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Recognizing majority-rule equilibrium in spatial voting games*

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Abstract. It is provably difficult (*NP*-complete) to determine whether a given point can be defeated in a majority-rule spatial voting game. Nevertheless, one can easily generate a point with the property that if any point cannot be defeated, then this point cannot be defeated. Our results suggest that majority-rule equilibrium can exist as a purely practical matter: when the number of voters and the dimension of the policy space are both large, it can be too difficult to find an alternative to defeat the status quo. It is also computationally difficult to determine the radius of the yolk or the Nakamura number of a weighted voting game.

1. Introduction

Majority rule equilibrium exists when there is a candidate who is *undominated*; that is, he cannot be defeated by any other candidate in a pairwise election[1]. Once in place, such a candidate can never be voted out of office. It need not be that an undominated candidate exists. For spatial voting games, when there is no undominated point the entire policy space collapses into a voting cycle and the process of social choice is vulnerable to manipulation [14, 20]. Accordingly, one of the most studied problems in the spatial theory of voting is to give simple conditions for the existence of a undominated point. (See [6, 7] for surveys.)

We show that it is computationally difficult - more precisely, *NP*-complete [9] - to determine whether a given point can be defeated in a spatial voting game.

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[1] Alternative terminology: *core* point, *Condorcet* candidate.

This provides strong theoretical evidence that there are no simple conditions by which to check that a given point is undominated. Indeed, assuming $NP \neq$ co-NP as is widely thought to hold [9], all such conditions must require space that increases exponentially in the size of the problem (the number of voters and the dimension of the policy space). Furthermore, this difficulty is inherent in even the simplest model of spatial voting, in which each voter is assumed to have Euclidean preferences, and so prefers candidates that are closer to his ideal point. A corollary of this result is that it is computationally difficult to determine the radius of the yolk [8, 13].

Our results are consistent with known conditions for the existence of an undominated point, all of which require computation that grows rapidly in the size of the problem. The definition of an undominated point cannot be tested directly since such a point must not be defeated by any conceivable alternative, of which there are an infinite number. Local geometric conditions for the existence of an undominated point were given by Plott [18]; however these conditions are sufficient but not necessary when the ideal points of some voters are coincident [6]. Conditions that are both necessary and sufficient were given by Slutsky [24], but these are not directly testable since they are not finite conditions. Subsequently, Davis, DeGroot, and Hinich [4] gave simpler conditions that are both necessary and sufficient for the existence of an undominated point. We state them in detail because we will use them to establish our results.

> A point v_0 is undominated if and only if any hyperplane containing v_0 divides the ideal points of the voters such that at least one-half lie on either closed side of the hyperplane.

As suggested by the geometry of this condition, the authors call a hyperplane which so divides policy space a *median hyperplane* and an undominated point a *total median.*

The total median condition is pleasing because it is purely geometrical. However, like the others, the total median condition is not directly testable: it requires that a property hold for *every* hyperplane containing v_0.

Hoyer and Mayer [10] suggested a natural generalization of the spatial model of voting that allows each voter to have a separate utility function with elliptical indifference contours. They also defind a "generalized total median", showed it to be undominated when it exists, and gave necessary and sufficient conditions for its existence. Again, as stated, these conditions are not directly testable.

The final word in this line of inquiry may have been given by McKelvey and Schofield, who established very general conditions for social equilibrium under arbitrary voting rules [15]. These conditions are satisfying because they are explicitly finite and so are directly testable. However, the work required to test these conditions increases exponentially in the number of voters since one must consider all possible coalitions.

Our results show that exponential work is apparently necessary, at least in the worst-case, to prove that a given point is undominated. This suggests that it can be impractical to decide whether majority-rule equilibrium holds – except, of course, when the instances are sufficiently small or specially structured. For the special case in which the number of voters is large and the dimension of the policy space is small, we observe that an algorithm due to Johnson and Preparata [11] gives a new test for equilibrium that, even though requiring exponential time in the worst case, is nevertheless considerably more practical than any other known.

Interestingly, it is easier in a sense to search for an undominated point than it is to test whether a particular point is undominated. Specifically, we give a fast procedure that produces a point with the following property: if any point is an undominated point, then this point is. Our algorithm uses specialized techniques of computational geometry to achieve remarkable speed. If the election is modelled by the ideal points of n voters in E^d, then our algorithm requires only $O(dn)$ computational steps. This is within a constant factor of the effort necessary simply to read the data.

Continuing the theme of discovering what might and might not be computable in practice, we also show that even conditions that are merely sufficient (but not necessary) for equilibrium, if very general, can be difficult to check. For example, a sufficient condition for majority-rule equilibrium in a weighted voting game is that the "Nakamura number" be sufficiently large; however, we show that it is *NP*-hard to determine the Nakamura number.

At the very least, our results say that, assuming $NP \neq$ co-NP, any computational procedure to check for majority rule equilibrium must be essentially an enumeration over an exponentially large set. It might be that such computational difficulties introduce new tactics into spatial voting games since players might not have sufficient time to perform the computation necessary to recognize whether a given point can be defeated. (Similar work has appeared in [1, 2, 3].)

Two notes on terminology: since "undominated point" and "total median" mean the same thing, we use the former term when emphasizing social choice issues and the latter term when emphasizing geometrical issues. Also, we use the "Big O" notation of computer science to indicate the asymptotic rate of growth of functions: we write that function $f(n)$ is "$O(g(n))$" when there exist integers n_0 and K such that $f(n) \leqq Kg(n)$ for all $n \geqq n_0$. Thus for large n, the function g will not underestimate f by more than a constant factor. See [12] for more details.

2. Computational complexity

For convenience of the reader we briefly and informally describe some terms from the theory of computational complexity. The interested reader should consult [9] for more information.

NP is the class of "yes/no" problems for which, if the answer to an instance is "yes", then there exists a polynomial-time proof of this fact. Thus once one has the proof in hand, thereafter one can quickly convince others, even though to find the proof originally might have required a very long time. For example, the question "Does there exist a point that can defeat x?" is in *NP* since a "yes" answer can be quickly proved by displaying a winning alternative and counting votes to verify that it beats x. (In this case, finding the proof means finding a winning alternative, which might be hard even if one exists.) *NP*-complete problems are those that are members of *NP* and, moreover, have the property that if there exists a polynomial-time algorithm to solve any one of them, then all problems in *NP* could be solved in polynomial time. It is thought that no *NP*-complete problem admits of polynomial-time solution. Thus the *NP*-complete problems are in a sense the most difficult problems in *NP*.

The class co-*NP* is complementary to *NP*; it is the class of problems for which a "no" answer can always be proved in polynomial time. For example, the question "Is x undominated?" is in co-*NP*. Co-*NP*-completeness is defined similarly to *NP*-completeness and has the same practical import as the more familiar *NP*-completeness. A problem that is co-*NP*-complete is *NP*-hard, and so is at least as hard as the *NP*-complete problems [9].

Finally, class *P* consists of those problems for which either a "yes" or a "no" answer can be proved in polynomial time.

It is widely believed that the problems of *P*, the *NP*-complete problems, and the co-*NP*-complete problems are all distinct, although this is not known for sure. At any rate, much theoretical evidence and computational experience confirms that for problems in *P* the computational effort to solve does not grow too quickly as the size of the instance increases. In short, problems in *P* can be solved in reasonable amounts of time. On the other hand, problems that are *NP*-or co-*NP*-complete are impractical to solve unless the instances are small or specially structured; the computational effort increases exponentially in the size of the problem, whatever the solution procedure. It is thought that this difficulty is inherent.

3. Is there an undominated point?

Ideally we would like to either find an undominated point or else prove that none exists. We partition this task into two simpler tasks: find a promising point; and test whether it is undominated. We shall show that finding is significantly easier than testing.

If we are given a set V of ideal points of voters in E^d, it is not obvious even where to look for a point that might be undominated. In the following we give an algorithm that will quickly generate a point with the property that if any point is undominated, then this point is. We refer to such a point as a *best bet.* The algorithm hunts for a "best bet" by constructing a special kind of hyperplane that must contain an undominated point if one exists.

Throughout the paper we make the allowance that the points of V need not be distinct, since several voters might have ideal points that are coincident. Also, in the remainder of this section we assume that the points of V span E^d. This is a reasonable restriction, since if the points do not span E^d, then in a sense some dimensions of policy space are superfluous since they are unnessary to distinguish among the voters.

Definition 1. A median hyperplane is unique for a set of points V if there exist no other median hyperplanes with the same slope.

Lemma 1. *If H is a unique median hyperplane for V and if v_0 is an undominated point, then v_0 lies on H.*

Proof. If v_0 does not lie on H, then the hyperplane passing through v_0 parallel to H is another median hyperplane, contradicting the uniqueness of H. □

Our strategy to find a "best bet" candidate will be to construct sufficiently many unique median hyperplanes that they jointly determine a candidate point. It is sufficient for our purposes – and essential to the speed of our algorithm – to do this in only two dimensions. Therefore we begin by restricting the problem to $d=2$. In E^2 we will construct two unique median hyperplanes, which by definition must intersect. By Lemma 1 the point of their intersection must be an undominated point if any point is undominated.

In our statement of the algorithm, Steps 1 and 2 contain the essential ideas; Step 3 handles "degenerate" cases, including when n is odd.

Algorithm GENERATE-POINT (2D): to produce a "best bet" candidate for undominated point

Step 1. Find H_1, the leftmost vertical median hyperplane, and H_2 the rightmost vertical median hyperplane.

Step 2. If $H_1 \neq H_2$ then there can be no points of V strictly between H_1 and H_2, so the hyperplane $(1/2)H_1 + (1/2)H_2$ strictly separates V into two disjoint sets, V_1 on the left and V_2 on the right. Construct the two hyperplanes that separate V_1 and V_2 and are tangent to both V_1 and V_2 (the "transverse tangents"). Return the intersection of these two hyperplanes as the candidate point.

Step 3. If $H_1 = H_2$, find the topmost and bottommost horizontal median hyperplanes.

3.1 If they are distinct, then perform Step 2 with them as H_1 and H_2;

3.2 Otherwise, return the intersection of the (unique) vertical median hyperplane and the (unique) horizontal median hyperplane.

Figures 1 and 2 illustrate the constructions of Steps 1 and 2.

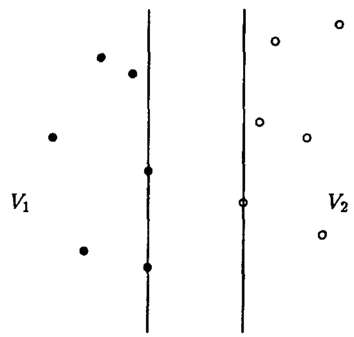

Fig. 1. The leftmost and rightmost median hyperplanes partition V

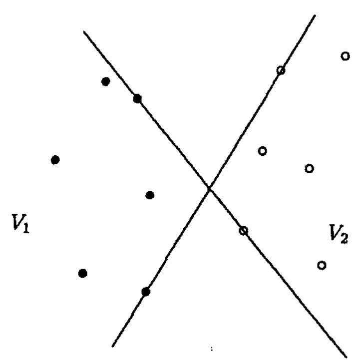

Fig. 2. The intersection of the transverse tangents to V_1 and V_2 is the "best bet" for undominated point

Theorem 1. *Algorithm* GENERATE-POINT (2D) *produces a point that is undominated if any point is.*

Proof. The algorithm can halt in either of two places. It can halt after Step 3.2 (as when n is odd), in which case it produces a point that satisfies the theorem. Alternatively, it can halt after Step 2 (as when n is even). In the latter case each of the hyperplanes that is a transverse tangent to V_1 and V_2 is a unique median hyperplane. This follows since each of V_1 and V_2 must contain exactly $n/2$ points so that each transverse tangent is a median hyperplane; and each is a unique median hyperplane because any distinct hyperplane that is parallel must have less than $n/2$ points strictly to one side. Therefore any undominated point must lie on the intersection of the transverse tangents. In addition, there must be two distinct transverse tangents since the points of V span the space. Therefore the transverse tangents must intersect at a unique point. □

Theorem 2. *Algorithm* GENERATE-POINT (2D) *can be implemented to run in time* $O(n)$.

Proof. The leftmost and rightmost vertical median hyperplanes can be found in time $O(n)$ using fast median-finding techniques such as described in [12].

The transverse tangents to V_1 and V_2 can be constructed by brute force in $O(n^2)$ time by considering the line segments formed by all pairs of points, one in V_1 and the other in V_2; the line segments of greatest and smallest angle determine the desired hyperplanes. However there is a significantly faster method based on linear programming. It requires only $O(n)$ time.

To find a hyperplane with boundary $y=mx+b$ that separates V_1 and V_2 and that is tangent to both V_1 and V_2, consider the following linear program, with variables m and b and with a constraint for each point $(x,y)\in V$.

$$\max m$$
$$y_i \leqq mx_i + b \quad (x_i, y_i) \in V_1 \ ;$$
$$y_j \geqq mx_j + b \quad (x_j, y_j) \in V_2 \ ;$$

The constraints require the hyperplane with boundary $y=mx+b$ to separate V_1 and V_2, and the objective function seeks such a hyperplane of largest slope.

Note that the linear program is feasible since the hyperplane $-(1/2)H_1-(1/2)H_2$ can be perturbed to make the slope finite, which gives values of m and b that satisfy the constraints. Furthermore, the linear program must have a finite solution since the constraints imply that for any $(x_i,y_i)\in V_1$ and $(x_j,y_j)\in V_2$, it must hold that

$$y_j - y_i \geqq (mx_j + b) - (mx_i - b) = m(x_j - x_i) \ .$$

But $x_j - x_i > 0$ since by construction all the points of V_1 are strictly to the left of the points of V_2. Therefore we have that

$$m \leqq \frac{y_j - y_i}{x_j - x_i}$$

and so the linear program is bounded from above. Thus the linear program has a feasible, finite optimum solution, which will determine one of the transverse tangents. Similarly we can pose the problem of finding the other transverse tangent as the following linear program.

$$\min m$$
$$y_i \geqq mx_i + b \quad (x_i, y_i) \in V_1 \ ;$$
$$y_j \leqq mx_j + b \quad (x_j, y_j) \in V_2 \ .$$

Furthermore, these two linear programs will have distinct solutions since the points of V span the space. Since each of these linear programs is restricted to two dimensions, we can solve them in time $O(n)$ by the specialized techniques of Megiddo [5, 16].

Finally, a candidate for undominated point can be found as the intersection of two hyperplanes in two dimensions within time $O(1)$. □

Now we are ready to give an algorithm to produce a "best bet" point for the general problem in d dimensions.

Algorithm GENERATE-POINT
Step 1. For $i = 1, 3, 5, \ldots, d-1$, project the points of V onto the plane determined by the ith and $i+1$st coordinates. The projected points must span the plane since the points of V span E^d. Therefore we can determine the ith and $i+1$st coordinates of the "best bet" point by calling Algorithm GENERATE-POINT (2D) on the projected points. (Note that if d is odd, Algorithm GENERATE-POINT (2D) is called for each $i = 1, 3, 5, \ldots, d-2, d-1$.)

Theorem 3. *Algorithm* GENERATE-POINT *runs in time* $O(dn)$.

Proof. At each of $\lceil d/2 \rceil$ iterations, the algorithm constructs a projection of V onto two dimensions, which requires $O(n)$ time, and then calls algorithm GENERATE-POINT (2D), which also requires $O(n)$ time. □

Finally, from the mechanics of the algorithm we have the following observation.

Corollary 1. *If all the points of V have rational coordinates, then if there exists an undominated point, there exists one with rational coordinates.*

Proof. If the number of voters is odd, then the point generated by our algorithm is identical to one of the points of V and so has rational coordinates. If the number of voters is even, then each coordinate is part of the solution of a linear program with rational data and is therefore rational. □

4. Is *this* point undominated?

4.1. The difficulty of knowing

While we are interested in recognizing an undominated point, it will be convenient to study the complementary problem of recognizing when a point is dominated. We formalize the complementary problem according to the usual conventions of complexity theory [9].

Recognizing A Dominated Point.

Given: A set V of the ideal points of n voters in E^d and an additional, distinguished point v_0, where the coordinates of all points are rational.

Question: Is there any point in E^d with rational coordinates that can defeat v_0 in a majority-rule election?

If there exists a polynomial-time algorithm to recognize an undominated point, then we could use it to recognize when a point is dominated. However the following theorem shows that it is highly unlikely that there exists a polynomial-time algorithm to answer *Recognizing A Dominated Point.*

Theorem 4. *Recognizing A Dominated Point is NP-complete*[2].

Proof. This problem is a member of the class *NP* since an answer of "yes" can be proven in polynomial time by giving an alternative point and confirming that it will defeat v_0.

To show that the problem is complete for *NP*, we show a polynomial-time reduction between it and the following problem that is known to be *NP*-complete [9].

Open Hemisphere.

Given: A finite set V of d-tuples of rational numbers, and a positive integer $K \leqq |V|$.

Question: Is there a d-tuple y of rational numbers such that $v \cdot y > 0$ for at least K d-tuples $v \in V$?

Now, as is traditional in complexity theory, we show how, given any instance of *Open Hemisphere*, one can, in polynomial-time, contrive an instance of *Recognizing Dominated Point* such that the answer to the former is "yes" if and only if the answer to the latter is "yes". Thus each problem is "as hard as" the other; and *Open Hemisphere* is known to be *NP*-complete [9, 11]. Furthermore, *Open Hemisphere* remains *NP*-complete even when K is restricted to be greater than $|V|/2$. (This follows from the straightforward observation that the answer to *Open Hemisphere* is always "yes" when $K \leqq \lceil |V|/2 \rceil$,.)

Given an instance of *Open Hemisphere*, interpret the points of V as the ideal points of voters. In addition, augment the election by adding the ideal points of p additional voters at the origin, where p is chosen so that $n = |V| + p$ is odd and $K = \lceil (|V| + p)/2 \rceil$. (This is always possible since $K > |V|/2$.) Let U be the set of ideal points of the p additional voters. The voters of U prefer the origin to any alternative.

Assume that in the contrived election the origin can be defeated by some other point y_0. Let $V' \subseteqq V$ be the voters who prefer y_0 to the origin; and for each $v \in V'$, let $B(v)$ be the open ball centered at v whose boundary contains the origin. Since each $v \in V'$ prefers y_0 to the origin, it must be that $y_0 \in \cap_{v \in V'} B(v)$. Thus $\cap_{v \in V'} B(v)$ is non-empty and open, and so there exists $y \in \cap_{v \in V'} B(v)$ with rational coordinates. Furthermore, since $y \in B(v)$ for each $v \in V'$, it must be that $v \cdot y > 0$ for the $v \in V'$, of which there are at least K. Thus if the answer to *Recognizing A Dominated Point* is "yes", then so is the answer to *Open Hemisphere*.

Now assume that there exists a point y with rational coordinates such that for at least K elements $v \in V$ it holds that $v \cdot y > 0$. Let $V' \subseteqq V$ be the points for which this is true. For each $B(v)$ there is an ε_v such that the point $\varepsilon_v y$ lies entirely within $B(v)$. Then all of the points λy, where $0 < \lambda \leqq (\min_{v \in V'} \varepsilon)$ must lie within $B(v)$ for each $v \in V'$. Let y_0 be such a point with rational coordinates. Then each voter in V' prefers y_0 to the origin, which is sufficient for y_0 to defeat the origin in a pairwise election. □

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[2] In fact this theorem is implicit in [11]; however the result seems unknown in the social choice literature.

That *Recognizing A Dominated Point* is in *NP* means that a "yes" answer can always be established by a short (polynomial-time) proof. One can quickly prove that a point is dominated by simply displaying an alternative and tallying the votes. However there is no apparent way by which a "no" answer can always be quickly proved. Indeed, unless $NP = \text{co-}NP$ there is in general no succinct demonstration that a point is undominated [9]. Thus even if one knows that a point is undominated, it might be impractical to convince anyone else.

Now we formalize the question of whether majority-rule equilibrium holds.

Majority-Rule Equilibrium.

Given: A set V of the ideal points of n voters in E^d, where the coordinates of all points are rational.

Question: Is there any point in E^d that is undominated?

The following result is strong theoretical evidence that it is computationally difficult to determine whether majority-rule equilibrium pertains.

Corollary 2. *Majority-Rule Equilibrium is co-NP-complete.*

Proof. The problem is in class co-*NP* since an answer of "no" can be verified in polynomial time by the following procedure. Use algorithm GENERATE-POINT to produce a "best-bet" point v_0 and then exhibit a point that defeats v_0. Complexity follows from Theorem 4. □

A second corollary shows that it is formally difficult to determine the radius of the "yolk" of a spatial voting game. This conclusion is consistent with the empirical observations of [13], wherein it is remarked that "even for a small committee determination of the location and size of the yolk is a formidable task".

Corollary 3. *It is NP-hard to determine the radius of the yolk.*

This follows since if there were a polymonial-time algorithm to determine the radius of the yolk, then one could test for equilibrium in guaranteed polynomial time by checking whether the radius is zero.

4.2. *A fast test for a special case*

While *Recognizing A Dominated Point* is *NP*-complete, nevertheless special cases of this question can always be answered quickly. For example, we now give a polynomial-time algorithm for the special case in which $v_0 \notin V$. The algorithm is based on the fact that a point v_0 is a total median when the points of V are arranged about v_0 according to a weak sort of radial symmetry in which angle but not distance matters. This idea is well-known [7, 18]. However we formalize it as a computational procedure that gives conditions that are both necessary and sufficient: a point that is distinct from the ideal points of all the voters is undominated if and only if the following algorithm says it is.

The basic idea of the algorithm is to match points that correspond under the necessary symmetry. Without loss of generality, let v_0 be the origin.

Algorithm. TEST: to determine whether v_0 is a total median of V

Step 0. Compute the polar coordinates of all points of V.

Step 1. For each point in V with coordinates $(r, \theta_1, \ldots, \theta_d)$, if there exists some other point in V with polar coordinates $(r', \theta_1 + \pi, \ldots, \theta_d + \pi)$, then match these two points and remove them from further consideration.

Step 2. If any point is unmatched, return FALSE; otherwise return TRUE.

Lemma 2. *The point $v_0 \notin V$ is a total median if and only if identified as such by Algorithm* TEST.

Proof. Sufficiency is obvious. To argue necessity, consider an instance for which Algorithm TEST returns FALSE. Then there must exist some unmatched point v that is distinct from v_0; let l be the line determined by v_0 and v. There can be no other unmatched points on l opposite v, since otherwise such a point could be paired with v.

Project all of the points onto a 2-dimensional plane chosen so that no projected point lies on the projection of l unless the original point lies on l. (This is always possible since there are only finitely many points.) Denote the projection of v_0 by $P(v_0)$, the projection of l by $P(l)$, and so on. Let V_1 and V_2 be the points strictly to each side of $P(l)$, and assume without loss of generality that $|V_1| > |V_2|$. Rotate $P(l)$ slightly about $P(v_0)$ so that $P(v)$ is to the same side of $P(l)$ as the points of V_1; then $P(l)$ determines a hyperplane for which one side – the side opposite that on which $P(v)$ lies – now contains strictly less than $|V|/2$ points, so that $P(v_0)$ is not a total median for the points of $P(V)$. But then v_0 cannot be a total median for the points of V. □

Lemma 3. *Algorithm* TEST *halts within $O(dn \log n)$ steps.*

Proof. The points can be sorted lexicographically on the angles of their polar coordinates within $O(dn \log n)$ steps. Now within the sorted list the points can be matched within $O(dn \log n)$ steps using binary search [12]. □

Since the Plott conditions are necessary for $v_0 \in V$ when $|V| = n$ is odd and v_0 is distinct from all other points of V, algorithm TEST applies in this case too. Thus the difficult instances of recognizing a total median must be confined to the problem in which $v_0 \in V$ and, in addition, either n is odd or else v_0 is coincident with some other point of V.

Algorithm TEST quickly recognizes an undominated point when equilibrium is unstable (that is, when small perturbations in the locations of the voter ideal points can destroy the equilibrium [22]). This reflects the severity of the Plott conditions: their restrictiveness supports easy testing, but this restrictiveness is naturally "brittle". We do not know whether all unstable equilibria can be quickly recognized nor whether there are interesting special cases in which stable equilibria can be easily recognized. Nevertheless, it is tempting to think that, at least in an informal sense, it is easier to recognize unstable equilibria because they are so much more highly constrained.

4.3. Computing the Nakamura number

If it is suspected that an undominated point exists, one might check sufficient conditions and hope they confirm the fact. It seems natural to apply the most general possible conditions sufficient to guarantee an undominated point since the test is more likely to give information. However, we will demonstrate a previously unsuspected tradeoff here: the more general (and therefore more powerful) the sufficient conditions, the more difficult they can be to invoke.

Consider the *Nakamura number*, which is defined for a more general model than we have considered heretofore: In a *weighted* voting game voter i has a weight p_i, and there is a *quorum rule* q such that a candidate is elected only if the sum of the weights of the voters who support him is at least q. Let $(q; p_1, \ldots, p_n)$ denote an instance of a weighted voting game with n voters. For a weighted voting game the Nakamura number N is the cardinality of the smallest set of winning coalitions with the property that the intersection of its members is empty [17, 21]. An interesting feature of the Nakamura number is that if $N \geqq d+2$, where d is the dimension of the policy space, then there exists an undominated point [17, 21]. (The converse is not true.) Thus, in a sense, the higher the Nakamura number of a weighted voting game, the more likely there is to be an undominated point. However, we show that it is probably not practical to depend on the Nakamura number to signal equilibrium; it is NP-hard to determine whether the Nakamura number is large enough.

We formalize the problem of computing the Nakamura number as follows.

Nakamura Number.

Given: A weighted voting game $(q; p_1, \ldots, p_n)$ and an integer $m \geqq 3$.

Question: Is the Nakamura number $N > m$?

We establish the complexity of *Nakamura Number* somewhat indirectly, by showing that the complementary problem is formally difficult. In the complementary version of the problem we ask whether the Nakamura number is small.

Complementary Nakamura Number.

Given: A weighted voting game $(q; p_1, \ldots, p_n)$ and an integer $m \geqq 3$.

Question: Is the Nakamura number $N \leqq m$?

Theorem 5. *Complementary Nakamura Number is NP-complete in the strong sense.*[3]

Proof. Complementary Nakamura Number is in *NP* since an answer of "yes" can be verified in polynomial time by exhibiting a set of no more than m coalitions and checking that each is winning and that no voter is a member of all the coalitions. Now we show that Complementary Nakamura Number is "as hard as" the following problem that is known to be *NP*-complete in the strong sense [9].

3-Partition. Given: A positive integer B, and a set of positive integers $\{p_1, p_2, \ldots, p_{3m}\}$ such that $\sum_{i=1}^{3m} p_i = mB$.

Question: Can $I = \{1, 2, \ldots, 3m\}$ be partitioned into m disjoint sets $\{I_j\}_{j=1}^{m}$ so that $\sum_{i \in I_j} p_i = B$ for each $1 \leqq j \leqq m$?

[3] This means that the problem remains difficult even when all numbers q, p_i are restricted to be no larger than some *a priori* bound. Thus the inherent difficulty can be attributed to the structure of the problem and not just the size of the numbers (voting weights and quorum rule) [9].

Given an instance of 3-Partition we contrive a voting problem that we will show to be equivalent: Let there be $n=4m$ voters, with weights defined as follows. First there is a group of $3m$ "weak" voters, with weights p_i, $i=1,\ldots,3m$. To this we add a group of m "strong" voters, with weights $p_{3m+1}=p_{3m+2}=\cdots=p_{4m}=2mB$. Finally, we set the quorum rule $q=2(m-1)mB+(m-1)B$. Notice that the quorum rule is such that a coalition of all the strong voters will win, but any set of $m-1$ strong voters needs additional support from the weak voters. Furthermore, no coalition containing fewer than $m-1$ of the strong voters can win.

We now argue that if the Nakamura number $N \leqq m$ for this election then there exists a 3-partition of I. Denote a coalition by the set of indices of the voters in that coalition. Note that if C is a winning coalition, then $i \notin C$ for at most one strong voter $i>3m$. Thus any collection of winning coalitions with empty intersection must include m coalitions, each one of which is missing exactly one distinct strong voter. Thus if $N \leqq m$, in fact $N=m$. Without loss of generality let these m coalitions $C_1, C_2, \ldots, C_m$ be indexed so that $(3m+j) \notin C_j$ but $(3m+i) \in C_j$ for $i \neq j$ and $i,j=1,2,\ldots,m$. Let $T_j=C_j \cap \{1,2,\ldots,3m\}$ be the set of indices of the weak voters of the coalition $C_j (j=1,2,\ldots,m)$. Since C_j is a winning coalition, $\sum_{i \in C_j} p_i \geqq q$, which implies that for all $j=1,2,\ldots,m$,

$$\sum_{i \in T_j} p_i \geqq (m-1)B \ . \tag{1}$$

However, since $\bigcap_{j=1}^{m} T_j = \emptyset$, for each i, $1 \leqq i \leqq m$, there is some T_j such that $i \notin T_j$. In other words, no weak voter can be in all coalitions. Thus, since each weak voter can be counted at most $m-1$ times and since $\sum_{i=1}^{3m} p_i = mB$,

$$\sum_{j=1}^{m} \sum_{i \in T_j} p_i \leqq m(m-1)B \ . \tag{2}$$

Now summing expression 1 over $j=1,2,\ldots,m$ gives

$$\sum_{j=1}^{m} \sum_{i \in T_j} p_i \geqq m(m-1)B \ ,$$

which, combined with expression 2 implies that expression 1 holds with equality. Thus, since each $p_i>0$, for each $i \in I$ there exists a unique j, $1 \leqq j \leqq m$, such that $i \notin T_j$.

Let $I_j = I - T_j$, the indices of the weak voters not in T_j. By the foregoing, the I_j, $1 \leqq j \leqq m$, form a partition of I. Furthermore, since expression 1 holds with equality; and since by definition of I_j, $\sum_{i \in I_j} p_i + \sum_{i \in T_j} p_i = mB$, we have that for each $1 \leqq j \leqq m$

$$\sum_{i \in I_j} p_i = B \ .$$

Thus $I_1, I_2, \ldots, I_m$ form a 3-partition of I, so that if $N \leqq m$ then the answer to the instance of 3-Partition is "yes".

Now suppose that there exists a 3-partition $I_1, I_2, \ldots, I_m$ of I. Reversing the development above, we construct $T_j = I - I_j$ and $C_j = \{3m+i \mid i=1,2,\ldots,m;$

$i \neq j\} \cup T_j$ for $j = 1, 2, \ldots, m$. Then $C_1, C_2, \ldots, C_m$ are winning coalitions and $\bigcap_{j=1}^{m} C_j = \emptyset$, so it must be that $N \leqq m$. □

Finally we conclude that it is computationally difficult to decide whether the Nakamura number is large.

Corollary 4. *Nakamura Number is co-NP-complete in the strong sense.*

Thus the Nakamura number, while interesting, might not be practically computable and so of no help in deciding whether equilibrium pertains.

5. Yes, but *is* there an undominated point?

In summary, the practical hunt for an undominated point might proceed as follows.

Step 1. Call algorithm GENERATE-POINT to quickly generate a "best bet" y.

Step 2. If $y \notin V$, then call algorithm TEST to determine quickly whether y is a dominant point.

Step 3. If $y \in V$, then apparently one must use an exponential-time algorithm. This step will be computationally infeasible for all but small or specially structured instances.

While Step 3 could be implemented as the algorithm implicit in [15], there is a potentially serious difficulty. These conditions apply to more general preferences than Euclidean; the price to be paid for such generality is that the algorithm requires work that increases exponentially in the number of voters. This can be impractical when the number of voters is large. However, when d is small (2 or 3), an attractive alternative is to realize Step 3 by straightforward adaptation of the algorithm of [11], which requires $O(n^{d-1} \log n)$ steps. This would be quite practical, even for large n. In addition, the algorithm of [11] can be extended in a straightforward manner to determine whether there exists an undominated point in a weighted voting game with Euclidean preferences and arbitrary supramajoritarian voting rule.

6. Conclusions

It is a desirable feature of a model, like the spatial theory of voting, that it not only enlarge our understanding by supporting powerful characterizations, but that it allow us to answer operational questions, like "Is there an undominated point?" This was suggested by Hoyer and Mayer [10], who wrote

> At least one area for concern in attempting to fit a mathematical model to a sociopolitical environment and then generalizing the model is whether or not you eventually put yourself out of business. Certainly our principal interest is locating winning political strategies. If...we significantly limit our ability to locate optimal strategies, then our effort is, for the most part, wasted.

We have shown that it is NP-complete – hence computationally difficult in the worst-case – to recognize a winning (actually, non-losing) strategy. It is an empirical question whether the formal complexity of recognizing majority-rule equilibrium "significantly limits our ability to locate optimal strategies". Experience in other fields such as operations research and computer science confirms that NP-complete problems quickly become impractically time-consuming to solve as the size of the problem grows. It is widely thought that this difficulty is inherent, so that no conceivable algorithm can guarantee fast solutions for theses problems [9].

Of course it might be that difficult instances of the problem are rare. Indeed it has often been observed in the social choice literature that undominated points are highly unlikely. However, this sense is a casual interpretation of a careful theorem such as that of Rubinstein, who showed that, relative to a certain topology, the set of voter profiles for which an undominated point exists is a "nowhere dense set" [19]. Whether an undominated point exists in practice depends on the process by which the ideal points of the voters are generated. For example, if ideal points are generated by sampling a uniform distribution over some bounded region of policy space, then indeed dominant points will be rare. On the other hand, consider an alternative process that generates ideal points by sampling a uniform distribution to determine a position in policy space and then sampling another distribution to determine how many voters have their ideal points at that position. Such a process can be made to produce voter profiles for which dominant points (and perhaps difficult recognition problems) are quite likely.

In any event our results have interesting theoretical implications for voting theory. For example, all algorithms to construct voting cycles [14, 20] must require exponential work in the worst-case and so might be impractical. (If any such algorithm could be guaranteed to run in polynomial-time, then one could use it to test quickly whether a point is dominated. This would mean that $P = NP$, in contradiction to the belief of complexity theorists [9]).

Our results further suggest that, as a practical matter, it can be computationally infeasible to make strategic use of intransitivities in social choice, since finding an intransitivity is equivalent to proving the non-existence of an undominated point. Other researchers have tried to escape the apparent social chaos of intransitivities by suitably restricting the model (see [23] for a survey). However, our results suggest – without any restrictions on the model – that *majority-rule equilibrium can exist as a purely practical matter: it can be too difficult to find an alternative to overthrow the status quo!* (Of course we do not suggest that the process of proposing alternatives need cease; rather just that it can take so long to find preferred alternatives that, for all practical purposes, society is stable.)

Such practical considerations can suggest interesting tactics. For example, a party could use our $O(dn)$ algorithm to choose quickly its platform, leaving to the opposition the computationally difficult (NP-complete) problem of finding a better alternative. Our results imply that the opposition cannot do significantly better than enumerative (or probabilistic) search over policy space to find a better alternative.

Finally, we note that from one point of view our results are firmly within the tradition of previous work in social choice and welfare: The famous theorems of Arrow, Gibbard, Satterthwaite, Gärdenfors, the "chaos" theorems of McKelvey and Schofield are all "worst-case" results that say that inconvenient things can happen; indeed the possibility is endemic to social choice. To this list we add computational inconvenience.

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