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Fuzzy preferences and Arrow-type problems in social choice*

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Abstract. There are alternative ways of decomposing a given fuzzy weak preference relation into its antisymmetric and symmetric components. In this paper I have provided support to one among these alternative specifications. It is shown that on this specification the fuzzy analogue of the General Possibility Theorem is valid even when the transitivity restrictions on the individual and the social preference relations are relatively weak. In the special case where the individual preference relations are exact but the social preference relation is permitted to be fuzzy it is possible to distinguish between different degrees of power of the dictator. This power increases with the strength of the transitivity requirement.

1. Introduction

The conventional literature on the problem of arriving at a social preference relation by aggregating individual preference relations assumes that both individual and social preference relations are *exact* [i.e. for any two alternatives x and y individuals as well as the society can *unambiguously* specify whether x is at least as good as y]. It is well-known that this type of aggregation faces Arrow-type problems.

In recent years economists have recognised that preference patterns (whether individual or social) can be *fuzzy* i.e. the rankings between alternatives can be ambiguous (see, for instance, Basu 1984). Social choice theorists have investigated whether the introduction of fuzzy preferences offers a solution to the Arrow type problem. Barrett et al. (1986) allows both individual and social preferences to be fuzzy and establishes the fuzzy analogues of the imposibility results of Arrow (1963) and Gibbard (1969). They, however, made the assumption that the domain of the aggregation rules consists of n-tuples of fuzzy strict preference relations.

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The problem of aggregating n-tuples of fuzzy *weak* preference relations into fuzzy *weak orderings* has been investigated by Dutta (1987). In the context of fuzzy binary relations the concept of transitivity can be defined in several alternative ways. Dutta shows that if we use a relatively strong version of transitivity, aggregation rules which satisfy the fuzzy analogues of Arrow's Independence and Pareto Conditions and whose ranges consist of fuzzy binary relations would be oligarchic but not necessarily dictatorial. Moreover, if we take a weaker version of transitivity, even oligarchies are avoided. (The dictatorship result is, however, retrieved if a condition of Positive Responsiveness is imposed)[1] . These results imply that the problem of aggregating individual preference relations into a social preference relation is significantly easier when the relations are permitted to be fuzzy.

One difficulty with taking a (fuzzy) weak preference relation ['at least as good as'] rather than a (fuzzy) strict preference relation ['better than'] as the primitive concept in a discussion of fuzzy binary relations is that unlike in the case of exact preferences there is no uniquely defined way in which a given weak preference relation (R) can be decomposed into its strict preference (P) and indifference (I) components. Dutta (1987) used a particular construction and justified it by deriving it from a set of axioms.

However, as Dutta notes, there are other ways of deriving P and I from a given R. In this paper I first show that there are grounds for not rejecting these other constructions. I then show that if we accept one of these other methods, the dictatorship result reappears even with relatively weak versions of the transitivity condition and without the use of the Positive Responsiveness Condition. In the special case where individual preference relations are exact but the social preference relation is permitted to be fuzzy I establish a hierarchy of results showing that while there is a dictator even under relatively weak transitivity restrictions, the power of the dictator increases with the strength of this restriction.

In Sect. 2 I discuss the particular way of deriving fuzzy strict preference relations from given fuzzy weak preference relations used in this paper. Section 3 contains the results relating to social choice. Section 4 presents some concluding observations.

2. Fuzzy weak and strong preference relations

Let X be a set of alternatives. X has at least 3 elements. A fuzzy binary[2] weak preference relation on X is a function $R: X^2 \to [0,1]$ while an exact binary weak preference relation is a fuzzy binary weak preference relation with the special property that $R(X^2) \subseteq \{0,1\}$.

If a given exact preference relation R is *reflexive* and *connected* there is an obvious and unique way of deriving its asymmetric and symmetric components P and I: For all $x, y \in X$, $P(x,y)=1$ if $R(x,y)=1$ and $R(y,x)=0$; $P(x,y)=0$ if $R(x,y)=0$ and $R(y,x)=1$. $I(x,y)=1$ if $R(xy)=R(y,x)=1$; $I(x,y)=0$ otherwise.

[1] Dutta (1987) also shows that if we insist that fuzzy social preference relations generate exact social choice, the dictatorship result will follow under certain conditions.

[2] In this paper we shall be concerned with binary relations only. Therefore, we shall often drop the qualification 'binary' without confusion.

In the case of a fuzzy weak preference relation R on X the following definitions are given in obvious analogy with the exact case:

Definition 1. R is *reflexive* if $R(x,x)=1$ for all $x \in X$. It is *connected* if for all $x, y \in X$, $R(x,y)+R(x,y) \geq 1$.

Definition 2. Relations P and I on X are called the *components* of R if $R=P \cup I$.

Definition 3. A relation P is called *antisymmetric* if $P(x,y)>0 \Rightarrow P(y,x)=0$ for all $x, y \in X$. A relation I is called *symmetric* if $I(x,y)=I(y,x)$ for all $x, y \in X$.

In the fuzzy case, the condition that $P \cap I = \phi$ is not acceptable since we should permit the existence of alternatives $x, y \in X$ such that both $P(x,y)$ and $I(x,y)$ are positive.

Unlike in the exact case, however, there is now no obvious and unique way of decomposing a given weak preference relation into its antisymmetric and symmetric components. Alternative ways have been suggested in the literature. One source of this non-uniqueness is the fact that the condition $R=P \cup I$, can be interpreted in different ways since in the case of fuzzy sets several alternative definitions of the union operator have been given. Dutta (1987) made use of the following definition:

Definition 4. If A and B are fuzzy sets, $A \cup B$ is defined by $[A \cup B](x)=\max[A(x), B(x)]$ for all $x \in X$. With the help of this definition, Dutta established the following proposition:

Proposition 2.1 (Dutta). *If R is a fuzzy binary relation such that (i) it is connected, (ii) $R=P \cup I$, (iii) P is antisymmetric, (iv) I is symmetric and (v) $R(x,y)=R(y,x) \Rightarrow P(x,y)=P(y,x)$, then for all $x, y \in X$*

$$P(x,y)=R(x,y) \quad \text{if} \quad R(x,y)>R(y,x)$$
$$=0 \quad \text{otherwise}$$

and $I(x,y)=\min[R(x,y), R(y,x)]$.

This proposition, then, provided a method of obtaining P and I from a given R. On this method $P(x,y)$ may be equal to $R(x,y)$ even when $I(x,y)>0$. Intuitively, however, $R(x,y)$ represents the extent to which x is weakly superior to y. If $P(x,y)$ and $I(x,y)$ are given the corresponding interpretations, it seems natural to expect that if $I(x,y)>0$, $R(x,y)>P(x,y)$. This is denied in Proposition 2.1. For instance, this proposition would rule out the existence of alternatives $x, y \in X$ such that $R(x,y)=1$, $R(y,x)=0.8$, $I(x,y)=I(y,x)=0.8$, $P(x,y)=0.2$ and $P(y,x)=0$ although apriori these numerical specifications do not seem to be logically wrong.

To get around this problem we look for an alternative way of defining the union operator. Yager has suggested the following generalised operator: For all $x \in X [A \cup B](x)=\min\{1, \{A(x)^p+B(x)^p\}^{1/p}\}$ where $p \geq 1$ (see Yager 1980, referred to in Zimmermann 1985, p 34). This operator converges to the max operator for $p \to \infty$. For $p=1$, it coincides with the 'bold union' and has been modelled by the bounded sum:

Definition 5. If A and B are fuzzy sets, for all $x \in X$

$$[A \cup B](x)=\min\{1, A(x)+B(x)\}.$$

There are still other definitions of the union operator in the literature. As emphasised by the fuzzy set theorists, the choice between the alternative definitions is to be decided with reference to the context (see, for instance, Zimmermann, pp 34–36). In the present context it seems to me that Definition 5 would be pertinent in that it fulfills the intuitive requirement stated above. It is this definition that I shall use in our propositions[3].

It turns out, however, that if we take this union operator, conditions *(i)–(v)* of Proposition 2.1 are unable to define P and I uniquely on the basis of a given R. To get a unique definition we add two further conditions.

Proposition 2.2. *Let R be a connected fuzzy weak preference relation satisfying the following conditions:*

(i) $R = P \cup I$ *where the union operator is defined as in Definition 5.*
(ii) $P(x, y) + I(x, y) \leq 1$ *for all* $x, y \in X$
(iii) I *is symmetric*
(iv) P *is antisymmetric*
(v) *For all* $x, y \in X$, $R(x, y) < 1$ *implies* $P(y, x) > 0$.

Then for all $x, y \in X$

$$P(x, y) = 1 - R(y, x) \tag{1}$$

and

$$I(x, y) = \min[R(x, y), R(y, x)] \ . \tag{2}$$

Proof. We first prove (2)
Conditions *(i)* and *(ii)* imply that for all $x, y \in X, R(x, y) = P(x, y) + I(x, y)$. Since $I(x, y) = I(y, x)$ for all $x, y \in X$ by *(iii)*, it follows that $R(x, y) \gtreqless R(y, x)$ according as $P(x, y) \gtreqless P(y, x)$.

Now, if $R(x, y) > R(y, x)$, $P(x, y) > P(y, x)$. Hence, $P(x, y) > 0$. By *(iv)* $P(y, x) = 0$. Hence, $R(y, x) = P(y, x) + I(y, x) = I(y, x) = I(x, y)$. Similarly, if $R(x, y) < R(y, x)$, $R(x, y) = I(x, y)$. If $R(x, y) = R(y, x)$, $P(x, y) = P(y, x)$. By *(iv)* $P(x, y) = P(y, x) = 0$. Hence $R(x, y) = I(x, y) = I(y, x) = R(y, x)$. This completes the proof of (2).

To prove (1) first note that if $R(y, x) = 1$, we have, by definition, $R(x, y) \leqq R(y, x)$. By an argument similar to that in the preceding paragraph, $P(x, y) = 0$. Hence, in this case $P(x, y) = 1 - R(y, x)$.

If $R(y, x) < 1$, by *(v)* we have $P(x, y) > 0$. By *(iv)* $P(y, x) = 0$. Again by *(v)*, $R(x, y) = 1$. Thus, $R(x, y) > R(y, x)$ implying that $I(x, y) = R(y, x)$. Hence, $1 = R(x, y) = P(x, y) + I(x, y) = P(x, y) + R(y, x)$, completing the proof of (1). Q.E.D.

In the rest of this paper we shall use Eq. (1) and (2).

Most of the theorems on social choice hinge crucially on the transitivity properties of the individual and the social preference relations. Applications of fuzzy set theory to choice theory have made use of various definitions of transitivity. Basu (1984) introduced the following definition:

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[3] Similarly, there are alternative definitions of an intersection of fuzzy sets in the literature. However, for my purposes in this paper I do not need to choose an intersection operator.

Definition 6.1. A fuzzy weak preference relation R on X^2 where X is a set of alternatives is transitive if for all $x, y, z \in X$, $R(x,z) \geq \frac{1}{2} R(x,y) + \frac{1}{2} R(y,z)$ for all $z \in X - \{x, y\}$ such that $R(x,y) \neq 0$ and $R(y,z) \neq 0$.

To distinguish this notion of transitivity from the others mentioned below we shall refer to it as T-transitivity[4].

Dutta (1987) introduced the following two definitions:

Definition 6.2. If R is a fuzzy weak preference relation on X^2, it is T_1-transitive if for all $x, y, z \in X$, $R(x,z) \geq \min\{R(x,y), R(y,z)\}$.

Definition 6.3. R is called *T_2 – transitive if for, all* $x, y, z \in X$, $R(x,z) \geq R(x,y) + R(y,z) - 1$.

A general notion of transitivity is that of *max-star transitivity* which is defined as follows:

Definition 6.4. R is transitive if for all $x, y, z \in X$, $R(x,z) \geq R(x,y) * R(y,z)$ where $*$ is a binary operation.

The binary operator in Definition 6.4 is often constrained to be a *triangular norm* (see Dutta 1987, also see Obchinnikov 1981).

The following two propositions are stated without proof.

Proposition 2.3. *T-transitivity $\Rightarrow$ T_1-transitivity $\Rightarrow$ T_2-transitivity.*

Proposition 2.4. *If a weak preference relation R is T_2-transitive, then for all $x, y, z \in X$, $[P(x,y) > 0$ and $P(y,z) > 0] \Rightarrow P(x,z) > 0$.*

3. Posibility results for social choice

Let N be a finite set of individuals with cardinality not less then 2. X is a set of alternatives. Let H, H_1 and H_2 be the set of reflexive and connected fuzzy weak preference relations satisfying T, T_1 and T_2 transitivities respectively.

We follow the literature in introducing the following definitions.

Definition 3.1. A fuzzy aggregation rule is a function $f: S^n \to S$ where S is a set of fuzzy relations over X.

Elments of S are denoted by R, R' etc. while those of S^n are denoted by $(R_1, R_2, \ldots, R_n)$, $(R_1', R_2', \ldots, R_n')$ etc.

Definition 3.2. Let f be a fuzzy aggregation rule.

(i) Independence of irrelevant alternatives (IIA): f satisfies IIA if for all $(R_1, \ldots, R_n)$, $(R_1', \ldots, R_n') \in S^n$ and for all distinct $x, y \in X$, $[R_i(x,y) = R_i'(x,y)$ and $R_i(y,x) = R_i'(y,x)$ for all $i \in N] \Rightarrow [R(x,y) = R'(x,y)$ and $R(y,x) = R'(y,x)]$.

(ii) Pareto condition (PC): f satisfies PC if for all $(R_1, \ldots, R_n) \in S^n$ and all distinct x, $y \in X$, $P(x,y) \geq \min_{i \in N} P_i(x,y)$ where P_i is the fuzzy strict preference relation derived from R_i and P is similarly derived from $R = f(R_1, \ldots, R_n)$.

Definition 3.3. Let f be a fuzzy aggregation rule.

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[4] This is a modification of the notion of 'max-min' transitivity. See Basu (1984, p 215, footnote 3).

(i) An individual $j \in N$ is called a *dictator* if for all distinct $x, y \in X$ and all $(R_1, \ldots, R_n) \in S^n$, $P_j(x, y) > 0 \Rightarrow P(x, y) > 0$.
(ii) A subset $M \subseteq N$ is *almost decisive over* $x, y \in X$ if for all $(R_1, \ldots, R_n) \in S^n$, $[P_j(x, y) > 0$ for all $j \in M$ and $P_j(y, x) > 0$ for all $j \notin M] \Rightarrow P(x, y) > 0$.

In the propositions below the set of fuzzy relations S will be identified with H or H_1 or H_2.

Proposition 3.1. *Let $f: H_2^n \rightarrow H_2$ be a fuzzy aggregation rule satisfying IIA and PC. Then, there exists a dictator.*

Proof. First note that if a group of individuals G is almost decisive between a pair $(x, y) \in X$, then it is decisive. The Proof which uses Proposition 2.4 is similar to the proof of the Field Expansion Lemma in the exact case (see Sen 1985).

Then we show that if G is a decisive group, a proper subset of G is a decisive group. Although the proof of this part is, again, similar to that of the Group Contraction Lemma in the exact case (see Sen 1985) we set it out here for later reference: let G be decisive. Let $G = G_1 \cup G_2$ where G_1 and G_2 are non-empty. Suppose that for all $i \in G_1$, $P_i(x, y) > 0$ and $P_i(y, z) > 0$; for all $i \in G_2$, $P_i(y, z) > 0$ and $P_i(z, x) > 0$; for all $i \in N - G$, $P_i(z, x) > 0$ and $P_i(x, y) > 0$ where $x, y, z \in X$. If $P(x, z) > 0$, G_1 is almost decisive over (x, z). Hence, it is decisive. If $P(z, x) > 0$, $P(x, z) = 0$. Hence, $R(z, x) = 1 - P(x, z) = 1$.

Now it follows that $P(y, x) > 0$ since if $P(y, x) = 0$, $R(x, y) = 1$ and T_2-transitivity would than imply that $R(z, y) = 1$ so that $P(y, z) = 0$ contradicting the fact that G is decisive. Thus, in this case G_2 is decisive. (The roles of *IIA* and PC are similar to their roles in the exact case.)

As in the exact case the proof of the proposition is completed by repeated application of this 'group contraction' result. Q.E.D.

Since T-transitivity $\Rightarrow$ T_1-transitivity $\Rightarrow$ T_2-transitivity, $H \subseteq H_1 \subseteq H_2$. Hence, we have the following

Corollary. *Proposition 3.1 remains valid if H_2 is replaced by either H or H_1.*

The proof of Proposition 3.1 hinges crucially on the fact that for all $x, y \in X$, $P(x, y) = 1 - R(y, x)$. This proof does not apply if, given R, we define P by the relation: for all $x, y \in X$, $P(x, y) = R(x, y)$ if $R(x, y) > R(y, x)$ and $P(x, y = 0$ otherwise. In fact, the counterexample constructed in Dutta (1987) for the case $S = H_1$, then serves as a counterexample to the proposition since $H_1 \subseteq H_2$.

It is important to consider the special case where the individual preference relations are exact but the social preference relation is permitted to be fuzzy because it can be considered to be the natural first step in extending the traditional exact framework. For this case our earlier definitions have to be restated. We shall now assume that individual exact preference relations are transitive. Denoting the set of transitive exact preference relations by U, we introduce the following definitions.

Definition 3.1′. A fuzzy aggregation rule is a function $f: U^n \rightarrow S$ where S is as in Definition 3.1.

Definition 3.2′. Let f be a fuzzy aggregation rule.

(i) f satisfies *IIA* if it satisfies condition IIA of *Definition 3.2.*
(ii) f satisfies *PC* if for all $(R_i, \ldots, R_n) \in U^n$ and for all distinct $x, y \in X$, $P(x, y) = 1$ if $P_i(x, y) = 1$ for all $i \in N$.

In modifying the definition of a dictator we face a complication. One can categorise an individual i as a dictator if for all $x, y \in X$, $P(x, y) > 0$ whenever $P_i(x, y) = 1$. It is easily checked that this definition of a dictator would reduce to the traditional definition if $S = U$ since in this case for all $x, y \in X$, 1 is the only permissible positive value of $P(x, y)$. However, it is possible to raise a question regarding the *power* of such a dictator. How powerful is individual i as a dictator if for all $x, y \in X$, $P(x, y)$ is positive but small, and $I(x, y)$ close to 1, whenever $P_i(x, y) = 1$?

To take account of this problem we distinguish between different degrees of dictatorships:

Definition 3.3′. Let f be a fuzzy aggregation rule.

(i) An individual $j \in N$ is a dictator if for all distinct $x, y \in X$ and for all $(R_1, \ldots, R_n) \in U^n$, $P_j(x, y) = 1 \Rightarrow P(x, y) > 0$.
(ii) An individual $j \in N$ is an *almost perfect dictator* if for all distinct $x, y \in X$, for all $(R_i, \ldots, R_n) \in U^n$ and for any positive number $\alpha < 1$, $P_j(x, y) = 1 \Rightarrow P(x, y) \geq \alpha$.
(iii) An individual $j \in N$ is a *perfect dictator* if for all distinct $x, y \in X$ and for all $(R_1, \ldots, R_n) \in U^n$, $P_j(x, y) = 1 \Rightarrow P(x, y) = 1$.

Clearly, a dictator is not necessarily either an almost perfect dictator or a perfect dictator. An almost perfect dictator is a dictator but not necessarily a perfect dictator.

Proposition 3.1. *Immediately implies the following:*

Proposition 3.1′. *If $f: U^n \to H_2$ is a fuzzy aggregation rule satisfying IIA and PC, there exists a dictator.*

However, there does not necessarily exist a perfect dictator.

Proposition 3.2. *There exists a fuzzy aggregation rule $f: U^n \to H_2$ which satisfies IIA and PC and which is not perfectly dictatorial.*

Proof. Consider the fuzzy aggregation rule f defined as follows:

For all $x \in X$ and for all $(R_1, \ldots, R_n) \in U^n$, $R(x, x) = 1$.

For all distinct $x, y \in X$ and for all $(R_1, \ldots, R_n) \in U^n$,

(i) $R(x, y) = 1$ and $R(y, x) = 0$ if $R_1(x, y) = 1$ and $R_i(y, x) = 0$ for all $i \in N$,
(ii) $R(x, y) = \beta$ and $R(y, x) = 1 - \beta$ where β is a constant number less than 1 if $R_1(x, y) = 1$ and $R_1(y, x) = 0$ and there exists $j \neq 1$ for which $R_j(y, x) = 1$.
(iii) $R(x, y) = 0$ and $R(y, x) = 1$ otherwise.

Since in all cases $R(x, y) + R(y, x) = 1$ for all $x, y \in X$, R is connected for all $(R_1, \ldots, R_n) \in U^n$. It is easily checked that f satisfies *IIA*. Case *(i)* shows that it satisfies *PC*. Case *(ii)*, where $P(x, y) = 1 - R(y, x) = \beta < 1$ shows that individual 1 is not a perfect dictator. Cases *(ii)* and *(iii)* show that no other individual can be a dictator.

It remains to be shown that for all $(R_1, \ldots, R_n) \in U^n$, $R \in H_2$. However, for later reference we prove the *stronger* proposition that $R \in H_1$ i.e., for all x, y, $z \in X$, $R(x, z) \geq \min[R(x, y), R(y, z)]$. Note that $R(x, y)$ can take any of the three values 1, β, or 0. Similarly for $R(y, z)$. The inequality can be established for all the combinations of values of $R(x, y)$ and $R(y, z)$. For instance, let

$R(x, y)=1$ and $R(y, z)=\beta$. Here $\min[R(x, y), R(y, z)]=\beta$. Also $R_i(x, y)=1$ for all $i \in N$ and $R_1(y, z)=1$ so that $R_1(x, z)=1$. Thus, $R(x, z)$ is either 1 or β. The inequality follows. The other cases are similar. Q.E.D.

However, if $S=H_1$, the existence of an almost perfect dictator follows. To show this we introduce the following definitions:

Definition 3.4. Let f be a fuzzy aggregation rule.

(i) A group of individuals $M \subseteq N$ is called *almost decisive of degree* α *over* (x, y) where $x, y \in X$ if for all $(R_1, \ldots, R_n) \in U^n$, $[P_i(x, y)=1$ *for all* $i \in M$ *and* $P_j(y, x)=1$ *for all* $j \notin M] \Rightarrow P(x, y) \geq \alpha$.
(ii) A group of individuals $M \subseteq N$ is called *decisive of degree* α if for all distinct $x, y \in X$ and for all $(R_1, \ldots, R_n) \in U^n$, $[P_i(x, y)=1$ for all $i \in M] \Rightarrow P(x, y) \geq \alpha$.

Proposition 3.3. *Let $f: U^n \rightarrow H_1$ be a fuzzy aggregation rule satisfying IIA and PC. Then, there exists a unique individual who is an almost perfect dictator.*

Proof. Let α be any positive number less than 1. The proof is obtained by (a) showing that if a group $M \subseteq N$ is almost decisive of degree α over some pair (x, y) where $x, y \in X$, it is decisive of degree α, (b) using this 'field expansion' result to show that if group G is decisive of order α, so is some proper subset of G, and (c) by repeated application of part (b). The proofs of (a) and (b) are similar to those of the corresponding results in the exact case. Uniqueness is easily checked. Q.E.D.

The example constructed in the proof of Proposition 3.2 can be used to establish the following:

Proposition 3.4. *There exists a fuzzy aggregation rule $f: U^n \rightarrow H_1$ which satisfies IIA and PC and which is not perfectly dictatorial.*

However, if $S=H$, the existence of a perfect dictator follows.

Definition 3.5. Let f be a fuzzy aggregation rule.

(i) A group $M \subseteq N$ is *almost perfectly decisive over* (x, y) where $x, y \in X$ if for all $(R_1, \ldots, R_n) \in U^n$ $[P_i(x, y)=1$ for all $i \in M$ and $P_i(y, x)=1$ for all $i \in N-M] \Rightarrow P(x, y)=1$.
(ii) M is *perfectly decisive* if for all $x, y \in X$ and for all $(R_1, \ldots, R_n) \in U^n$, $[P_i(x, y)=1$ for all $i \in M] \Rightarrow P(x, y)=1$.

Proposition 3.5. *Let $f: U^n \rightarrow H$ be a fuzzy aggregation rule satisfying IIA and PC. Then, there exists a perfect dictator.*

Proof. The structure of the proof is similar to that of the proof of Proposition 3.3. We indicate the proof of the 'Group contraction' part. Let M be a decisive group. Let $M=M_1 \cup M_2$ where M_1 and M_2 are nonempty. Suppose that for some x, y, $z \in X$.

$$P_i(x, y)=1 \quad \text{and} \quad P_i(y, z)=1 \quad \text{for all} \quad i \in M_1 ;$$

$$P_i(y, z)=1 \quad \text{and} \quad P_i(z, x)=1 \quad \text{for all} \quad i \in M_2 ;$$

$$P_i(z, x)=1 \quad \text{and} \quad P_i(x, y)=1 \quad \text{for all} \quad i \in N-M .$$

If $P(x,z)=1$, M_1 is almost perfectly decisive over (x,z). Hence, it is a decisive group. If $P(x,z)<1$, $R(z,x)>0$. Hence, we have $R(z,y)\geq\frac{1}{2}\ R(z,x)+\frac{1}{2}R(x,y)>0$, contradicting the fact that M is decisive so that $P(y,z)=1$ and $R(z,y)=1-P(y,z)=0$. Hence, $R(x,y)=0$. (Note that, by the definition of T-transitivity, the weak inequality does not apply in this case). Hence $P(y,x)=1$. Thus M_2 is almost decisive over (y,x). Hence, it is decisive. Q.E.D.

Thus, when the individual preference relations are exact the power of the dictator increases if the transitivity property of the social preference relation is strengthened. T_2, T_1 and T-transitivity restrictions on the social preference relation imply, respectively, the existences of a dictator, an almost perfect dictator and a perfect dictator.

It is interesting to ask whether similar results can be obtained when the domain of f is H_1^n and H^n instead of U^n. If the definitions of a dictator, an almost perfect dictator and a perfect dictator in *Definition 3.3′* are kept unchanged these similar results are immediately established. However, in these cases the definition that is accepted in the literature declares individual j as a dicator if $P_j(x,y)>0$ implies $P(x,y)>0$ for all $x,y\in X$ and for all profiles of individual preference relations. In this case, if we wish to define, say, a perfect dictator, a natural definition will assign that status to individual j if $P_j(x,y)=\alpha$ implies $P(x,y)=\alpha$ for all values of α lying between 0 and 1, for all x, $y\in X$ and for all profiles of individual preference relations. However, in this strong form the result is not true. In fact, it is not difficult to construct counterexamples.

4. Summary and conclusion

There are alternative ways of decomposing a given fuzzy weak preference relation into its antisymmetric and symmetric components. In this paper I have advocated one among these alternative specifications. I have shown that on this specification the fuzzy analogue of the General Possibility Theorem is valid even when the transitivity restrictions on the individual and the social preference relations are relatively weak.

In the special case where the individual preference relations are exact but the social preference relation is permitted to be fuzzy, it is possible to distinguish between different degrees of power of the dictator. This power increases with the strength of the transitivity requirement.

I conclude that the problem of aggregating individual preference relations into social preference relations in an 'acceptable' way remains difficult even when we permit the individual and the social weak preference relations to be fuzzy and the transitivity restriction on them to be relatively weak.

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