Conjectures and Unsolved Problems

In this column *Social Choice and Welfare* will present conjectures and unsolved problems dealing with social choice theory. Suggestions, unsolved problems, conjectures and solutions should be sent to the column editor: Jerry S. Kelly, Department of Economics, Syracuse University, Syracuse, New York: 13244-1090, USA

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10. Craven's conjecture

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Suppose that there are at least three alternatives on which everyone has a *strong* preference ordering. If a social ordering is constructed by simple majority vote, this ordering may not have a maximal element. We can avoid this by restricting the preferences that individuals are allowed to have. Sen and Pattanaik [3] showed over 20 years ago that a necessary[1] and sufficient condition for the existence of a maximal element is that for each triple of alternatives at least one of the following three constraints is satisfied:

Value restriction: There is some alternative, x, in the triple and some position, i ($i=1, 2$ or 3), such that x is never in position i in anyone's ordering (restricted to the triple).

Limited agreement: There are two alternatives, x, y, in the triple such that everyone orders x and y the same way.

Extremal restriction: For each x, z in the triple, if (in his restricted ordering) someone has x first and z last, then anyone else who puts z first must put x last.

Given n alternatives, there are $n!$ different strong preference orderings; these make up the set S. Associated with each set C of $n!/[(n-3)! \cdot 3!]$ constraints (one constraint for each triple) there are subsets T of S consisting only of orderings that are "admissible", that satisfy all the constraints. Of course, if T is a set of admissible orderings, so is any subset of T. Accordingly, we look for subsets that are large. For any set C of constraints we look for largest subsets T^C of S consisting only of admissible orderings. (Because of extremal restriction, these largest $T^C s$ may not be unique.) Now suppose that we vary the set C of constraints, looking for the set with the largest T^C. We put the question: What set of constraints maximizes the number of admissible preference orderings and what is that number?

[1] See also Kelly [2].

John Craven, in a forthcoming social choice textbook [1], conjectures that the maximum number is 2^{n-1}. He illustrates this with an example on four alternatives, $\{a, b, c, d\}$. There are four different triples: $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$, and $\{a, c, d\}$. On each of the first three triples, we impose the constraint: b is not ranked last in anyone's ordering (restricted to the triple). On the fourth triple, we impose the constraint: d is not ranked last in anyone's ordering (restricted to the triple). Of the $24 = 4!$ possible strong orderings

abcd *abdc* **acbd** **acdb** **adbc** **adcb**
bacd *badc* **bcad** *bcda* *bdac* *bdca*
cabd **cadb** **cbad** *cbda* **cdab** **cdba**
dabc **dacb** *dbac* *dbca* **dcab** **dcba**

16 orderings (emboldened) violate at least one of these constraints. There remains a set of $8 = 2^{4-1}$ orderings satisfying all the constraints.

Our first result generalizes this example. On the set of n alternatives, $\{x_1, x_2, \ldots, x_n\}$, consider the following set of $n!/[(n-3)! \cdot 3!]$ constraints: for each $k, 3 \leqq k \leqq n$, x_k is not last in anyone's ordering restricted to a triple containing x_k from $\{x_1, x_2, \ldots, x_k\}$. Call this the "basic" set of constraints on $\{x_1, x_2, \ldots, x_n\}$.

Theorem. *There are 2^{n-1} admissible orderings for the set of basic constraints.*

Proof. We use induction on the number of alternatives. For our basis step, we take $n = 3$; the set of alternatives is $\{x_1, x_2, x_3\}$. There is one constraint: x_3 does not appear last in anyone's order. Of the six possible orderings on three alternatives, $4 = 2^{3-1}$ satisfy this constraint: $x_1x_3x_2$, $x_3x_1x_2$, $x_2x_3x_1$ and $x_3x_2x_1$.

Now suppose that the claim is true for $n-1$. For the nth alternative, x_n, we introduce $(n-1)\cdot(n-2)/2$ new basic constraints: x_n can not appear last in any triple in which it occurs. There are by the induction assumption 2^{n-2} orderings of $\{x_1, x_2, \ldots, x_{n-1}\}$ satisfying the basic constraints on that set. For each of those orders on $n-1$ alternatives, create two new orderings on n alternatives: one with x_n as the new first and one with x_n inserted as the new second. These 2.2^{n-2} orderings satisfy all the basic constraints. It is straightforward to see that only these orderings satisfy all the constraints. □

Remark 1. Clearly this is not the only set of constraints with 2^{n-1} admissible orderings. Changing "last" to "first" in the definition of "basic" would create another.

Remark 2. The same argument shows that if for any n the bound of 2^{n-1} could be exceeded, then it could be exceeded for any larger n.

The rest of this paper shows that we have found a "local" maximum in the sense that if we change from the set of basic constraints to a new set that differs for only *one* triple, then the number of orderings satisfying all the constraints is no greater than 2^{n-1}.

To illustrate this idea, let us return to Craven's example. When all four of Craven's basic constraints are imposed, there are exactly eight admissible orderings:

abdc badc bcda bdac bdca cbda dbac dbca .

Suppose that we remove the constraint that "b is not last in $\{a, b, d\}$". Then two of the orderings previously rejected satisfy all the remaining constraints:

adbc and *dabc*

Combining these two with the eight that had already satisfied the basic constraints we have an "augmented" set of ten orderings. Now the claim of local maximization says that imposing any *new* constraint on $\{a, b, d\}$ will throw out at least two orderings from the augmented set, leaving us with at most eight orderings.

Example 1. Suppose that we impose a new value restriction: b can not be second in anyone's ordering restricted to $\{a, b, d\}$. This would reject three orderings from the augmented set:

abdc dbac dbca

all from the original eight.

Example 2. Suppose that we impose a limited agreement: everyone prefers b to d. This would reject four orderings from the augmented set: *dbac* and *dbca* from the original eight plus *adbc* and *dabc* that had been originally rejected.

Example 3. Suppose that we impose extremal restriction on $\{a, b, d\}$. Arrange the ten orderings of the augmented set according to the way $\{a, b, d\}$ is ordered:

1: *abdc*	2: *adbc*	3: *badc*
4: *bcda cbda bdac bdca*	5: *dabc*	6: *dbca dbac*

With the extremal restriction constraint, choosing an ordering from any one of these groups to be admissible means rejecting *all* the orderings from two other groups. Suppose, for example, that we declare the ordering from group 3 to be admissible; it orders $\{a, b, d\}$ as: bad. Extremal restriction leads us to reject orderings with *dba* and *adb*, i.e., the orderings in groups 6 and 2. By rejecting at least two groups, each with at least one ordering, we leave at most $8 = 2^{4-1}$ admissible.

Part 1. To get started now on a more general analysis, let's break our discussion into two parts. In the first part we drop one of the *last* constraints, one of the constraints that x_n not be last in a triple in which it appears. We drop the constraint that x_n not be last in $\{x_i, x_j, x_n\}$.

Case 1. x_{n-1} is not one of x_i or x_j.

We wish to show that in this case dropping the constraint does not increase the number of admissible orderings. Hence imposing one new constraint on this triple either leaves the number at 2^{n-1} or reduces it.

Imagine an ordering that violates the constraint that x_n not be last in $\{x_i, x_j, x_n\}$ but satisfies all the rest of the original constraints. But x_{n-1} is not last in $\{x_i, x_j, x_{n-1}\}$; this implies that x_n is also last in both $\{x_i, x_{n-1}, x_n\}$ and $\{x_j, x_{n-1}, x_n\}$, contrary to our assumption that all other original constraints are satisfied. Thus there is no such ordering.

Case 2. The triple is of the form $\{x_i, x_{n-1}, x_n\}$

If we drop the one constraint that x_n is not last in $\{x_i, x_{n-1}, x_n\}$ what orderings previously rejected are now admissible? Clearly, in such an ordering, x_n must be last in $\{x_i, x_{n-1}, x_n\}$ but not last in any other triple. So the ordering must start with initial segment

$$x_i x_{n-1} x_n \dots \tag{A1}$$

or

$$x_{n-1} x_i x_n \dots \tag{A2}$$

Examining the constraints that x_{n-2} must not be last from any triple from $\{x_1, x_2, \dots, x_{n-2}\}$, we see that x_{n-2} must either *be* x_i or else x_{n-2} must be in the fourth position in the initial segment. There are then four possibilities:

$$x_{n-2} x_{n-1} x_n \dots \tag{A11}$$

$$x_{n-1} x_{n-2} x_n \dots \tag{A21}$$

$$x_i x_{n-1} x_n x_{n-2} \dots \tag{A12}$$

$$x_{n-1} x_i x_n x_{n-2} \dots \tag{A22}$$

For each of the initial segments of types A11 and A21, the *rest* of the ordering will have to satisfy all the original basic constraints on $\{x_1, x_2, \dots, x_{n-3}\}$. There are by our theorem 2^{n-4} such orderings. So there are $2 \cdot 2^{n-4} = 2^{n-3}$ orderings of types A11 and A21 previously rejected now admissible.

In types A12 and A22, x_{n-3} must either *be* x_i or must be in the fifth position in the initial segment. There are then four possibilities:

$$x_{n-3} x_{n-1} x_n x_{n-2} \dots \tag{A121}$$

$$x_{n-1} x_{n-3} x_n x_{n-2} \dots \tag{A221}$$

$$x_i x_{n-1} x_i x_n x_{n-2} x_{n-3} \dots \tag{A122}$$

$$x_{n-1} x_i x_n x_{n-2} x_{n-3} \dots \tag{A222}$$

For types A121 and A221, we will get $2 \cdot 2^{n-5} = 2^{n-4}$ orderings previously rejected but now admissible. At the next stage, in A122 and A222, x_{n-4} must either be x_i or must be in sixth place. Continuing, we find the total number of orderings previously rejected but now admissible to be:

$$N = 2^{n-3} + 2^{n-4} + \dots + 2^{n-(i+1)} = 2^{n-2} - 2^{n-(i+1)}.$$

These N orderings together with the original 2^{n-1} admissible orderings make up the *augmented* set of orderings.

When we now impose a *new* constraint on $\{x_i, x_{n-1}, x_n\}$ and count the number of orderings in the augmented set that violate the new constraint, the result may depend on the kind of new constraint imposed. So we must examine three cases.

Case 1: Value restriction. There are eight new value restrictions that can be imposed:

1. x_n is not first;	5. x_{n-1} is not third;
2. x_n is not second;	6. x_i is not first;
3. x_{n-1} is not first;	7. x_i is not second;
4. x_{n-1} is not second;	8. x_i is not third.

For each of these, we must calculate the number of orderings in the augmented set that violate the new constraint. Notice that in Example 1 it was possible to reject only orderings from the original admissible set; it was not necessary to look at the new augmenting orderings. In the general case, when imposing a new value restriction constraint, we can work just with the smaller set of originally admissible orderings.

Here we will not give details for all eight possibilities but work only with one that illustrates the kind of analysis required. So suppose we consider imposing the new value restriction: x_i is not first. What orderings, previously admissible, must now be rejected? In such an ordering, x_i must be first in $\{x_i, x_{n-1}, x_n\}$. No other alternatives (except x_{n-1}) can precede x_n. These constraints limit us to orderings with initial segments like:

$$x_i x_n x_{n-1} \cdots \quad \text{(B1)}$$

$$x_i x_{n-1} x_n \cdots \quad \text{(B2)}$$

Either x_{n-2} *is* x_i, or x_{n-2} occurs fourth in the initial segment:

$$x_{n-2} x_n x_{n-1} \cdots \quad \text{(B11)}$$

$$x_{n-2} x_{n-1} x_n \cdots \quad \text{(B21)}$$

$$x_i x_n x_{n-1} x_{n-2} \cdots \quad \text{(B12)}$$

$$x_i x_{n-1} x_n x_{n-2} \cdots \quad \text{(B22)}$$

There are 2^{n-4} orderings of type (B11) and another 2^{n-4} of type (B21), for a total of $2 \cdot 2^{n-4} = 2^{n-3}$. Turning to (B12) and (B22), either x_{n-3} *is* x_i, or x_{n-3} occurs fifth in the initial segment:

$$x_{n-3} x_n x_{n-1} x_{n-2} \cdots \quad \text{(B121)}$$

$$x_{n-3} x_{n-1} x_n x_{n-2} \cdots \quad \text{(B221)}$$

$$x_1 x_n x_{n-1} x_{n-2} x_{n-3} \cdots \quad \text{(B122)}$$

$$x_i x_{n-1} x_n x_{n-2} x_{n-3} \cdots \quad \text{(B222)}$$

There are 2^{n-5} orderings of each of types (B121) and (B221), for a total of 2^{n-4}. Continuing, we must now reject

$$2^{n-3} + 2^{n-4} + \ldots + 2^{n-(i+1)} = N.$$

Case 2: Limited agreement. On the triple $\{x_i, x_{n-1}, x_n\}$ there are six strong orderings which a limited agreement constraint could fix:

1. $x_i > x_n$;	4. $x_{n-1} > x_i$;
2. $x_i > x_{n-1}$;	5. $x_n > x_i$;
3. $x_{n-1} > x_n$;	6. $x_n > x_{n-1}$.

Again, instead of providing details for all six possibilitis, we illustrate by considering the case where we drop the original constraint that x_n not be last on $\{x_i, x_{n-1}, x_n\}$ in anyone's ordering and instead require: all orderings must satisfy $x_i > x_n$. What orderings in the augmented set must now be rejected? Clearly to violate the new constraint x_n must be ahead of x_i. Among the orderings previously admissible, x_n must *not* be last in $\{x_i, x_{n-1}, x_n\}$. These constraints limit us to orderings with initial segments like:

$$x_{n-1} x_n \cdots x_i \cdots \quad \text{(C1)}$$

$$x_n x_{n-1} \cdots x_i \cdots \quad \text{(C2)}$$

$$x_n x_i x_{n-1} \dots \tag{C3}$$

Of types (C1) and (C2), there are $2 \cdot 2^{n-3} = 2^{n-2} > N$.

Case 3: Extremal restriction. Recall that the N orderings that were excluded under the basic constraints but which were brought into the augmented set when we dropped the basic constraints "x_n is not last in $\{x_i, x_{n-1}, x_n\}$" had to be of the form

$$x_i x_{n-1} x_n \dots \tag{A1}$$

or

$$x_{n-1} x_i x_n \dots \tag{A2}$$

Half of the N were of type (A1) and half of type (A2). If we leave in even one ordering of type (A1), we must reject $N/2$ ordering of the form

$$x_n x_i x_{n-1} \dots . \tag{N1}$$

Hence we reject $N/2$ of the form (A1) or $N/2$ of the form (N1). Similarly, we must reject $N/2$ orderings of the form (A2) or $N/2$ of the form

$$x_n x_{n-1} x_i \dots \tag{N1}$$

In any case, we reject at least N altogether.

Part 2. Now we must remember that we have so far, in Part 1, only dealt with the case where we dropped one of the *last* constraints, one of the constraints that x_n not be last in a triple in which it appears. Suppose that instead that we drop the constraint that x_{n-1} in not last in some triple from $\{x_1, x_2, \dots, x_{n-2}, x_{n-1}\}$ and replace it by some other constraint on this same triple. By the kind of analysis we just completed, there are $T \leqq 2^{n-2}$ orderings on $\{x_1, x_2, \dots, x_{n-2}, x_{n-1}\}$ satisfying all the constraints on triples from this set. So we must be rejecting $(n-1)! - T$ orderings. For each of these, any of the n possible insertions of x_n still must be rejected. This shows the rejection already of $n[(n-1)! - T]$ orderings on $\{x_1, x_2, \dots, x_{n-1}, x_n\}$. In addition we must now reject any of the previous accepted T that have x_n inserted in positions $3, 4, \dots, n$. That is, of the n places among the previous $n-1$ alternatives that we could insert x_n, 2 are OK, $n-2$ lead to rejection. Altogether, we must reject at least

$$n((n-1)! - T) + (n-2)(T) = n! - 2T \geqq n! - 2 \cdot 2^{n-2} = n! - 2^{n-1}$$

so at most 2^{n-1} may remain. So the result for a constraint on x_n can be pushed back to a constraint on x_{n-1}. Using this argument as a pattern, we can push the result back to any basic constraint.

There remain two obvious unsolved problems. The first is to extend our local result to Craven's global claim. The second is to work with weak as well as strong orders (but this is probably very difficult).

References

1. Craven J: Social choice. Cambridge University Press (forthcoming)
2. Kelly JS (1974) Necessity conditions in voting theory. J Econ Theory 8: 149–160
3. Sen AK, Pattanaik PK (1969) Necessary and sufficient conditions for rational choice under majority decision. J Econ Theory 1: 178–202