_2 \{y, z\} R_2 \{x\} R_2 \emptyset R_2 \{x, y\} R_2 \{y\}.$$

Then $f(R_1, R_2) = \{x\}$. Since $\{x, y, z\} R_1 \{x\}$, $\{x, y, z\} R_2 \{(x\}$, and $\{x, y, z\} \in \bigcap_{i \in N} \{B_1(R_i), B_2(R_i), B_3(R_i)\}$, f does not satisfy 3-partial efficiency. □

Proof of Remark 2. We show this by way of an example. Let $K = \{x, y\}$, $N = \{1, 2\}$. Define $\mathfrak{R}_0 = \{R_{xy}, R_x, R_y, R_0\}$ by

$$\{x, y\} R_{xy} \{x\} R_{xy} \{y\} R_{xy} \emptyset, \quad \{x\} R_x \emptyset R_x \{x, y\} R_x \{y\},$$

$$\{y\} R_y \{x, y\} R_y \emptyset R_y \{x\}, \quad \emptyset R_0 \{x\} R_0 \{y\} R_0 \{x, y\}.$$

Then $\mathfrak{R}_0 \subset \mathscr{S}_0$, and $\mathfrak{R} = (R_0)^2 \subset \mathscr{S}$. Define $f: \mathscr{S} \to 2^K$ by

$$f(R) = \begin{cases} B_1(R_1) & \text{if } \bigcap_{i \in N} \{B_1(R_i), B_2(R_i)\} \neq \emptyset, \\ \{x\} & \text{otherwise.} \end{cases}$$

Note that f satisfies 2-partial efficiency. Let $R_1 = R_y$, and $R_2 = R_0$, then we have $f(R_y, R_0) = \{x\}$. Since $f(R_0, R_0) = \emptyset$, voter 1 can become better off by representing R_0 instead of R_y, so that f does not satisfy strategy-proofness on $\mathscr{S}$. Since $\emptyset R_1 \{x\}$, $\emptyset R_2 \{x\}$, and $\emptyset \in \bigcap_{i \in N} \{B_1(R_i), B_2(R_i), B_3(R_i)\}$, f does not satisfy 3-partial efficiency. □

Proof of Theorem 2. By definition, Pareto efficiency implies 4-partial efficiency.

Let $f: \mathscr{S} \to 2^K$ be a voting rule which satisfies strategy-proofness and 4-partial efficiency, the f satisfies unanimity. By the Possibility Theorem, f is voting by committees. We show that if $|K| = 2$, the 4-partial efficiency is equivalent to Pareto efficiency, and that if $|K| \geq 3$, then 4-partial efficiency implies dictatorship.

Case 1. $|K| = 2$. We have $2^{|K|} = 4$. Then

$$\forall R \in \mathscr{S}, \bigcap_{i \in N}\{B(R_i), B_2(R_i), B_3(R_i), B_4(R_i)\} = 2^K.$$

By 4-partial efficiency, $f(R) \in \mathscr{P}(R)$.

Case 2. $|K| \geq 3$. Suppose $|N| = 2$. Since 4-partial efficiency implies partial efficiency, it follows by Lemma 1 that f is voting by stable committee. Since $|N| = 2$, f must satisfy dictatorship. Hence f satisfies Pareto efficiency.

Suppose $|N| \geq 3$. Let $x, y \in K$, and $\{N_{\mathrm{I}}, N_{\mathrm{II}}, N_{\mathrm{III}}\}$ be a partition of N such that $R_i = R_{\mathrm{I}}$ for all $i \in N_{\mathrm{I}}$, $R_i = R_{\mathrm{II}}$ for all $i \in N_{\mathrm{II}}$, and $R_i = R_{\mathrm{III}}$ for all $i \in N_{\mathrm{III}}$.

Claim. *$N_{\mathrm{I}} \notin \mathscr{W}_x$ & $N_{\mathrm{II}} \notin \mathscr{W}_y \Rightarrow N_{\mathrm{III}} \in \mathscr{W}_z$ for all $z \in K\setminus\{x, y\}$.*

The proof below follows a part of the proof of Barberà et al. (1991), Theorem 4. There exists $z \in K\setminus\{x, y\}$. Define $R_{\mathrm{I}}, R_{\mathrm{II}}, R_{\mathrm{III}} \in \mathscr{S}_0$ as follows:

$$\{x\}\, R_{\mathrm{I}}\, \{x, y\}\, R_{\mathrm{I}}\{x, z\}\, R_{\mathrm{I}}\{x, y, z\}\, R_{\mathrm{I}}\, \emptyset\, R_{\mathrm{I}}\, \{y\}\, R_{\mathrm{I}}\{z\}\, R_{\mathrm{I}} \ldots,$$

$$\{y\}\, R_{\mathrm{II}}\, \{y, z\}\, R_{\mathrm{II}}\{x, y\}\, R_{\mathrm{II}}\{x, y, z\}\, R_{\mathrm{II}}\, \emptyset\, R_{\mathrm{II}}\, \{z\}\, R_{\mathrm{II}}\{x\}\, R_{\mathrm{II}} \ldots,$$

$$\{z\}\, R_{\mathrm{III}}\, \{y, z\}\, R_{\mathrm{III}}\{y, z\}\, R_{\mathrm{III}}\{x, y, z\}\, R_{\mathrm{III}}\, \emptyset\, R_{\mathrm{III}}\, \{x\}\, R_{\mathrm{III}}\{y\}\, R_{\mathrm{III}} \ldots$$

Since f is voting by committees, we have $f(R) \subset \bigcup_{i \in N} B(R_i) = \{x, y, z\}$. Since $N_{\mathrm{I}} \notin \mathscr{W}_x$ and $N_{\mathrm{II}} \notin \mathscr{W}_y$, we have $x \notin f(R)$ and $y \notin f(R)$. If $N_{\mathrm{III}} \notin \mathscr{W}_z$, we have $z \notin f(R)$, so that $f(R) = \emptyset$. Since $\{x, y, z\}\, R_{\mathrm{I}}\, \emptyset$, $\{x, y, z\}\, R_{\mathrm{II}}\, \emptyset$, and $\{x, y, z\}\, R_{\mathrm{III}}\, \emptyset$, this contradicts 4-partial efficiency. Here $N_{\mathrm{III}} \in \mathscr{W}_z$.

Since voting by committees satisfies strategy-proofness, and 4-partial efficiency implies partial efficiency, it follows by Lemma 1 that f is represented as a simple game $\mathscr{W}$ which is proper and strong. By the above claim, we also have

$$\text{for all partitions } \{N_{\mathrm{I}}, N_{\mathrm{II}}, N_{\mathrm{III}}\} \text{ of } N, \quad N_{\mathrm{I}} \notin \mathscr{W}\ \&\ N_{\mathrm{II}} \notin \mathscr{W} \Rightarrow N_{\mathrm{III}} \in \mathscr{W}. \qquad (*)$$

Since $\mathscr{W}$ is nonempty, there exists $r = \min\{|M| \mid M \in \mathscr{W}\}$.

Suppose $r = |N|$. Divide N into two disjoint nonempty coalitions N_{I} and N_{II}. By the minimality of N in $\mathscr{W}$, we have $N_{\mathrm{I}} \notin \mathscr{W}$ and $N_{\mathrm{I}} \notin \mathscr{W}$. This contradicts strength. Then $r \leq |N| - 1$.

Suppose $2 \leq r \leq |N| - 1$. Let $M \in \mathscr{W}$ be such that $|M| = r$. Divide M into two disjoint nonempty coalitions M_{I} and M_{II}. By the minimality of M in $\mathscr{W}$, we have $M_{\mathrm{I}} \notin \mathscr{W}$ and $M_{\mathrm{II}} \notin \mathscr{W}$. Then, by (*), $N\setminus M = N\setminus(M_{\mathrm{I}} \cup M) \in \mathscr{W}$. By properness, we have $M \notin \mathscr{W}$. This contradicts $M \in \mathscr{W}$. Then $r \leq 1$.

Since $\emptyset \notin \mathscr{W}$, we have $r \geq 1$. We therefore have $r = 1$, so that f satisfies dictatorship. Hence it satisfies Pareto efficiency. □

Proof of Remark 3. We show this by way of an example. Let $K = \{x, y, z\}$, $N = \{1, 2\}$, and $\mathscr{W}_x = \{\{1, 2\}\} = \mathscr{W}_y$, $\mathscr{W}_z = \{\{1\}, \{2\}\}$. Define the proper subset $\mathfrak{R}_0$ of $\mathscr{S}_0$ by

$$\begin{aligned}\mathfrak{R}_0 = \{R_0 \in \mathscr{S} \mid\ & B_1(R_0) = \{x, y, z\} \Rightarrow |B_t(R_0)| \leq |B_{t+1}(R_0)|,\\ & B_1(R_0) = \{x, z\} \text{ or } \{y, z\} \Rightarrow B_2(R_0) = \{z\},\\ & B_1(R_0) = \{x\} \text{ or } \{y\} \text{ or } \{z\} \Rightarrow B_2(R_0) = \emptyset,\\ & B_1(R_0) = \emptyset \Rightarrow |B_t(R_0)| \geq |B_{t+1}(R_0)|\}.\end{aligned}$$

Let $\mathfrak{R} = (\mathfrak{R}_0)^2$ and $f\colon \mathfrak{R} \to 2^K$ be voting by committees with simple games $\mathscr{W}_x$, $\mathscr{W}_y$, and $\mathscr{W}_z$. Then f satisfies strategy-proofness. It can be verified that f satisfies

4-partial efficiency. Let $R_*, R_0 \in \Re$ satisfy

$$\{x, y, z\}\, R_*\, \{x, y\}\, R_*\, \{x, z\}\, R_* \{y, z\}\, R_* \{x\}\, R_* \{y\}\, R_* \{z\}\, R_*\, \emptyset,$$

$$\emptyset\, R_0 \{x\}\, R_0 \{y\}\, R_0 \{z\}\, R_0 \{x, y\}\, R_0 \{x, z\}\, R_0 \{x, y, z\}.$$

Let $R_1 = R_0$, and $R_2 = R_*$, then $f(R_1, R_2) = \{z\}$. Since $\{x\}\, R_1 \{z\}$, and $\{x\}\, R_2 \{z\}$, f does not satisfy Pareto efficiency. □

Proof of Remark 4. We show this by way of an example. Let $K = \{x, y, z\}$, $N = \{1, 2\}$. Define $f: \mathscr{S} \to 2^K$ by

$$f(R) = \begin{cases} B_1(R_1) & \text{if } \bigcap_{i \in N} \{B_1(R_i), \ldots, B_4(R_i)\} \neq \emptyset, \\ \{z\} & \text{otherwise.} \end{cases}$$

Note that f satisfies 4-partial efficiency. Define $R_*, R_0 \in \mathscr{S}_0$ as in the proof of Remark 3. Let $R_1 = R_*$, and $R_2 = R_0$, then $f(R_*, R_0) = \{x\}$. Let $R_{00} \in \mathscr{S}_0$ be such that

$$\{x, y, z\}\, R_{00} \{x, y\}\, R_{00} \{x, z\}\, R_{00} \{x\}\, R_{00} \{x, z\}\, R_{00} \{y\}\, R_{00} \{z\}\, R_{00}\, \emptyset.$$

Since $\{x\} \in \bigcap_{i \in N} \{B_1(R_i), B_2(R_i), B_3(R_i)\}$, we have $f(R_{00}, R_0) = \{x, y, z\}$. Hence voter 1 can become better off by representing R_{00} instead of R_*, so that f does not satisfy strategy-proofness on $\mathscr{S}$. As we have checked in the proof of Remark 3, $\{x\}$ is not a Pareto efficient outcome, so that f does not satisfy Pareto efficiency. □

Proof of Remark 5. We can show this by way of an example. Let $K = \{x, y, z\}$, $N = \{1, 2, 3\}$, and $\mathscr{W} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. Define R_1, R_2, and R_3 by

$$\{x, y\}\, R_1 (y)\, R_1 \{x\}\, \emptyset\, R_1 \{x, y, z\}\, R_1 \{y, z\}\, R_1 \{x, z\}\, R_1 \{z\},$$

$$\{y, z\}\, R_2 \{y\}\, R_2 \{z\}\, R_2\, \emptyset\, R_2 \{x, y, z\}\, R_2 \{x, y\}\, R_2 \{x, z\}\, R_2 \{x\},$$

$$\{x, z\}\, R_3 \{y\}\, R_3 \{x\}\, R_3\, \emptyset\, R_3 \{x, y, z\}\, R_3 \{y, z\}\, R_3 \{x, y\} R_3 \{z\}.$$

Then $R_1, R_2 \in \mathscr{S}_0$. However, $R_3 \notin \mathscr{S}_0$ since $B(R_i) = \{x, z\}$ and $\emptyset\, R_3 \{z\}$. Suppose that $f: \Re \to 2^K$ is voting by stable committee with the simple game $\mathscr{W}$, $\Re_1 = \mathscr{S}_0$, $\Re_2 = \mathscr{S}_0$, and $\Re_3 = \mathscr{S}_0 \cup \{R_3\}$. Then $f(R_1, R_2, R_3) = \{x, y, z\}$. Since $\{y\}$ is ranked second and preferred to $\{x, y, z\}$ by every voter, f does not satisfy partial efficiency. □

6. Conclusion

We have considered the problem of assigning a combination of objects to a society. We have proposed several notions of efficiency and studied their implications for the existence of strategy-proof voting rules on the domain of separable preferences.

We have shown that on this domain, our efficiency notions can be categorized as unanimity, partial efficiency, and Pareto efficiency, in order of weakness. In addition, we have proven that strategy-proofness and each one of the efficiency conditions can be used to characterize a class of voting rules: voting by committees, voting by stable committee, and dictatorial rules, respectively. These results tell us how the efficiency conditions reflect the structure of decisiveness in voting rules, and that on the domain of separable preferences, partial efficiency is as close to efficiency one can get under strategy-proofness and nondictatorship.

References

1. Barberà S, Peleg B (1990) Strategy-proof voting schemes with continuous preferences. Soc Choice Welfare 7: 31–38
2. Barberà S, Sonnenschein H, Zhou L (1989) Voting by quota and committee. Southern European Economics Discussion Series 64
3. Barberà S, Sonnenschein H, Zhou L (1991) Voting by committees. Econometrica 59: 595–609
4. Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41: 587–602
5. Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge
6. Peleg B (1984) Game theoretic analysis of voting in committees. Cambridge University Press, Cambridge
7. Satterthwaite MA (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10: 187–217
8. Schmeidler D, Sonnenschein H (1978) Two proofs of the Gibbard–Satterthwaite theorem on the possibility of a strategy-proof social choice function. In: Gottinger HW, Leinfellner W (eds), Decision theory and social ethics. Redel, Dordrecht, pp 227–234