THE CONTRIBUTION OF OBSERVATIONS OF SATELLITES TO THE DETERMINATION OF THE EARTH'S GRAVITATIONAL POTENTIAL

A. H. COOK

Standards Division, National Physical Laboratory, Teddington, Middlesex, England

(Received April 2, 1963)

Abstract. The history of the determination of the external gravitational potential of the Earth is sketched briefly. A discussion of the principles by which the potential may be derived from the observations of changes in the orbits of artificial satellites is followed by outlines of the principal theories and by detailed consideration of the formal differences between them that arise from differences in the ways that the orbits are described and it is shown that those formulae on which most of the numerical results depend are equivalent in the principal theories. The usual methods of treatment break down in certain special conditions and the analysis of these cases is also considered although they are not of great practical importance in the derivation of the potential; similarly a short account is given of the behaviour of a satellite having an orbital angular velocity commensurable with the spin angular velocity of the Earth. Methods by which satellites are observed are mentioned and the main numerical results on the external potential of the Earth are discussed critically. Finally the results are compared with those derived from observations of gravity on the surface of the Earth and the application of the results to problems of geodesy, of the physical state of the Earth and of the motion of the Moon are described.

1. Introduction

The external gravity field of the Earth has been the subject of intensive study from the time of NEWTON onwards. NEWTON himself was the first to attempt to relate the values of gravity as measured at the surface of the Earth to the motion of the Moon in her orbit about the Earth and it was he who first argued that the Earth must be flattened at the poles, a view which when challenged by CASSINI led to the series of measurements of geodetic arcs beginning with those in Lapland and Peru and continuing to the great arc between Dunkerque and Barcelona that established the correctness of NEWTON's view, that gave birth to the Metric system and that redound for ever to the credit of French science. Although NEWTON's arguments had been based in part on considerations of the distribution of matter inside the Earth, CLAIRAUT laid the foundations of modern theories of the Earth's field by showing that the external potential, the shape of the equipotential surface bounding the Earth and the value of gravity upon it, were all related in a way that was independent of the distribution of matter within the Earth and this theory was later extended by Sir G. DARWIN and CALLANDREAU to terms of the order of the square of the polar flattening and at the same time DARWIN showed how the flattening could be related to the moment of inertia about the polar axis. This last result has in recent years assumed considerable importance because when it is combined with the values of the elastic properties of the interior of the Earth derived from seismic studies, it enables the distribution of density as a function of radius to be estimated.

The actual Earth is not bounded by a spheroidal equipotential surface as supposed in this theoretical work and for purposes of accurate surveying and map making and for more detailed studies of the interior, account must be taken of the irregularities in the surface, in gravity and in the potential. Sir G. G. STOKES in a celebrated paper, first showed how this might be done and the theory has been greatly extended since his time, especially in the years before the first artificial satellites were launched. The problem is one of solving LAPLACE's equation given values of the potential on the surface of the Earth. Usually in such boundary value problems, values of the potential or of its derivatives are given on a surface of known form but in the problem of the Earth, the shape of the surface is not known, at least not with any precision, apart from measurements of potential itself which lie at the basis of spirit levelling. Thus the boundary value problem of the Earth's external field is to determine the form of the physical surface from measurements of both potential and gravity upon it, a problem of some complexity which has certainly not been explored in the same depth as the more familiar boundary value problems. There is moreover a great observational problem: the data, values of gravity and potential, have so far been measured at rather erratically distributed sites, the seas, over three-quarters of the Earth's surface, being but sparsely covered because of the technical difficulties of making observations upon them.

A great simplification was effected in these studies when it became possible to observe the external potential directly through its effects on the movements of artificial satellites. It had indeed been realised before any satellite was launched (BLITZER *et al.*, 1956) that the variation of potential with latitude, corresponding to the ellipticity of the meridian, the polar flattening, that is, would produce changes in the orbit of a satellite from which this part of the potential could be inferred*, but such studies were only properly initiated when the first Sputnik satellites were observed from England by KING-HELE and his collaborators and from Czechoslovakia by BUCHAR and his associates and evidence was obtained that the ellipticity could indeed be found in this way and that it was notably different from the value inferred from measurements of gravity on the surface. It was not long before these results were confirmed and extended and within two years of the launching of the first satellite, it was possible to find the polar flattening of the Earth with an accuracy some twenty times greater than it was possible to attain with surface gravity measurements, while subsequently it has been possible to derive other, much smaller components of the field that are of doubtful statistical significance in the surface gravity values.

With the results so far obtained, estimates of the shape of the Earth can be improved, knowledge of the physical state of the interior of the Earth extended and the way prepared for obtaining like knowledge of the Moon, but perhaps of as great importance is the notable simplification in the ways of thinking about the Earth's field that has come about as the result of being able to measure it directly instead of having to infer it from a difficult boundary value problem.

* The external potential of other planets may be derived from observations of their natural satellites.

The plan of this article is that the general principles by which the field can be derived from satellite motions are first discussed and then more detailed accounts are given of the principal theories, some reference being made to circumstances in which the theories may fail. The observations and their reduction and the results obtained

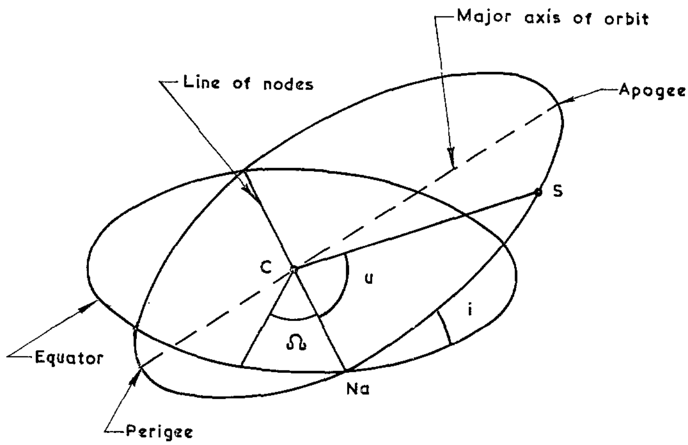


Fig. 1a. Geometry of orbit. Plane of orbit relative to equator. C = Centre of mass of Earth.

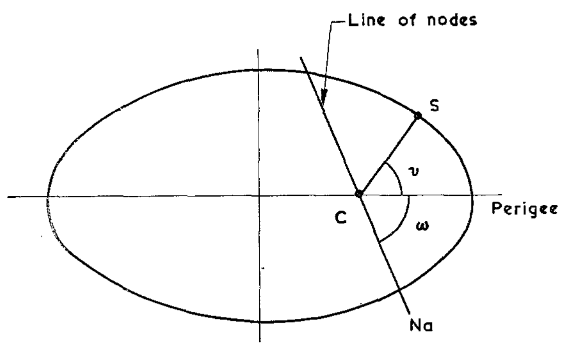


Fig. 1b. Geometry of orbit. Angles in plane of orbit. C = Centre of mass of Earth.

are then reviewed and finally some consideration is given to the significance and application of the results.

The gravity field of the Earth is very nearly that of a point mass, the largest departures, those due to the polar flattening, being about one part in a thousand while all others are of the order of one part in a million or less, and so the orbits of artificial satellites about the Earth are very nearly ellipses executed in a fixed plane with the centre of mass of the Earth at one focus. The main part of this article is, in one way or another, concerned with small changes in such orbits or in their orientation in space.

It will therefore be useful to conclude this section by definining some parameters of the elliptical orbit and other parameters that will be frequently used in the rest of the paper.

An elliptical orbit about a centre of attraction C is shown in Figure 1a. The orientation of the plane of the orbit in space is defined by reference to a fixed plane, the equator for satellites of the Earth. The two planes intersect along the *line of nodes* and the angle between them is called the *inclination*, i. If CX is some direction fixed in space and if N_a is the *ascending node*, the one at which the satellite passes from south to north across the equator, then the angle between CX and CN_a measured in the same direction as the motion of the satellite is called the *longitude of the node* and is denoted by ☊. If S is the position of the satellite, the angle SCN_a is called the *argument of latitude*, u, the name coming from the fact that the latitude of the satellite, β say, is given by

$$\sin\beta = \cos u \sin i\,.$$

Angles in the plane of the orbit are shown in Figure 1b. P is the position of *pericentre* (*perigee* for the Earth). The angular distance of the satellite from pericentre is called the *true longitude* or *true anomaly* and will be denoted by v in this article – there is a variety of usage. The angular position of pericentre measured from the ascending node is called the *longitude of pericentre* and is denoted by ω.

The parameters so far defined fix the direction of the satellite in relation to axes fixed in space. The remaining parameters concern the shape and size of the orbit. As usual the semi-major axis of the ellipse will be called a and the eccentricity, $[(a^2 - b^2)/a^2]$, will be called e. The semi-latus rectum, $a(1 - e^2)$, is usually denoted by p in celestial mechanics.

By Kepler's third law, the mean angular velocity of the satellite in its orbit, n, is given by

$$n^2 a^2 = \mu$$

where the law of force is $-\mu/r^2$.
The mean anomaly, M, is defined as nt.
The external potential of the Earth will be written in the form

$$-\frac{GM}{r}\left[1 - \sum_n \left(\frac{R}{r}\right)^n J_n P_n(\cos\theta) - \sum_n \sum_m \left(\frac{R}{r}\right)^n P_n^m(\cos\theta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]$$

where r is the radius vector from the centre of the Earth,
θ is the co-latitude, equal to $\frac{1}{2}\pi - \beta$
λ is the longitude
R is a radius giving the scale factor for distance. It is commonly taken to be the equatorial radius of the Earth (6378 km) and will be so understood when numerical values are given for the J coefficients but for some purposes it is more convenient to use the mean radius (6371 km).

M is the mass of the Earth and G the constant of gravitation, the product, GM being written as μ.

(Recommendations of Commission 7 of the International Astronomical Union).

$P_n(\cos\theta)$ is a LEGENDRE polynomial;

$$P_n(z) = \frac{1}{2_n^n!}\frac{\mathrm{d}_n}{\mathrm{d}_n^z}(z^{2-1})^{\frac{1}{2}}$$

and $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ for example; $P_n^m(\cos\theta)$ is an associated LEGENDRE polynomial:

$$P_n^m(z) = (z^2 - 1)^{\frac{1}{2}m}\frac{\mathrm{d}^m P_n(z)}{\mathrm{d}z^n}.$$

There are other definitions of $P_m(z)$ that differ by a normalising factor (see Appendix).

2. General Principles

Since the external gravitational potential of the Earth satisfies LAPLACE's equation, $\nabla^2 V = 0$, in the mass-free region outside the Earth, it has the following properties: (1) it is an harmonic function and so can be continued uniquely from one region to an adjacent one; (2) it is uniquely determined by the boundary values on the surface of the Earth; (3) outside the sphere of convergence that encloses all the matter of the Earth it may be developed as a convergent series of spherical harmonics with terms proportional to the inverse powers of the geocentric radius vector.

If the shape of the Earth were known and either gravity or potential were known at all points of the surface, then LAPLACE's equation could be solved throughout all the exterior space. Spherical harmonics could not be used to construct the solution because the Earth's surface, on which the boundary values are given, is not a sphere and numerical methods would have to be employed in the neighbourhood of the surface. Spherical harmonics can only be used when either gravity or potential is given at all points of a sphere enclosing all matter and if the potential on such a sphere enclosing the Earth can be calculated from the values on the physical surface of the Earth, then the field beyond that sphere can be developed in spherical harmonics. Such, in principal is the procedure for determining the external potential as a boundary value problem.

In fact the procedure is not at present practicable because the shape of the Earth's surface is not known independently of the potential on it (as found by geodetic spirit levelling) and therefore geodesists have been led to study a boundary value problem not met with elsewhere in potential theory, although there are analogous problems in fluid dynamics. That problem is to determine the form of the surface from measured surface values of both potential and gravity – it will be recalled that if both potential and gravity are given on a *known* surface, the boundary value problem is overdetermined. The question was first discussed by STOKES (1849) who considered an equipotential surface on which gravity was known; subsequently methods have been given for solving the more general case (MOLODENSKY, 1948, LEVALLOIS, 1957) and although the problem and its solution have not been fully studied, it is believed that the problem is correctly set and that a unique solution exists (ARNOLD, 1959). Presumably if that is so, LAPLACE's equation may be solved outside the surface.

The determination of the potential from observations of artificial satellites involves, by contrast, a direct measurement of the potential at points along the track of the satellite. Since the acceleration of the satellite is equal to the negative gradient of the potential, measurements of the acceleration should enable the potential to be determined. Such a direct procedure is for two reasons, not generally practical. First, the acceleration is predominantly due to the potential $-GM/r$ corresponding to a spherically symmetrical Earth of mass M, and the components of interest, those corresponding to the polar flattening of the Earth and other smaller departures from spherical symmetry, do not exceed one thousandth of the dominant term in the potential, in the face of which it would be difficult to determine them accurately. Secondly there are relatively large, rather irregular, accelerations due to air resistance which again mask the accelerations that it is desired to find. Furthermore, the direct measurement of acceleration would require a network of observatories with accurately known positions. It is true that such a network is now being built up but the dynamical studies of artificial satellites that so far have produced the bulk of information have not depended on it and the direct measurement of acceleration is not practical except in certain special cases.

Because the potential of the Earth is dominated by the part $-GM/r$, the orbit of a satellite is very nearly a Keplerian ellipse with one focus at the centre of mass of the Earth and the additional terms in the potential give rise to small departures from the ellipse. Certain of these departures increase steadily with time or vary with a period of many weeks and can therefore be measured accurately by simple observations continued over a long time, and the components of the potential that produce these departures can be determined very accurately. Other components give rise to departures that vary with a period of a day or less and they cannot be found accurately from simple observations. It will now be shown that the development of the potential in spherical harmonics is a natural one to adopt in the analysis of satellite orbits because the symmetry properties of the harmonics correspond to the distinct classes of perturbation of the orbit.

Let the potential be written in the form given at the end of Section 1.

The non-dimensional coefficients, J_n, C_{nm} and S_{nm}, describe the departures of the potential from spherical symmetry. J_2 is related to the moments of inertia of the Earth:

$$J_2 = \frac{C - \frac{1}{2}(A + B)}{MR^2}$$

where C is the moment about the polar axis and A and B are moments about perpendicular axes in the equatorial plane.

The terms of the series are symmetrical about the polar axis of the Earth if they are independent of the longitude and are proportional to the LEGENDRE coefficients

$P_n(\cos\theta)$, while otherwise there is no axial symmetry. Of the axi-symmetric terms, those with even order LEGENDRE coefficients are symmetrical about the equator and those with coefficients of odd order are anti-symmetric about the equator.

J is about 10^{-3} and all other coefficients are about 10^{-6} or less.

Now consider a satellite in an orbit about the Earth.

At any point in the orbit the force acting on the satellite may be resolved into rectangular components, S, T and W, of which S is directed to the centre of the Earth, T is perpendicular to S and in the plane of the orbit and W is perpendicular to the other two. (Figure 2). Then in general the parameters that describe the orbit are not constants because the orbit is not a constant ellipse and the rates of change of the parameters may be shown to be related to the components S, T and W as follows (see, for example, SMART, 1953, p. 221 and below in this Section)

$$\dot{a} = \frac{2}{n(1-e^2)^{\frac{1}{2}}}\left[Se\sin v + \frac{b^2}{ar}T\right]^*,$$

$$\dot{e} = \frac{(1-e^2)^{\frac{1}{2}}}{na}\left[S\sin v + T(\cos E + \cos v)\right],$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}}}Wr\cos u,$$

$$\dot{\Omega} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}Wr\sin\mu,$$

$$\dot{\omega} = 2\sin^2\tfrac{1}{2}i\cdot\dot{\Omega} + \frac{(1-e^2)^{\frac{1}{2}}}{nae}\left[-S\cos v + T\left(1+\frac{ar}{b^2}\right)\sin v\right].$$

These equations are a form of LAGRANGE's planetary equations, the derivation of which is given later in this section.

It will be seen that the sign of $\dot{\Omega}$ is the same as the sign of $W\sin u$. Suppose first that the potential is symmetrical about the equator so that W is directed always either towards or away from the equator so that it has one sign in the northern hemisphere and the opposite sign in the southern hemisphere. Sin u likewise has one sign in the northern hemisphere (positive) and the other sign in the southern hemisphere and so the sign of $\dot{\Omega}$ is always the same; on the average, therefore, the nodes move continuously in the one direction although at a variable speed because the product $Wr\sin u$ will vary with the position of the satellite in the orbit. It can easily be seen that the motion of perigee due to W is also on the average in the one direction if W is derived from a potential symmetrical about the equator. If, on the other hand, the potential is anti-symmetric about the equator, the product $Wr\sin u$ has different signs in the two hemispheres and there is no net change of Ω over a long period. The change over one

* b is the semi-minor axis of the orbit.

revolution of the satellite is however not necessarily zero because the magnitude of $Wr \sin u$ will in general be different in the two hemispheres. In the case of a satellite about the ellipsoidal Earth, perigee has a steady motion on account of the polar flattening and so conditions repeat after one revolution of perigee; in consequence an anti-symmetric component of the potential produces a component in the motion of the nodes, and also of perigee, with a period equal to the period of rotation of perigee. Such terms are known as *long period terms* and the steady motions are called *secular*.

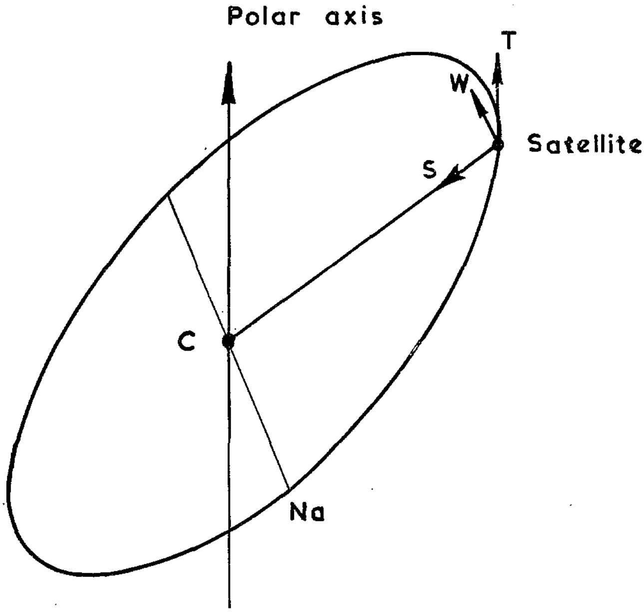


Fig. 2. Resolution of forces into orthogonal components.

So far the rotation of the Earth has been ignored and this is equivalent to supposing that the potential is independent of longitude. It can be seen that the effect of a term in the potential that depends on longitude will have a period of one day and in general will give rise to no secular or long periodic change in the orbit.

It will now be clear why spherical harmonics are so convenient for describing the potential, for their symmetry properties correspond exactly to the division of the potential according to the type of change, secular, long-periodic or short-periodic, in node and perigee.

In the remainder of this section some general results on the dynamics of orbits will be set out in preparation for the more detailed discussion of methods of solving the equations of motion to be presented in subsequent sections, and the problem of describing a changing orbit will also be considered.

In Cartesian co-ordinates, the equations of motion of a particle in a field of potential V are

$$\ddot{x} = -\frac{\partial V}{\partial n}$$

$$\ddot{y} = -\frac{\partial V}{\partial y}$$

$$\dot{z} = -\frac{\partial V}{\partial z}$$

or

$$\ddot{\mathbf{r}} = -\operatorname{grad} V.$$

One integral of these equations is the familiar integral of energy. Form the secular product of both sides with $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = -\dot{\mathbf{r}}\cdot\operatorname{grad} V.$$

Suppose only that V is constant in time so that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \dot{x}\frac{\partial V}{\partial x} + \dot{y}\frac{\partial V}{\partial y} + \dot{z}\frac{\partial V}{\partial z} = \dot{\mathbf{r}}\cdot\operatorname{grad} V.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}^2) = -\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

or denoting the kinetic energy by T,

$$T + V = \text{constant}.$$

The only condition is that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \dot{\mathbf{r}}\cdot\operatorname{grad} V$$

grad V but if the potential of the Earth depends on longitude, this condition is not satisfied in inertial co-ordinates and so in general the energy is not an integral invariant.

The other integrals that may be invariant are those of angular momentum. To obtain them form the vector product of each side of the equation of motion with $\mathbf{r}$:

$$\mathbf{r}\wedge\ddot{\mathbf{r}} = -\mathbf{r}\wedge\operatorname{grad} V.$$

Now

$$\ddot{x}_j x_i - \ddot{x}_i x_j = \frac{\partial}{\partial t}(\dot{x}_j x_i - \dot{x}_i x_j)$$

so that

$$\mathbf{r} \wedge \ddot{\mathbf{r}} = \frac{\partial}{\partial t}(\mathbf{r} \wedge \dot{\mathbf{r}})$$

and therefore

$$\frac{\partial}{\partial t}(\mathbf{r} \wedge \dot{\mathbf{r}}) = -\mathbf{r} \wedge \operatorname{grad} V.$$

It follows that the component of angular momentum in any direction is constant if the component of $\mathbf{r} \wedge \operatorname{grad} V$ in that direction is zero and that the total angular momentum is constant if grad V is parallel to $\mathbf{r}$.

The only solution of LAPLACE's equation that vanishes at great distances and the gradient of which is parallel to $\mathbf{r}$ is $1/r$ and so it is only for this potential that the total and all resolved components of the angular momentum are constant.

If the potential is symmetrical about an axis, as is nearly the case for the Earth, then grad V is in the plane of a meridian, $\mathbf{r} \wedge \operatorname{grad} V$ is perpendicular to the meridional plane, it has no component parallel to the axis of symmetry and the component of angular momentum parallel to the axis of symmetry is a constant of the motion. This is nearly true for the Earth.

These results may be obtained from a slightly different point of view by using spherical polar co-ordinates, r, θ, λ. The equations of motion are

$$\ddot{r} - r\dot{\theta}^2 - v\sin^2\theta \cdot \dot{\lambda}^2 = -\frac{\partial V}{\partial r},$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\lambda}^2 \sin\theta\cos\theta = -\frac{\partial V}{r\partial\theta},$$

$$\frac{1}{r\sin\theta} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(r^2\sin^2\theta \cdot \dot{\lambda}) = -\frac{\partial V}{r\sin\theta \cdot \partial\lambda}.$$

Now suppose that the potential is independent of λ. Then $\partial V/\partial\lambda$ is zero and $r^2 \sin^2 \theta. \dot{\lambda}$ is constant.

Now the components of the vector $\mathbf{r}$ are $(r, 0, 0)$ and those of $\dot{\mathbf{r}}$ are $(\dot{r}, r\dot{\theta}, r\dot{\lambda}\sin\theta)$ and therefore the angular momentum vector, $\mathbf{r} \wedge \dot{\mathbf{r}}$, has components $(0, r^2\dot{\lambda}\sin\theta, r^2\dot{\theta})$. Taking the z-axis to be that for which $\sin\theta = 0$, the z-component of angular momentum is

$$(r^2\,\dot{\lambda}\sin\theta)\,.\,\sin\theta,$$

and so is constant.

The magnitude of the angular momentum is $r^2(\dot{\lambda}^2\sin^2\theta + \dot{\theta}^2)^{\frac{1}{2}}$ and is constant only if $\sin\theta$ is constant in addition to $\dot{\lambda}$ being constant. But if $\sin\theta$ is constant, $\dot{\theta}$ is zero and we may transform to new co-ordinates in which $\theta = \pi/2$. In this new system, the angular momentum is $r^2\dot{\lambda}$ and so is a constant. Again, the total angular momentum is constant only if the potential is a function of r alone.

The orbit of a particle moving in a potential proportional to $1/r$ is an ellipse with one focus at the centre of attraction and for such an orbit the energy and the total angular momentum are constants. Since the orbits of particles about the Earth must be very nearly Keplerian ellipses, it is natural to look for a method of solution of the

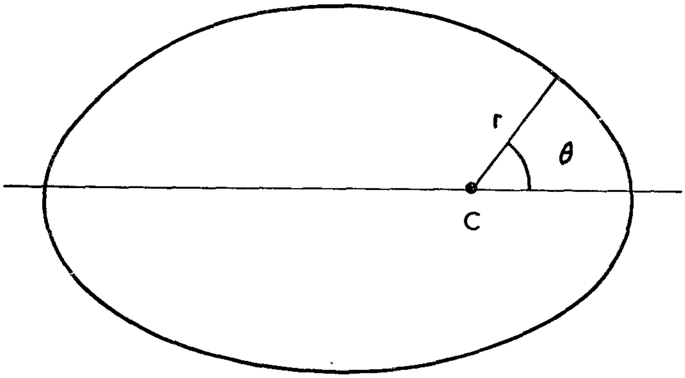


Fig. 3. Polar co-ordinates in plane of orbit.

equations of motion that makes use of the fact that energy and angular momentum are very nearly constants. The Hamiltonian formulation of the equations of motion provides such a method. Before showing how it may be applied to almost elliptical orbits, some results for the elliptical orbit must be stated.

Taking polar co-ordinates in the orbital plane with the centre at the centre of attraction (Figure 3), the equations of motion are

$$\ddot{\mathbf{r}} - r\dot{\theta}^2 = -\frac{\partial V}{\partial r} = -\frac{\mu}{r^2},$$

$$r^2\dot{\theta} = \text{const} = c_1 .$$

The last equation is the equation of angular momentum. It also says that the rate at which area is swept out by the radius vector of the particle is a constant ($\frac{1}{2}c_1$). But the mean rate at which area is covered is $4\pi\, ab/T$, where a is the semi-major axis, b is the semi-minor axis, and T is the period of revolution. Writing $\mathbf{n}$ for the mean orbital angular velocity, (mean motion) and remembering that $b = a(1 - e^2)^{\frac{1}{2}}$, where e is the eccentricity, it follows that

$$na^2(1 - e^2)^{\frac{1}{2}} = c_1 .$$

The solution of the two equations of motion is

$$r = \frac{c_1^2/\mu}{1 + e\cos(\theta - \omega)}$$

where ω is the angular distance of perigee from the zero of θ. The semi-latus rectum of the ellipse represented by this equation is c_1^2/μ and so

$$c_1^2 = \mu a(1 - e^2).$$

Thus

$$\mu = n^2 a^3.$$

Now the total energy of the system is $\mu/2a$ that is, $\frac{1}{2}n^2a^2$, and is a constant. Two of the invariants of the elliptical orbit are therefore

$$\mu/2a \qquad \text{and} \qquad \{\mu a(1 - e^2)\}^{\frac{1}{2}}.$$

The third follows from the fact that the projection of the angular momentum on to the equatorial plane is to be a constant.

Hence

$$\{\mu a(1 - e^2)\}^{\frac{1}{2}} \cos i = \text{const.}$$

In the Hamiltonian formulation of dynamics, the configuration of the system is described by generalised co-ordinates and corresponding generalised momenta, usually denoted by q_r and p_r respectively. The Hamiltonian F is the function $T - V$ and the equations of motion take the canonical form

$$\dot{q}_r = \frac{\partial F}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial F}{\partial q_r}$$

If, for example, the q_r are the Cartesian co-ordinates, x, y, z,

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \tfrac{1}{2}m(p_x^2 + p_y^2 + p_z^2); \\ F &= \tfrac{1}{2}m(p_x^2 + p_y^2 + p_z^2) - V \end{aligned}$$

and both q_r and p_r change with time.

It has been seen however that there are three quantities, two angular momenta and the energy, that are constant in the elliptical motion and it is therefore natural to seek a set of co-ordinates corresponding to these momentum variables. The means of doing so are provided by the Hamilton-Jacobi partial differential equation.

Suppose that q_r and p_r are a set of conjugate canonical variables and that Q_r and P_r are another conjugate set. Then it may be shown that the condition for the Q_r, P_r variables to be a canonical set is that

$$\sum_r (P_r \, \mathrm{d}Q_r - p_r \, \mathrm{d}q_r)$$

should be a perfect differential, dS (PLUMMER, 1960). The transformation from the one set to the other is then effected by equations such as

$$P_r = \frac{\partial S}{\partial Q_r}, \qquad p_r = -\frac{\partial S}{\partial q_r}.$$

Such transformations are known as *contact transformations.*

In general, S will be a function of time as well as of the other variables and in that case

$$P_r = -\frac{\partial K}{\partial Q_r}, \qquad Q_r = \frac{\partial K}{\partial P_r}$$

where

$$K = F + \frac{\partial S}{\partial t}.$$

Now K will be zero if

$$F\left(q_r, \frac{\partial S}{\partial q_r}\right) + \frac{\partial S}{\partial t} = 0,$$

and then P_r and Q_r will be constants, denoted by α_r and β_r respectively.

Thus we have the two systems of parameters in terms of which the dynamical system may be described, namely α_r and

$$\beta_r = -\frac{\partial S}{\partial \alpha_r},$$

and q_r and

$$p_r = \frac{\partial S}{\partial q_r}.$$

Since the α_r, β_r are constants, the problem posed has been solved provided S can be found.

If F does not contain t explicitly, a situation often met with in celestial mechanics,

$$S = -\alpha_n t + S'$$

and

$$F\left(q_r, \frac{\partial S'}{\partial q_r}\right) = \alpha_n.$$

This is the HAMILTON-JAKOBI differential equation in the form it takes when the Hamiltonian does not contain the time explicitly. The solutions for the constants and the equations of transformation are then:

$$-\beta_r = \frac{\partial S'}{\partial \alpha_r}, \qquad p_r = \frac{\partial S'}{\partial q_r}$$
$$t - \beta_n = \frac{\partial S'}{\partial \alpha_n}, \qquad p_n = \frac{\partial S'}{\partial q_n}$$

If, now a Hamiltonian F_0 is known for which a set of canonical constants can be found by solving the Hamiltonian-Jacobi equation, and if the Hamiltonian F of a perturbed system can be written in terms of these constants, then in the perturbed motion, the constants will change at rates that are given by

$$\dot{\alpha}_r = \frac{\partial K}{\partial \beta_r} \qquad \dot{\beta}_r = -\frac{\partial K}{\partial \alpha_r},$$

where

$$K = F - F_0$$

and

$$F_0 + \frac{\partial S}{\partial t} = 0.$$

In practice, the HAMILTON-JACOBI equation must be solved by separation of the variables in such a way that S is the sum of components each a function of one co-ordinate alone, for this corresponds to decomposition of momentum into three perpendicular components.

The form of the potential for which such separation is possible in spherical polar co-ordinates will now be derived. In this system,

$$F = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\lambda}^2) - V(r,\theta,\lambda),$$

assuming unit mass,

$$p_r = \dot{r},$$
$$p_\theta = r^2\dot{\theta},$$
$$p_\lambda = r^2 \sin^2\theta\,\dot{\lambda},$$

and

$$F = \tfrac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\lambda^2\right) - V(r,\theta,\lambda).$$

The HAMILTON-JACOBI equation is therefore

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \lambda}\right)^2 - 2V = \text{const.}$$

Now suppose that $S = S_r + S_\theta + S_\lambda$
where S_r is a function of r only, and similarly for the other terms. Accordingly

$$S_r'^2 + \frac{1}{r^2}S_\theta'^2 + \frac{1}{r^2\sin^2\theta}S_\lambda'^2 - 2V = \text{const} = 2\alpha_1.$$

It follows that S_λ' can only be a function of r and θ whereas it is required to be a function of λ alone. It must therefore be a constant:

$$S_\lambda' = \alpha_3$$

and so

$$r^2\sin^2\theta\,S_r'^2 + \sin^2\theta\,S_\theta'^2 - 2Vr^2\sin^2\theta = 2\alpha_1 r^2\sin^2\theta - \alpha_3^2,$$

or

$$r^2 S_r'^2 + S_\theta'^2 - 2Vr^2 = 2\alpha r^2 - \alpha_3/\sin^2\theta.$$

This equation can be separated if

$$V = -\frac{1}{r^2}[f(r) + g(\theta)],$$

for then

$$r^2 S_r'^2 + 2f(r) = 2\alpha_1 r^2 + \alpha_2^2,$$

and

$$S_\theta'^2 + 2g(f) = \alpha_2^2 - \frac{\alpha_3^2}{\sin^2\theta}.$$

If V is to be a solution of LAPLACE's equation, the only possible functions are

$$f(r) = \mu r,$$
$$g(\theta) = k P_1(\cos\theta),$$

and then

$$S_r'^2 - \frac{2\mu}{r} = 2\alpha_1 + \frac{\alpha_2^2}{r},$$

$$S_\theta'^2 + 2kP_1(\cos\theta) + \frac{\alpha_3^2}{\sin^2\theta} = \alpha_2^2.$$

It will be seen that once again the conclusion is that constants of the motion can exist only if the potential is of the form $1/r$, for the harmonic P_1 (cos θ) merely corresponds to a co-ordinate displacement along the axis of symmetry, as in fact the earlier discussion also shows.

Separability in another co-ordinate system will be discussed later (Section 3.4).

If $k = 0$,

$$(S_\lambda')^2 = \alpha_3^2,$$
$$(S_\theta')^2 = \alpha_2^2 - \alpha_3^2/\sin^2\theta,$$

and

$$(S_r')^2 = 2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r},$$

and so

$$S = \int_{r_0}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{\frac{1}{2}} dr + \int_0^\theta \left(\alpha_2^2 - \alpha_3^2/\sin_2^2\theta\right)^{\frac{1}{2}} d\theta + \alpha_3\lambda.$$

The solutions for the constants β_r are then:

$$t - \beta_1 = \frac{\partial S}{\partial \alpha_1} = \int_{r_0}^{r}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{-\frac{1}{2}} dr,$$

$$-\beta_2 = \frac{\partial S}{\partial \alpha_2} = -\int_{r_0}^{r}\frac{\alpha_2}{r^2}\left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{-\frac{1}{2}} dr + \int_0^\theta \alpha_2\left(\alpha_2^2 - \frac{\alpha_3^2}{\sin^2\theta}\right)^{-\frac{1}{2}} d\theta,$$

$$-\beta_3 = \frac{\partial S}{\partial \alpha_3} = \lambda - \int_0^\theta \frac{\alpha_3}{\sin^2\theta}\left(\alpha_2^2 - \frac{\alpha_3^2}{\sin^2\theta}\right)^{-\frac{1}{2}} d\theta.$$

The geometrical and dynamical significance of the constants will now be developed.

Let r_0 be the radius vector at perigee. Then $t - \beta_1$ is zero at perigee, so β_1 is the time of passage through perigee.

Now since

$$\dot{r} = \frac{\partial S}{\partial r},$$

$$\dot{r}^2 = \frac{2\mu}{r} - \frac{\alpha_2}{r^2} + 2\alpha_1,$$

and hence $\dot{r}$ may be written as

$$2\alpha_1 (r - r_1)(r - r_2)/r^2 .$$

But $\dot{r}$ is zero at perigee and apogee when $r = a(1 - e)$ and $a(1 + e)$ respectively, so that

$$r_1 = a(1 - e), \quad r_2 = a(1 + e).$$

Thus

$$\mu = -2a\alpha_1 \qquad \text{or} \qquad \alpha_1 = -\mu/2a$$

and

$$\alpha_2 = \{\mu a (1 - e^2)\}^{\frac{1}{2}}.$$

Now let $\alpha_3/\alpha_2 = \cos i$.

Then

$$\left.\begin{aligned} -\beta_2 &= f_1(r) + \sin^{-1}(\sin\theta/\sin i) \\ -\beta_3 &= \lambda - \sin^{-1}(\tan\theta/\tan i) \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \sin\theta &= \sin i \sin\{f_1(r) + \beta_2\} \\ \tan\theta &= \sin i \sin(\lambda + \beta_3). \end{aligned}\right\}$$

The last equation shows that the orbit lies in a plane making an angle i with the equatorial plane and that β_3 is the longitude of the node.

The equation for $\sin\theta$ shows that $f_1(r) - \beta_2$ is the angle between the radius vector and the line of nodes, and since $f_1(r)$ vanishes at perigee, β_2 is the angle ω between node and perigee.

Thus the constants β_1, β_2 and β_3 have been identified with elements of the elliptic orbit and it is easy to see that α_1 is the total energy, α_2 the total angular momentum and α_3 the angular momentum about the polar axis.

The complete set is then

$$\begin{aligned} \alpha_1 &= -\mu/2a & \beta_1 &= \tau \\ \alpha_2 &= \{\mu a(1 - e^2)\}^{\frac{1}{2}} & \beta_2 &= -\omega \\ \alpha_3 &= \{\mu a(1 - e^2)\}^{\frac{1}{2}} \cos i & \beta_3 &= -\Omega \end{aligned}$$

τ is the time of passage through perigee.

It is now seen that not only do the longitudes of the node and perigee have an observational importance in that they can readily be observed and secular changes in them can be determined with high accuracy, but in addition they are of particular importance in the dynamics of the elliptical orbit.

Instead of the quantity τ it is sometimes more convenient to use the mean anomaly $M = n(t - \tau)$. The resulting set of constants is due to DELAUNY

$$\begin{array}{ll} l = M & L = (\mu a)^{\frac{1}{2}} \\ g = \omega & G = \{\mu a(1-e^2)\}^{\frac{1}{2}} \\ h = \Omega & H = \{\mu a(1-e^2)\}^{\frac{1}{2}}\cos i\,. \end{array}$$

Lastly when the eccentricity is very small so that perigee cannot be defined, or when the inclination is very small and the node cannot be defined, the following elements are useful:

$$\begin{array}{rl} l + g + h & L \\ g + h & G - L \\ h & H - G\,. \end{array}$$

In the general problem of the orbit of a satellite about the actual Earth, the potential differs slightly from $-GM/r$ and the elements which are constants for the $1/r$ potential vary slightly in the actual case. The actual Hamiltonian may be written as

$$F = F_0 + F_1$$

where

$$F_0 = T + \mu/r\,,$$

and F_1 is the part of the Hamiltonian representing the difference of the actual potential from μ/r.

Then since

$$\frac{\partial F_0}{\partial G} = \dot{g} = 0\,, \qquad \frac{\partial F_0}{\partial g} = -\dot{G} = 0\,, \text{etc.},$$

the equations of motion are

$$\dot{g} = \frac{\partial F_1}{\partial G}\,, \qquad \dot{G} = -\frac{\partial F_1}{\partial g} \qquad \text{and so on.}$$

This general procedure was followed by DELAUNY in his theory of the Moon and BROUWER (1959) has used it in his theory of satellite motion. In common with all methods that start from the exact elliptical orbit, it has the disadvantage that the perturbations that have to be calculated are not very small and it would be more convenient if an exact solution could be found closer to the actual orbit. It has been seen that in spherical polar co-ordinates, there is no other potential that leads to such a solution if it is to satisfy LAPLACE's equation but by dropping that requirement GARFINKEL (1959) has obtained an exact solution closer to the observed orbits.

The Hamiltonian equations of motion may be transformed into equations that give the rates of change of the elements (a, e, ε, Ω, ω, i) of the elliptical orbit. These elements are not a canonical set of variables but have the advantage that they are more nearly the quantities that are observed. The equations for the rates of change are due to LAGRANGE and have been extensively used in the study of artificial satellites. They of course express the departures from an orbit obtained with a $1/r$ potential and therefore involve the difference of the actual potential from this form, expressed as a function of the elliptical elements, just as the difference must be expressed as a function of the canonical constants when the Hamiltonian equations of motion are used.

Suppose the solution to be expressed in elements ε_i of the elliptical orbit. These elements are functions of the canonical constants, (α_r, β_r).
Hence

$$\dot{\varepsilon}_i = \sum_r \frac{\partial \varepsilon_i}{\partial \alpha_r} \dot{\alpha}_r + \sum_r \frac{\partial \varepsilon_i}{\partial \beta_r} \dot{\beta}_r .$$

Expressing this in terms of the function

$$K = H - H_0 ,$$

$$\dot{\varepsilon}_i = \sum_r \left(\frac{\partial \varepsilon_i}{\partial \alpha_r} \cdot \frac{\partial K}{\partial \beta_r} - \frac{\partial \varepsilon_i}{\partial \beta_r} \cdot \frac{\partial K}{\partial \alpha_r} \right).$$

But K is also a function of the ε_j:

$$\dot{\varepsilon}_i = \sum_r \sum_j \left(\frac{\partial \varepsilon_i}{\partial \alpha_r} \cdot \frac{\partial K}{\partial \varepsilon_j} \cdot \frac{\partial \varepsilon_j}{\partial \beta_r} - \frac{\partial \varepsilon_i}{\partial \beta_r} \cdot \frac{\partial K}{\partial \varepsilon_j} \cdot \frac{\partial \varepsilon_j}{\partial \alpha_r} \right)$$

$$= \sum_j \{\varepsilon_i , \varepsilon_j\} \cdot \frac{\partial K}{\partial \varepsilon_j} ,$$

where $\{\varepsilon_i, \varepsilon_j\}$ is known as a *Poisson bracket*:

$$\{\varepsilon_i , \varepsilon_j\} = \sum_r \left(\frac{\partial \varepsilon_i}{\partial \alpha_r} \cdot \frac{\partial \varepsilon_j}{\partial \beta_r} - \frac{\partial \varepsilon_i}{\partial \beta_r} \cdot \frac{\partial \varepsilon_j}{\partial \alpha_r} \right).$$

The equations for the rates of change of an elliptic orbit are known as *Lagrange's equations* and may be derived in a variety of ways (e.g. PLUMMER, 1960, p. 146 *f.* SMART, 1953, p. 55 ff.). Here it will be shown how they follow simply from the Hamiltonian equations.

With the canonical set

$$\begin{aligned} L &= (\mu a)^{\frac{1}{2}} \quad , & L &= M\,; \\ G &= \{\mu a(1-e^2)\}^{\frac{1}{2}} \quad , & g &= \omega\,; \\ H &= \{\mu a(1-e^2)\}^{\frac{1}{2}} \cos i\,, & h &= \Omega\,; \end{aligned}$$

and a potential $\mu/r + R$,
we have, for example,

$$\dot{\Omega} = -\frac{\partial R}{\partial H}.$$

But

$$H = G \cos i\,,$$

and so

$$\frac{\partial}{\partial H} = -\frac{1}{G \sin i}\frac{\partial}{\partial i}.$$

Also

$$G = na^2(1-e^2)^{\frac{1}{2}},$$

and therefore

$$\dot{\Omega} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}} \sin i}\frac{\partial R}{\partial i}.$$

Again,

$$\cos i = \frac{H}{G}$$

and

$$\sin i \frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{1}{G}\dot{H} + \frac{H}{G^2}\dot{G} = -\frac{1}{G}\frac{\partial R}{\partial \Omega} + \frac{H}{G^2}\frac{\partial R}{\partial \omega},$$

so that

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cos i}{na^2(1-e^2)^{\frac{1}{2}} \sin i}\frac{\partial R}{\partial \omega} - \frac{1}{a\,n^2(1-e^2)^{\frac{1}{2}} \sin i}\frac{\partial R}{\partial \Omega}.$$

Next since

$$L = (\mu a)^{\frac{1}{2}},$$

$$\frac{\mathrm{d}L}{\mathrm{d}a} = \tfrac{1}{2}\mu^{\frac{1}{2}}a^{-\frac{1}{2}}.$$

Thus

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}L}\cdot\dot{L} = \frac{\mathrm{d}a}{\mathrm{d}L}\cdot\frac{\partial R}{\partial M} = \frac{2}{na}\frac{\partial R}{\partial M}.$$

For $\dot{e}$, we have $(1 - e^2)^{\frac{1}{2}} = G/L$ and therefore

$$\frac{e\dot{e}}{(1-e^2)^{\frac{1}{2}}} = \frac{1}{L}\frac{G}{L}\dot{L} - \frac{\dot{G}}{L}$$

$$= \frac{1}{L}(1-e^2)^{\frac{1}{2}}\frac{\partial R}{\partial M} - \frac{1}{L}\frac{\partial R}{\partial \omega},$$

or

$$\dot{e} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e}\frac{\partial R}{\partial \omega}.$$

The remaining equations are more readily derived by using the Poisson brackets. Thus

$$\dot{\omega} = \sum_i \{\omega, \varepsilon_i\}\frac{\partial R}{\partial \varepsilon_i}$$

and the only non-zero Poisson brackets are

$$\{\omega, e\} = (1-e^2)^{\frac{1}{2}}/e\mu a$$

and

$$\{\omega, i\} = -\cos i/\{\mu a(1-e^2)\}^{\frac{1}{2}}\sin i$$

Hence

$$\dot{\omega} = \frac{(1-e^2)^{\frac{1}{2}}}{na^2 e}\cdot\frac{\partial R}{\partial e} - \frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}}\cdot\frac{\partial R}{\partial i}.$$

Lastly, in a similar way,

$$\dot{M} = n - \frac{1-e^2}{na^2 e}\cdot\frac{\partial R}{\partial e} - \frac{2}{na}\cdot\frac{\partial R}{\partial a}.$$

Although the Hamiltonian methods give the deepest insight into the dynamics of the motion of a satellite, it is not necessary to use them in order to derive an orbit and it is possible to proceed to integrate the equations of motion in suitable co-ordinate systems as they stand. KING-HELE (1958) has used spherical polar co-ordinates and as will be seen (Section 3.1), his method has some similarity to the lunar theory of DE PONTECOULANT, who however used cylindrical co-ordinates. The most effective lunar theory is that of HILL and BROWN who used a set of Cartesian co-ordinates rotating with the Sun and this also has been applied to the problem of artificial satellites (BROUWER, 1958).

It is convenient to point out here why the theories of the Moon, although very similar to those of artificial satellites, cannot be applied directly to that problem. In the first place, the main disturbing function in the theory of the Moon is the attraction of the Sun and a principal problem is to determine the constant term in the Moon's semi-major axis arising therefrom. There is no similar problem in artificial satellite theory where the major part of the disturbing function is due to the ellipticity of the figure of the Earth and the effect of the Sun (and of the Moon) is very small. The second difference is that in the lunar problem the inclination and eccentricity of the orbit are both small whereas those of the orbits of artificial satellites cover a wide range of values, including some that are singular. For both these reasons rather different approximations have to be used in the theories. Another difference of some interest is that the lunar problem, being a three-body problem never admits an exact solution, whereas the artificial satellite problem can, in certain cases, be solved exactly in terms of analytical functions.

It has already been emphasised that if the Hamiltonian equations of motion are formulated in spherical polar co-ordinates, then the only solution of LAPLACE's equation that allows the HAMILTON-JACOBI equation to be solved by separation of the variables and a set of canonical *constants* to be derived, is the potential μ/r, and the principal additional term in the potential,

$$-\mu J_2 \frac{R^2}{r_3} P_2(\cos\theta)$$

has to be treated as a perturbation. But if oblate confocal spheroidal co-ordinates are used, then VINTI (1959) has shown that the HAMILTON-JACOBI equation is separable for a potential that can incorporate the whole of the $P_2(\cos\theta)$ term. The solution is obtained in elliptic functions and in the particular case of an equatorial orbit it is fairly straightforward to obtain the exact solution, (KING-HELE, 1958).

The overall result of theories of the motion of artificial satellite is that the node and perigee change steadily, that long and short periodic fluctuations are superimposed on the steady changes, that the steady, or secular, rates of change may be written in the form

$$\dot{\Omega}_s = a_2 J_2 + a_4 J_4 + a_6 J_6 + \ldots + a_{22} J_2^2\,;$$
$$\dot{\omega}_s = b_2 J_2 + b_4 J_4 + b_6 J_6 + \ldots + b_{22} J_2^2\,;$$

and that the long period changes may be written as similar series.

The coefficients, a_n, b_n ... in these series are functions of the elements of the orbit, in particular of the inclination of the orbit to the equator, and it follows that if observations can be made on orbits with many different values of inclination, it should be possible to estimate numerical values for at least some of the parameters J_n. This is the basis of almost all the studies of the Earth's external gravity field that have been made using observations of the orbits of artificial satellites.

However the gravitational attraction of the Earth is not the only force acting on an artificial satellite and for the full accuracy of the observations to be exploited, corrections must be applied to the observed motions to allow for the other forces to which the satellite is subject. Of these, the largest is that due to the drag of the atmosphere but it can to a large extent be ignored because it acts almost entirely along the tangent to the orbit. The main effect of the drag is on the semi-major axis and the eccentricity of the orbit, both of which decrease, and it has no first order effect on the motion of the node or inclination. Second order effects do in fact occur because the drag is not exactly perpendicular to the radius vector in elliptical orbits and because winds in the atmosphere may give rise to small non-tangential components (G. E. COOK, 1961).

The other forces are conservative and Hamiltonian theory may be applied; in fact because the forces are small it is convenient to use the Lagrangian equations for the variation of the elliptical elements. All satellites are subject to the attraction of the Sun and the Moon which in general give rise to secular and long-periodic terms in the motions of node and perigee. The effects are greater the greater the distance of the satellite from the Earth and there is no difficulty in principle in their calculation to the first order (G. E. COOK, 1962).

Radiation pressure from the Sun can be treated in a rather similar way to the luni-solar attraction. Again it is more important for the more distant satellites, and like atmospheric drag, it is greater the greater the ratio of the area of the satellite to the mass. (BRYANT, 1961).

Lastly, the effect of using relativistic equations of motion should be mentioned: the correction is very small, about 0.005 deg/y in the motion of perigee, and may be ignored. (KING-HELE, 1958).

To illustrate the simple first order use of the Lagrangian and Hamiltonian equations of motion and to show the form of the dependence of the motion of the node and perigee on inclination for the $P_2(\cos\theta)$ term in the potential, the secular terms will now be derived.

Take first the secular motion of the node due to the J_2 term in the potential, for which the additional part of the Hamiltonian is

$$F_1 = \mu J_2 \frac{R^2}{r^3} \cdot P_2(\cos\theta).$$

The use of canonical variables will be illustrated and therefore the disturbing function must be put in terms of the canonical constants of the ellipse. Now (Figure 1a)

$$\cos\theta = \sin u \sin i$$

and

$$\cos i = \frac{H}{G}$$

so that

$$P_2(\cos\theta) = \tfrac{1}{2}\left\{3\sin^2 u\left(1 - \frac{H^2}{G^2}\right) - 1\right\}.$$

Hence

$$\dot{\Omega} = \dot{h} = -\frac{\partial H_1}{\partial H} = \frac{3}{2}\mu J_2 \frac{R^2}{r^3}\frac{H}{G^2}$$

in the mean, since the mean value of $\sin^2 u$ is $\frac{1}{2}$.

Also,

$$\frac{1}{r^3} = \frac{(1 + e\cos v)^3}{a^3(1 - e^2)^3},$$

the mean value of which is

$$\frac{1 + \frac{3}{2}e^2}{a^3(1 - e^2)^3}.$$

Lastly $\mu^2 = na^3$ where n is the mean motion of the satellite and so the average value of $\dot{\Omega}$ is

$$\tfrac{3}{2} n J_2 \left(\frac{R}{a}\right)^2 \cos i$$

ignoring terms of order e^2.

The use of the Lagrangian equations will be illustrated by deriving the secular motion of perigee. The appropriate equation is (see above)

$$\dot{\omega} = \frac{(1 - e^2)^{\frac{1}{2}}}{na^2 e} \cdot \frac{\partial R}{\partial e} - \frac{\cot i}{na^2(1 - e^2)^{\frac{1}{2}}} \frac{\partial R}{\partial i},$$

where the disturbing function has been written as R. R must in this instance be expressed in terms of the elliptic elements. As before

$$\cos\theta = \sin u \sin i$$

and

$$\frac{1}{r^3} = \frac{(1 + e\cos v)^3}{a^3(1 - e^2)^3}.$$

Hence

$$\frac{\partial R}{\partial i} = \tfrac{3}{2}\mu J_2 \frac{R^2}{a^3} \cdot \frac{(1 + \frac{3}{2}e^2)}{(1 - e^2)^3} \sin i \cos i,$$

taking the mean value of sin u.

Again

$$\frac{\partial R}{\partial e} = \frac{\mu J_2 R^2}{a^3}(\tfrac{3}{2}\sin^2 i \cdot \sin^2 u - \tfrac{1}{2})\frac{\partial}{\partial e}\frac{1 + 3e\cos v + 3e^2\cos^2 v + e^3\cos^2 v}{(1 - e^2)^3}$$

$$= \frac{3\mu J_2 R^2}{a^3}(1 - e^2)^2(1 + \tfrac{1}{2}e^2)(\tfrac{3}{4}\sin^2 i - \tfrac{1}{2})$$

on taking mean values of trigonometrical functions.

Thus the mean rate of change is

$$\dot{\omega} = \frac{\mu J_2 R^2}{n a^5} 3(1-e^2)^{\frac{5}{2}}(1+\tfrac{1}{2}e^2)(\tfrac{3}{4}\sin^2 i - \tfrac{1}{2}) - \frac{3\mu J_2 R^2}{2 n a^5 (1-e^2)^{\frac{3}{2}}}\cos^2 i$$

$$= \tfrac{3}{4} n J_2 \left(\frac{R}{a}\right)^2 (1 - 5\cos^2 i)$$

when e^2 can be neglected.

These results show, as was said before, that the dependence of ☊̇ and $\dot{\omega}$ on J_2 are given by functions of the parameters of the orbit and it is easily seen that the most important factor is the function of $\cos i$. ☊̇ simply increases steadily as i decreases with a maximum value of 9°.5 per day for a close satellite on the equator, but $\dot{\omega}$ has a maximum of 19° per day for equatorial orbits and goes through zero when $\cos^2 i = 1/5$, that is, at 63°.45. The behaviour of a satellite in an orbit with this critical inclination has some interesting features that will be mentioned below (Section 4).

From a purely algebraical point of view, the mean values of ☊̇ and $\dot{\omega}$ differ from zero because the mean value of $\sin^2 u$ is not zero but $\frac{1}{2}$. All even order spherical harmonics contain even powers of $\sin u$ which have non-zero mean values and so give rise to non-zero rates of change and secular motions of node and perigee. Spherical harmonics of odd order, on the other hand, containing odd powers of $\sin u$ have zero mean values and they give rise to values of ☊̇ and $\dot{\omega}$ with periods of the revolution of perigee.

A question that has given some difficulty is the definition of the elements of the elliptic orbit when the actual orbit is not an exact ellipse. Dynamically, the canonical constants L, G, H, l, g, h, are defined by the ellipse that has the same radius vector and velocity vector as the actual satellite at any instant since physically it is products of these two vectors that give the energy and angular momentum constants and so if the vectors are to be specified by elliptical elements, the elements must be those appropriate to each point of the path and in general will change from point to point. The elements that correctly describe the actual radius and velocity vectors are called *Osculating Elements*. They vary continuously round a non-elliptical orbit. For example, i is the inclination to the equator of the plane defined by $\mathbf{r}$ and $\dot{\mathbf{r}}$ and in general this plane rotates about $\mathbf{r}$, leading to changes in i.

Because the osculating elements are changing continuously, it is not convenient to express the rates of change of the elements in terms of them. Instead, values of the elements at some specific point in the orbit (usually the ascending node) may be chosen, or some average value of the element may be used.

The distinctions are not important in small terms but they do affect the terms involving J_2, leading to different forms of the J_2^2 terms in ☊̇ and $\dot{\omega}$. Great care must therefore be taken in using the various theories of satellite motion and also in making sure that the observation procedure and reductions do in fact give the parameters employed in the theory (MASSEVIČ, 1961).

3. Theories of the Motion of a Satellite About an Axi-Symmetric Planet

In this section, accounts are given of the more important theories describing the motion of a satellite in an axi-symmetric potential field. This is the most important topic in the study of satellites because, as has been pointed out in Section 2, it is the coefficients of the zonal harmonics in the potential that can most readily and most accurately be found from observation; it is also the topic that has been most thoroughly studied theoretically. The theories to be described in this section are those of the greatest theoretical interest or those that have been most extensively used in the analysis of observations. Questions of singularities are deferred to Section 4.

The purpose of these theories discussed is to obtain explicit literal expressions for the effects of each harmonic component, the most convenient form into which to cast the theory when values of the harmonics are to be derived from the observations. It is, however, possible to develop semi-numerical theories in which the object is to derive the co-ordinates of the satellite numerically as a function of time. One such theory is that developed by HERGET and MUSEN (1958) on the basis of HANSEN's theory of the Moon. O'KEEFE *et al.* (1959) have shown how it is possible to apply this theory also to determine zonal harmonics by proceeding by numerical iteration until agreement is reached between the observed and calculated positions of a satellite.

3.1 INTEGRATION OF THE EQUATIONS OF MOTION IN POLAR CO-ORDINATES KING-HELE'S THEORY

KING-HELE's theory (KING-HELE, 1958) is important because it was the first that was used to derive exact information from satellite orbits, and has continued to be used subsequently for that purpose. It is stated in rather different terms from the theories that use LAGRANGE's equations and it will therefore be important to compare its results with those of other theories. The distinguishing characteristics of KING-HELE's method are the way in which the orbit is specified and the fact that the variation of the radius vector is obtained directly and in these two respects it has some similarity to DE PONTECOULANT's lunar theory in which, however, instead of spherical polar co-ordinates, cylindrical co-ordinates are used.

An important point of KING-HELE's theory is the specification of the inclination of the orbit. It will be clear from the part played by the angular momentum vector in Hamiltonian theory, that the significant quantity dynamically is the instantaneous inclination of the plane containing the radius vector and the velocity vector of the satellite, that is the inclination of the osculating ellipse. It is also evident that in general this inclination will vary with the position of the satellite in the orbit and an important question that will be discussed in Section 3.2 is the definition of a mean inclination, and other elements, from which short periodic variations are removed. KING-HELE, on the other hand, takes a reference plane with a fixed inclination to the equator and forces it to rotate so that it shall always contain the satellite. This means that the rotation of this plane will not be the same as that of the osculating plane and therefore that the corresponding motions of the nodes will differ. Another difference from conventional celestial mechanics is that the angular position of perigee is measured from the position of maximum latitude instead of from the node.

The geometry is shown in Figure 4. Π is the plane containing the centre of the Earth, O, and the satellite S. $Oxyz$ are axes fixed in direction, Oz being directed northwards along the polar axis. $Ox'\,y'\,z$ are rotating axes. Oz coincides with Oz in the fixed system, while Ox' is the direction in which Π cuts the plane of the equator. The angle xOx' is denoted by Ω measured in the opposite direction to the motion of the satellite. The, fixed, inclination of the plane Π is called α. The spherical polar

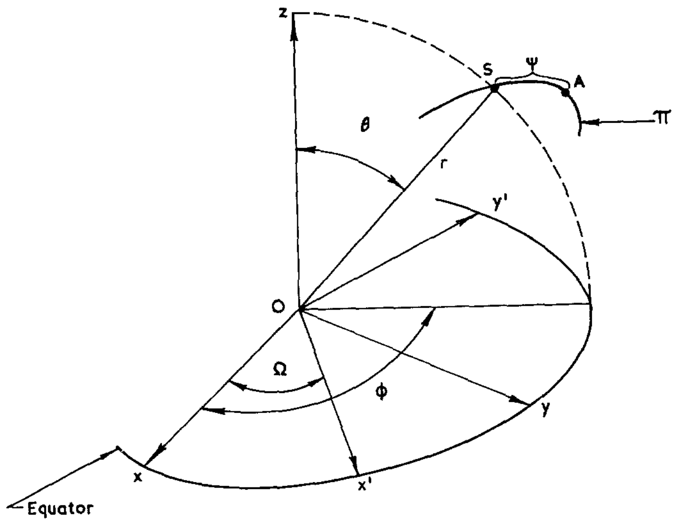


Fig. 4. Satellite geometry – KING-HELE's parameters.

co-ordinates of the satellite, (r, θ, ϕ) are as shown in the Figure, measured relative to the fixed axes. A is the point of maximum latitude of the satellite, β is the angle between this point and perigee and ψ is the angle between A and S.

Then:

$$\cot\theta = \tan\alpha\sin(\phi + \Omega),$$
$$\cos\psi = \operatorname{cosec}\alpha\cos\theta,$$
$$\sin\psi = -\sin\theta\cos(\phi + \Omega).$$

The equations of motion are

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = \frac{\partial V}{\partial r},$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2 = \frac{1}{r}\frac{\partial V}{\partial\theta},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\sin^2\theta\dot{\phi}) = \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}.$$

Writing

$$V = \frac{\mu}{R}\left\{\frac{R}{r} - \left(\frac{R}{r}\right)^3 J_2 P_2(\cos\theta) - \left(\frac{R}{r}\right)^5 J_4 P_4(\cos\theta)\right),$$

the equations become

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = -\frac{\mu^2}{r^2} + \frac{3\mu R^2}{r^4} J_2 P_2(\cos\theta) + \frac{5\mu R^4}{r^6} J_4 P_4(\cos\theta)$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r_2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2 = \frac{\mathrm{d}}{\mathrm{d}t}(r^2\sin^2\theta\dot{\phi}) = 0.$$

If the potential is just μ/r, the solution is

$$\frac{1}{r} = \frac{1}{p}(1 + e\cos(\psi - \beta)),$$

where e and β are constants, and the object of the theory is to find a solution of the form

$$\frac{1}{r} = \frac{1}{p}\{1 + e\cos(\psi - \beta) + J_2 v_1 + J_2 e v_2 + 0\,(J_2^2)\}$$

where v_1 and v_2 are functions to be found. The work is restricted to orbits with eccentricity not greater than 0.04; terms up to the order of the fourth power of the eccentricity are retained.

The first step is to show that the equation for θ leads to a steady change in Ω given by

$$\dot{\Omega} = +\tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2 \cos\alpha + 0(J_2 e)$$

to the order of J^3. To the same order,

$$\frac{1}{r} = \frac{1}{p}\left\{1 + e\cos(\psi - \beta) + \tfrac{3}{4}J_2\left(\frac{R}{p}\right)^2(5\cos^2\alpha - 3 + \tfrac{1}{3}\sin^2\alpha\cos 2\psi) + 0(J_2 e)\right\}.$$

This solution does not show any change in β and it is therefore necessary to include terms of the order of e^3. It is first shown that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\psi} = +\tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2[1 - e\lambda + 0(J_2)]\cos\alpha$$

where

$$\lambda = \tfrac{1}{6}\{3\cos(\psi + \beta) + \cos(3\psi - \beta)\}\operatorname{cosec}^2\psi.$$

There is a difficulty here because when β is a multiple of π, λ becomes infinite, implying that the satellite momentarily leaves the plane Π, a difficulty that KING-HELE supposes could be avoided by suitable re-definition.

The third order equation for the radius vector leads to

$$\frac{\mathrm{d}\beta}{\mathrm{d}\psi} = -\tfrac{3}{4}J_2\left(\frac{R}{p}\right)^2(5\cos^2\alpha - 1) + 0(Je)$$

and

$$\frac{1}{r} = \frac{1}{l}\{1 + e\cos(\psi - \beta) + J_2 v_1 + J_2 e v_2\}$$

with

$$v_1 = \tfrac{3}{4}\left(\frac{R}{p}\right)^2(5\cos^2\alpha - 3 + \tfrac{1}{3}\sin^2\alpha\cos^2\psi)$$

as above and

$$v_2 = \frac{5}{24}\left(\frac{R}{p}\right)^2\sin^2\alpha\cos(3\psi - \beta).$$

To obtain the effect of J_4 terms of order e^4 have to be included and in particular,

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\psi} = -\tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2\{1 - e\lambda - J_2\Lambda + 0(J_2^2)\}\cos\alpha$$

where λ was given above and

$$\Lambda = \frac{1}{4}\left(\frac{R}{p}\right)^2\{3 + 5\sin^2\alpha(1 + 2\sin^2\psi)\} - \frac{5}{12}\frac{J_4}{(J_2)^2}\cdot\left(\frac{R}{p}\right)^2\{6 - 7\sin^2\alpha(2 - \sin^2\psi)\}.$$

So far the motions of the node and perigee have been obtained in terms of the angle ψ, the true anomaly of the satellite in its orbit.

Using the angular momentum integral,

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \mu\frac{1}{r^2}\left\{1 + \tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2\sec\alpha\sin^2\theta\right\} + 0(J_2 e)$$

and thus, for example, the first order variation of Ω is given by

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\tfrac{3}{2}J_2\frac{\mu p}{r^2}\cdot\left(\frac{R}{p}\right)^2\cos\alpha.$$

King-Hele expresses the rates of rotation of node and perigee as follows:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -n\left(\frac{R}{r}\right)^2\cos\alpha\left[\tfrac{3}{2}J_2 + \tfrac{9}{4}J_2^2\left(\frac{R}{r}\right)^2(\tfrac{19}{12}\sin^2\alpha - 1) - \tfrac{15}{16}J_4\left(\frac{R}{r}\right)^2(7\sin^2\alpha - 4)\right],$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \tfrac{3}{4}n\left(\frac{R}{r}\right)^2J_2(5\cos^2\alpha - 1),$$

where r is the harmonic mean value of r with respect to ψ. The above formulae have been changed slightly from those originally given by KING-HELE in order to conform to the notation used elsewhere in this paper.

KING-HELE's method has been extended by BRENNER and LATTA (1960) who have allowed the inclination of the plane Π to vary, and by PETTY and BREAKWELL (1960) who have introduced in addition a small displacement of the satellite out of the plane Π. PETTY and BREAKWELL also discuss the behaviour of the satellite for inclinations close to

$$\cos^{-1}(5)^{-\frac{1}{2}}$$

when the secular motion of perigee vanishes.

STRUBLE (1961) has developed a theory that has some affinities with that of KING-HELE. He works in spherical polar co-ordinates and seeks a solution of the equations of motion of the same form as KING-HELE looks for, but he differs from KING-HELE in taking the reference plane to be defined by the radius vector and the velocity vector of the satellite so that the inclination is now not constant, and instead of the true longitude of the satellite, he uses a related angle that somewhat simplifies that equations of motion. STRUBLE discusses in particular the behaviour of the satellite when the inclination has the value $\cos^{-1}(1/5)^{\frac{1}{2}}$.

MUSEN (1961) has developed a theory in terms of the true longitude with the especial object of applying it to numerical integration.

MESSAGE (1960) has discussed the relation between the constant inclination of the reference plane chosen by KING-HELE and the changing inclination of the osculating plane. Let $Oxyz$ be a moving frame of reference in the plane Π of fixed inclination, so that Ox is the intersection of this plane with the equator, Oy the northerly line of greatest slope in the plane, and Oz the direction perpendicular to the plane. Then the position vector of the satellite in terms of this frame of reference is

$$\mathbf{r} = (-r\sin\psi,\ r\cos\psi, 0).$$

Now let $OXYZ$ be an inertial frame of reference with the plane OXY being the equatorial plane; the angle Ω in KING-HELE's theory is the angle between Ox and OX, measured in the opposite direction to ψ.

Then

$$\begin{aligned} X &= \quad x\cos\Omega + (y\cos\alpha - z\sin\alpha)\sin\Omega \\ Y &= -x\sin\Omega + (y\cos\alpha - z\sin\alpha)\cos\Omega \\ Z &= \quad y\sin\alpha + x\cos\alpha. \end{aligned}$$

Accordingly the position vector of the satellite in the inertial frame is

$$\begin{aligned} \mathbf{r} = (&r\sin\psi\cos\Omega + r\cos\psi\cos\alpha\sin\Omega, \\ &r\sin\psi\sin\Omega + r\cos\psi\cos\alpha\cos\Omega, \\ &r\cos\psi\sin\alpha). \end{aligned}$$

The velocity of the satellite referred to the moving frame is

$$(\mathbf{v}) = (-\dot{r}\sin\psi - r\dot{\psi}\cos\psi\,, \dot{r}\cos\psi - r\dot{\psi}\sin\psi\,, 0)$$

and relative to the fixed frame it is

$$\mathbf{v} = (\mathbf{v}) + \mathbf{\Theta} \wedge \mathbf{r} + O(Je)$$

where

$$\mathbf{\Theta} = (0, 0, -\dot{\Omega}).$$

Since the osculating plane is defined by the position and velocity vectors, **r** and **v**, the unit normal to it is $\mathbf{n} = \mathbf{N}/N$ where

$$\mathbf{N} = \mathbf{r} \wedge \mathbf{v} = \mathbf{r} \wedge (\mathbf{v}) + r^2\,\mathbf{\Theta} - \mathbf{r}(\mathbf{r}\cdot\mathbf{\Theta}) + O(Je, \Theta^2)$$

and

$$\mathbf{N}^2 = (\mathbf{r} \wedge \mathbf{v})^2 + 2r^2\{\mathbf{r} \wedge (\mathbf{v})\}\cdot\mathbf{\Theta} + O(Je, \Theta^2).$$

But referred to the moving frame,

$$\mathbf{r} \wedge (\mathbf{v}) = (0, 0, r^2\dot{\psi})$$

and so in the inertial frame

$$\mathbf{r} \wedge (\mathbf{v}) = (-r^2\dot{\psi}\sin\alpha\sin\Omega\,, -r^2\dot{\psi}\sin\alpha\cos\Omega\,, r^2\dot{\psi}\cos\alpha)$$

and therefore

$$N^2 = r^4\dot{\psi}^2 - 2r^4\dot{\psi}\Omega\cos\alpha + O(Je, \dot{\Omega}^2)$$

or

$$N = r^2\dot{\psi}\left(1 - \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\cdot\cos\alpha\right) + O\left(Je, \left(\frac{\mathrm{d}\,\Omega}{\mathrm{d}\psi}\right)^2\right).$$

Thus

$$\mathbf{n} \equiv -\sin\alpha\sin\Omega\left(1 + \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\cos\alpha\right) - \cos\psi\sin\alpha\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\right)(\sin\psi\cos\Omega - \cos\psi\cos\alpha\sin\Omega),$$

$$-\sin\alpha\cos\Omega\left(1 + \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\cos\alpha\right) + \cos\psi\sin\alpha\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}(\sin\psi\sin\Omega + \cos\psi\cos\alpha\cos\Omega),$$

$$\cos\alpha\left(1 + \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\cos\alpha\right) - \frac{\mathrm{d}\Omega}{\mathrm{d}\psi} + \cos^2\psi\sin^2\alpha\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}, + O\left(Je, \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\right)^2\right).$$

Now in terms of the inclination of the osculating plane and the longitude of the ascending node,

$$\mathbf{n} = (\sin i\sin\Omega\,, -\sin i\cos\Omega\,, \cos i).$$

For the two expressions for **n** to agree,

$$i = \alpha + \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\sin\alpha\sin^2\psi + O\left(Je, \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\right)^2\right)$$

and

$$☊ = -\,\Omega - \frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\cos\psi\sin\psi + O\left(Je, \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\psi}\right)^2\right).$$

But

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\psi} = \tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2\cos\alpha$$

so that

$$i = \alpha + \tfrac{3}{4}J_2\left(\frac{R}{p}\right)^2\sin 2\alpha\sin^2\psi = O(Je)$$

and

$$☊ = -\,\Omega - \tfrac{3}{4}J_2\left(\frac{R}{p}\right)^2\cos\alpha\sin 2\psi + O(Je).$$

The mean value of i is accordingly

$$i_0 = \alpha + \tfrac{3}{8}J_2\left(\frac{R}{p}\right)^2\sin 2\alpha + O(Je)$$

and KING-HELE's results for the time rate of change of Ω transform to

$$\dot{☊} = -\frac{2\pi}{T_N}\cos i_0\,\tfrac{3}{2}\left[J_2\left(\frac{R}{p}\right)^2 + \tfrac{9}{4}J_2^2\left(\frac{R}{p}\right)^4\left(\tfrac{23}{6}\sin^2 i_0 - \tfrac{5}{2}\right) - \tfrac{15}{16}J_4\left(\frac{R}{p}\right)^4(4 - 7\sin^2 i_0)\right],$$

where T_N is the nodal period, the time between successive passages through the ascending node.

MESSAGE points out that it is not in fact possible to define a plane of constant inclination such that the satellite always remains in it, as KING-HELE had indeed conjectured, and he observes that such a plane can only be defined if the maximum northerly and southerly latitudes attained by the satellite are the same, which of course they are not in an elliptical orbit. The variation in α will then be of order J_2e, and MESSAGE shows that

$$\alpha = \alpha_0 - \tfrac{1}{2}eJ_2\left(\frac{R}{p}\right)^2\sin 2\alpha_0\cos(\psi + \beta).$$

A very similar treatment to that of KING-HELE, has been given by HALL and GAWLOWICZ (1962). It applies to circular orbits but includes the effect of J_3.

3.2 THEORIES USING THE EQUATIONS OF VARIATION OF THE ELLIPTIC ELEMENTS

One derivation of these equations was given in Section 2 and methods of solving the equations will now be described. The main problem of manipulation is to express the disturbing function,

$$R = V + \mu/r,$$

in terms of the elliptic elements in such a way as to bring out the salient features of the problem. It has already been seen in Section 2 how this is to be done for the first order secular terms and consideration will now be given to the problems that arise in going to higher order terms.

KOZAI (1959) has used the equations of variation in the form given in Section 2, taking R to be the sum of zonal harmonics of order 2, 3 and 4. He writes R as the sum of four parts, first order secular, R_1, second order secular, R_2, long periodic, R_3 and short periodic, R_4:

$$R_1 = \frac{3}{2}\frac{\mu J_2 R^2}{a^3}\left(\tfrac{1}{3} - \tfrac{1}{2}\sin^2 i\right)\eta^{-3}$$

$$R_2 = \frac{35}{8}\frac{\mu J_4 R^4}{a^5}\left(\tfrac{3}{35} - \tfrac{3}{7}\sin^2 i + \tfrac{3}{8}\sin^4 i\right)\left(1 + \tfrac{3}{2}e^2\right)\eta^{-7}$$

$$R_3 = -\tfrac{3}{2}\mu J_3 \frac{R^3}{a^4}\left(\tfrac{5}{4}\sin^2 i - 1\right)e\eta^{-5}\sin i \sin\omega$$
$$+ \tfrac{35}{8}\mu\frac{J_4 R^4}{a^5}\left(\tfrac{9}{28} - \tfrac{3}{8}\sin^2 i\right)e^2\eta^{-7}\sin^2 i\cos 2\omega$$

$$R_4 = \frac{3}{2}\frac{\mu J_2 R^2}{a^3}\left(\frac{a}{r}\right)^3\left[\left(\tfrac{1}{3} - \tfrac{1}{2}\sin^2 i\right)\left\{1 - \left(\frac{r}{a}\right)^3\eta^{-3}\right\} + \right.$$
$$\left. + \tfrac{1}{2}\sin^2 i\cos 2\mu\right]$$

where

$$\eta = (1 - e^2)^{\frac{1}{2}}.$$

The first-order secular parts of the changes in the elements are simply obtained by direct substitution of R in LAGRANGE's equations:

$$\bar{\omega} = \omega_0 + \tfrac{3}{4}J_2\left(\frac{R}{p}\right)^2(4 - 5\sin^2 i)\bar{n}t,$$

$$\bar{\Omega} = \Omega_0 - \tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2\bar{n}t\cos i,$$

$$\bar{M} = M_0 + \bar{n}t.$$

$$\bar{n} = n_0 + \tfrac{3}{2}J_2\left(\frac{R}{p}\right)^2 n_0\left(1 - \tfrac{3}{2}\sin^2 i\right)(1 - e^2)^{\frac{1}{2}},$$

p is the semi-latus rectum of the orbit and $n_0^2 a^3 = \mu$.

Short periodic variations of the first order are obtained by inserting R^4 into the equations and then integrating by using the following relation to transform from time t to true longitude v:

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}M}\mathrm{d}M = \frac{1}{n}\left(\frac{r}{a}\right)^2 (1-e^2)^{-\frac{1}{2}}\,\mathrm{d}v\,.$$

For instance

$$\begin{aligned}\mathrm{d}i_s &= \frac{\cos i}{n^2a^2(1-e^2)\sin i}\int\left(\frac{r}{a}\right)^2\frac{\partial R_4}{\partial\omega}\,\mathrm{d}v\\ &= \tfrac{3}{8}J_2\left(\frac{R}{p}\right)^2\sin 2i\,\{\cos 2(v+\omega)+e\cos(v+2\omega)+\\ &\quad+\tfrac{1}{3}e\cos(3v+2\omega)\}\,.\end{aligned}$$

In deriving the long periodic perturbations and the secular perturbations of the second order, account must be taken of the fact that the elements in the expressions on the right sides of the equations of variation are not constant. Thus if the variation of an element E_i is given by

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} = f_i\,,$$

the function f_i may be expanded in a power series so that

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} = (f_i)_0 + \sum_j\left(\frac{\partial f_i}{\partial E_j}\right)_0\mathrm{d}E_j$$

and accordingly, on integrating by parts,

$$\mathrm{d}E_i = \int_0^t (f_i)_0\,\mathrm{d}t + \sum_j\left[F_{ij}\,\mathrm{d}E_j\right]_0^t - \sum_j\int_0^t F_{ij}\frac{\mathrm{d}f_j}{\mathrm{d}t}\,\mathrm{d}t$$

with

$$F_{ij} = \int\left(\frac{\partial f_i}{\partial E_j}\right)\mathrm{d}t\,.$$

Now it can be shown that i and e can contain no secular terms and KOZAI also shows that a can contain neither secular nor long-periodic terms and with the aid of these facts he is able to derive the long periodic variations of i and e, the latter from the condition that the angular momentum integral gives

$$\{a(1-e^2)\}^{\frac{1}{2}}\cos i = \text{const}\,.$$

The complete first order expressions for a, i and e are now available and may be inserted in the equation for $\mathrm{d}E_i$ to give the long-periodic and second-order secular changes of node and perigee.

Very general expressions for the first order changes in the elements (i.e. those obtained on the assumption that the right sides of LAGRANGE's equations are constants) have been derived by GROVES (1960). He takes the most general term of the expansion of the potential in spherical harmonics and expands it in terms of the inclination, the true longitude and the argument of perigee. He next transforms that expansion to one in terms of inclination, mean anomaly and argument of perigee and shows how to differentiate it with respect to the elements. General results are then derived for the first-order variations of the elements and explicit results are given for the zonal harmonics of order 2, 3 and 4. GROVES also obtains certain results for the motion of a satellite about an Earth with tesseral harmonics in the potential when the mean motion of the satellite is commensurable with the spin angular velocity of the Earth, a topic that is dealt with in more detail in Section 4.

The Lagrangian equations can be written in an alternative form in terms of the three perpendicular components of the disturbing force acting on the satellite:

$$\dot{a} = \frac{2a^2 e \sin v}{(\mu p)^{\frac{1}{2}}} S + \frac{2a^2}{(\mu p)^{\frac{1}{2}}} T,$$

$$\dot{\Omega} = \frac{r \sin \mu}{(\mu p)^{\frac{1}{2}} \sin i} W$$

$$\dot{\omega} = \frac{1}{e}\left(\frac{p}{\mu}\right)^{\frac{1}{2}}\left[-S \cos v + T\left(1 + \frac{r}{p}\right)\sin v - We\frac{r}{p}\sin v \cot i\right]$$

$$\dot{e} = \frac{p^{\frac{1}{2}} \sin v}{\mu^{\frac{1}{2}}} S + \frac{r(e + 2\cos v + e\cos^2 v)}{(\mu p)^{\frac{1}{2}}} T$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r \cos \mu}{(\mu p)^{\frac{1}{2}}} W$$

where S is the component along the radius vector, T the component perpendicular to S in the osculating plane, and W the component perpendicular to S and T. (PLUMMER, 1960; MERSON, 1961).

If R, the disturbing function, is independent of longitude, then

$$S = \frac{\partial R}{\partial r}$$

$$T = -\left(\frac{\cos u \sin i}{r \sin \theta}\right)\frac{\partial R}{\partial \theta}$$

$$W = -\left(\frac{\cos i}{r \sin \theta}\right)\frac{\partial R}{\partial \theta}.$$

If now, R is given as a series of zonal harmonics, then S, T and W may be written down in terms of r, u and the inclination i of the osculating plane. The solution of the equations of variation is carried out as before, the first order terms being obtained by direct integration and the second order terms by taking account of the first order changes of the elements on the right sides of the equations.

MERSON (1961) and ŽONGOLOVIČ (1960, 1962) have each given very complete treatments on these lines. Both transform from time to the argument of latitude, u, as the independent variable and so since the nodal or draconic period is defined as the time in which u increases by 2π, their results are in the form of the changes of the elements in one draconic period, just as are those of KING-HELE, whereas KOZAI's rates of change are with respect to time. Both give expressions (on which they agree) for the rates of change in terms of the osculating elements at the node.

Since the osculating plane is constrained to rotate with the velocity vector and since the motion of a satellite shows short periodic fluctuations, the inclination of the plane, the longitude of the node and other elements of the orbit show short periodic fluctuations. Since there are ways of describing the position of a satellite other than by means of the osculating elements, these short periodic variations in the elliptic elements are not necessarily essential to the description of the behaviour of the satellite and MERSON and ŽONGOLOVIČ both deal with the problem of defining other elements analogous to the osculating elements and equally suitable for representing the motion of a satellite but from which the short periodic fluctuations shall be removed so far as possible.

ŽONGOLOVIČ and PELLINEN (1962) take the expression for the element E in the form

$$E = E_0 + J_2 E'u + J_2 \sum_0^m (\sigma_m \cos mu + \rho_m \sin mu) + O(J_2^2)$$

and then define the mean value of the element as

$$\mathbf{E} = E_0 + J_2 E'(M + \omega) + \frac{J_2}{2\pi} \int_0^{2\pi} \sum_0^m (\sigma_m \cos mu + \rho_m \sin mu) + O(J_2^2).$$

They then derive the relations between these mean elements and the osculating elements at any value of the true anomaly and in particular for the values at the ascending node and hence they can express the secular and long periodic changes of the orbit in terms of their mean elements.

MERSON deals with the problem of defining elements that shall have minimum short periodic components. Corresponding to any element ζ he defines a smoothed element ζ' such that

$$\zeta' = \zeta + \gamma_\zeta J_2 .$$

Now the osculating element may generally be written as

$$\zeta = \zeta_0 + \delta\zeta_{\text{sec}} + \delta\zeta_{\text{per}}$$

and MERSON tries to find a γ_ζ such that

$$\gamma_\zeta J_2 = c_\zeta J_2 - \delta\zeta_{\text{per}}$$

where $\delta\zeta_{per}$ is the periodic component of ζ and c_ζ = const + $O(J_2)$, for then the periodic components will have been removed from the changes in ζ. It is not in fact possible to carry out this program completely (as has already been seen in the discussion of KING-HELE's definition of the inclination), but MERSON finds that smoothed elements with minimal periodic terms can be found and he obtains expressions for the secular and long-periodic changes in these optimum smoothed elements.

3.3 USE OF THE CANONICAL EQUATIONS OF MOTION

BROUWER (1959), GARFINKEL (1959) and KOZAI (1962) have developed theories of satellite motions starting from the canonical constants for an elliptical orbit in the form given by DELAUNAY:

$$\begin{aligned} L &= (\mu a)^{\frac{1}{2}} & l &= M \\ G &= L(1-e^2)^{\frac{1}{2}} & g &= \omega \\ H &= G\cos i & h &= \Omega\,. \end{aligned}$$

The object of the theory is to transform to new variables L' ... in such a way that the Hamiltonian written in terms of these new variables is as nearly as possible a function of L', G' and H' only, for then these parameters will be constants and the corresponding co-ordinates, l', g' and h' will be linear functions of time.

The transformation is effected by means of a solution, S, of the HAMILTON-JACOBI equation, following a method due to VON ZEIPEL.

In terms of the variables L, G, ..., F can be written out explicitly, as has already been seen. The L, G, ... are subject to various periodic changes and the object of the transformation is to remove these changes from the elements and put them into the form of F, which will now consist of an infinite series instead of a finite set of terms. Since the problem cannot be solved exactly, the determining function S will also be expressed as an infinite series.

Let the Hamiltonian as a function of $(L, \ldots, l, \ldots)$ be called F:

$$F = F_0 + F_1$$

where here, as in the sequel, the power of J_2 in any function is denoted by the subscript.

Let $S = S_0 + S_1 + S_2$, the determining function, be the solution of the HAMILTON-JACOBI equation, and let the Hamiltonian as a function of the variables $(L', \ldots, l', \ldots)$ be written

$$F^* = F_0^* + F_1^* + F_2^*.$$

The object is to find a form of F^* independent of l' and h', remembering that F is already independent of h since V is axi-symmetric.

S_0 is chosen to be the identical transformation:

$$S_0 = L'l + G'g + H'h\,.$$

Now

$$F(L, G, H, l, g, -) = F^*(L', G', H', -, g, -)$$

and so,

$$F_0(S_l) + F_1(S_l, S_g, S_h, g, -)$$

$$= F_0^* + F_1^*(L', G', H', -, S^{G'}, -)$$

$$+ F_2^*(L', G', H', -, S^{G'}, -),$$

where letter subscripts denote partial differentiation when it is helpful to use that notation:

$$S_l = \frac{\partial S}{\partial l}.$$

The various terms are expanded in TAYLOR series and the parts of the same order in J_2 on the two sides of the equation are equated:
order zero:

$$F_0(L') = F_0^*(L')$$

one:

$$S_{1l}\frac{\partial F_0}{\partial L'} + F_1 = F_1^*$$

two:

$$S_{2l}\frac{\partial F_0}{\partial L'} + \tfrac{1}{2}S_{1l}^2\frac{\partial^2 F_0}{\partial L'^2} + \frac{\partial F_1}{\partial L'}\cdot S_{1l} + \frac{\partial F_1}{\partial G'}S_{1g}$$

$$= F_2^* + F_{1g}^*\cdot\frac{\partial S_1}{\partial G'}.$$

BROUWER chooses for the unperturbed Hamiltonian that corresponding to the potential μ/r, whereas GARFINKEL (see also STERNE, 1958; GARFINKEL, 1958) starts from a potential that allows the HAMILTON-JACOBI equation to be separated but which is not a solution of LAPLACE's equation for in this way the perturbations that have to be dealt within the transformations are smaller. The potential that GARFINKEL chooses is

$$V_0 = -\frac{\mu}{r} + \frac{3kc_1}{r^2}(\sin^2\theta - c_2)$$

with

$$\mu = 1 - 6kc_3$$

a potential that gives a pseudo-elliptical orbit with secular variations of the order of k_2 when

$$c_1 = 1/a(1-e^2), \qquad c_2 = \cos^2 i$$
$$c_3 = (3\cos^2 i - 1)(1-e^2)^{\frac{1}{2}}/4p^2 .$$

BROUWER separates F_1 into a secular part, F_{1s}, independent of l, and a periodic part, F_{1p}, that is a function of l.

Now

$$F = \frac{\mu^2}{2L^2} - \frac{\mu J_2}{r}\left(\frac{R}{r}\right)^2 P_2(\cos\theta),$$

or, writing

$$j_2 = J_2\left(\frac{R}{a}\right)^2,$$

$$F = \frac{\mu^2}{2L^2} + \frac{\mu_4 j_2 a^2}{L^6}\left[\left(-\frac{1}{2} + \frac{7}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3} + \right.$$
$$\left. + \frac{3}{2}\left(1 - \frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos 2(g+v)\right].$$

Put

$$\frac{a^3}{r^3} = \frac{L^3}{G^3} + \sum 2P_j \cos jl \equiv \frac{L^3}{G^3} + \sigma_1$$

and

$$\frac{a^3}{r^3}\cos 2(g+v) = \sum Q_j \cos(2g + jl) \equiv \sigma_2 ,$$

when

$$F_{1s} = \frac{\mu^4 j_2 a^2}{L^3 G_1^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)$$

and

$$F_{1p} = \frac{\mu^4 j_2 a^2}{L^6}\left\{\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\sigma_1 + \tfrac{3}{2}\left(1 - \frac{H^2}{G^2}\right)\sigma_2\right\}.$$

Hence

$$\frac{\partial S_1}{\partial l} = \frac{\mu^2 j_2 a^2}{L^3}\left\{\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\sigma_1 + \tfrac{3}{2}\left(1 - \frac{H^2}{G^2}\right)\sigma_2\right\}$$

and so S_1 may now be determined by integrating this expression with respect to l, and then all the primed variables can be found:

$$L = L' + \frac{\partial S_1}{\partial l}, \qquad l = l' - \frac{\partial S_1}{\partial L,},$$

$$G = G' + \frac{\partial S_1}{\partial g}, \qquad g = g' - \frac{\partial S_1}{\partial G,},$$

$$H = H' \qquad h = h' - \frac{\partial S_1}{\partial H'}.$$

In terms of the primed variables, the Hamiltonian is

$$F^* = \frac{\mu^2}{2L'^2} + \frac{\mu^4 j_2 a^2}{L'^3 G'^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right) + F_2^*$$

and with

$$F_2^* = F_{2s}^* + F_{2p}^*,$$

all the terms can be found from the preceding work.

Now take a new determining function, S^*, transforming to variables L'', ...

$$S^* = L''l' + G''g' + H''h' + S_1^*(L'', G'', H'', g')$$

and let F^{**} be the Hamiltonian in terms of the new variables, $L'' \cdots$.

Once again, since $F^* = F^{**}$,

$$F_0^* + F_1^*\left(L'', G'' + \frac{\partial S_1^*}{\partial g,}, H''\right) + F_{2s}^* + F_{2p}^* = F_0^{**} + F_1^{**} + F_2^{**}$$

giving

$$F_0^* = F_0^{**}, \frac{\partial F_1^*}{\partial G''} \cdot \frac{\partial S_1^*}{\partial g'} + F_{2p}^* = 0$$

$$F_1^* = F_1^{**}, F_{2s}^* = F_2^{**}$$

since l'' has been eliminated from F^{**}.

$\partial F_1^*/\partial G''$ and F_{2p}^* are known from the preceding work and so $\partial S_1^*/\partial g'$ is also known. Then directly

$$G' = G'' + \frac{\partial S_1^*}{\partial g'},$$

and by integrating with respect to g', S_1^* is found. Then as before, the new variables (l'', g'', h'') can be calculated. Finally, F^{**} can be written down as a function of L'', G'', H'' only and hence the rates of change of l'', g'', h'' can be found, giving the secular changes of these elements to the order of $J.^2$

Higher harmonics are readily added to the Hamiltonian F_2^* since only first order terms in J_3, J_4, ..., are needed.

GARFINKEL's procedure is very similar to that of BROUWER and their results are in agreement.

KOZAI (1962) has continued BROUWER's treatment to include terms of order J_2^3.

3.4 SOLUTION TO THE EQUATIONS OF MOTION IN SPHEROIDAL CO-ORDINATES

It is known that the external potential of a body bounded by a spheroid of revolution (to which the potential of the Earth approximates quite closely) has a particularly simple form when expressed in spheroidal co-ordinates (see for example, PIZETTI, 1894; A. H. COOK, 1959) and it is therefore natural to ask if there is also a simple solution of the equations of satellite motion in these co-ordinates that might enable the effect of at least the J_2 term in the potential to be treated more exactly than in spherical polar co-ordinates. VINTI (1959) showed that there is indeed such a solution.

The working in this section will be given in terms of the co-ordinates, η, v, λ, which are related to Cartesian co-ordinates as follows:

$$\begin{aligned} x &= c\cosh\eta\sin v\cos\lambda\,, \\ y &= c\cosh\eta\sin v\sin\lambda\,, \\ z &= c\sinh\eta\cos v\,. \end{aligned}$$

λ is the azimuthal angle and v is the eccentric angle of the meridional section.

VINTI used a slightly different set:

$$\begin{aligned} x &= c\{(\xi^2+1)(1-\eta^2)\}^{\frac{1}{2}}\cos\phi \\ y &= c\{(\xi^2+1)(1-\eta^2)\}^{\frac{1}{2}}\sin\phi \\ z &= c\xi\eta\,. \end{aligned}$$

There is some slight algebraic advantage in the form used here and it is rather easier to relate the potential in this set of co-ordinates to that in spherical polar co-ordinates.

The line element, $\mathrm{d}s$, is given by

$$\mathrm{d}s^2 = h_1^2\,\mathrm{d}\eta^2 + h_2^2\,\mathrm{d}v^2 + h_3^2\,\mathrm{d}\lambda^2\,,$$

with

$$\begin{aligned} h_1^2 = h_2^2 &= c^2(\sinh^2\eta + \cos^2 v) \\ h_3^2 &= c^2\cosh^2\eta\sin^2 v\,. \end{aligned}$$

The kinetic energy for unit mass is

$$T = \tfrac{1}{2}\{h_1^2\dot\eta^2 + h_2^2\dot v^2 + h_3^2\dot\lambda^2\}\,,$$

or in terms of the momenta,

$$T = \frac{1}{2c^2}\left[\frac{p_\eta^2 + p_v^2}{\sinh^2\eta + \cos^2 v} + \frac{p_\lambda^2}{\cosh^2\eta\sin^2 v}\right].$$

As in the rest of this section, the potential is supposed to be symmetrical about the polar axis, that is, it is a function of η and v only, and the HAMILTON-JACOBI equation for the system is therefore

$$\frac{1}{2c^2}\left[\frac{\left(\frac{\partial S}{\partial \eta}\right)^2+\left(\frac{\partial S}{\partial v}\right)^2}{\sinh^2\eta+\cos^2 v}+\frac{\left(\frac{\partial S}{\partial \lambda}\right)^2}{\cosh^2\eta \sin v}\right]+V(\eta, v)=\alpha_1 .$$

This is a form of the equation which, according to the theory of STÄCKEL, is known to be soluble by separation of the variables for a suitable form of the potential (see ISZAK, 1960). That potential will now be determined in the same way as that which permits separation in spherical polar co-ordinates was found above.

Let S, the determining function, be the sum of terms which are functions of η, v or λ alone:

$$S=S_1(\eta)+S_2(v)+S_3(\lambda).$$

Denoting differentials by primes, the HAMILTON-JACOBI equation then becomes

$$\frac{1}{2c^2}\left[\frac{S_1'^2+S_2'^2}{\sinh^2\eta+\cos^2 v}+\frac{S_3'^2}{\cosh^2\eta \sin^2 v}\right]+V(\eta, v)=\alpha_1 .$$

Evidently S_3' must be a constant, α_3 say.

Now

$$\sinh^2\eta+\cos^2 v=\cosh^2\eta-\sin^2 v$$

and so

$$S_1'^2+S_2'^2+\alpha_3^2\left(\frac{1}{\sin^2 v}-\frac{1}{\cosh^2\eta}\right)+2c^2V(\cosh^2\eta-\sin^2 v)$$

$$=2c^2\alpha_1(\cosh^2\eta-\sin^2 v).$$

It is easily seen that this equation is separable if

$$V=(\cosh^2\eta-\sin^2 v)^{-1}\{f(\eta)+g(v)\}.$$

But

$$\cosh^2\eta-\sin^2 v=\sinh^2\eta+\cos^2 v$$

$$=-(\mathrm{i}\sinh\eta+\cos v)(\mathrm{i}\sinh\eta-\cos v)$$

and

$$\frac{1}{\mathrm{i}\sinh\eta-\cos v}=\sum(2n+1)Q_n(\mathrm{i}\sinh\eta)\cdot P_n(\cos v),$$

where $Q_n(\mu)$ is a LEGENDRE function of the second kind, defined by

$$Q_n(\mu) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(\mu)}{\mu - u}\,\mathrm{d}u\,. \quad \text{(HOBSON, 1931, p. 58)}$$

The general solution of LAPLACE's equation in spheroidal co-ordinates that vanishes at infinity is

$$Q_n(\mathrm{i}\sinh\eta)\cdot P_n(\cos\nu)$$

and so $(\mathrm{i}\sinh\eta - \cos\nu)^{-1}$ is such a solution, as is $(\mathrm{i}\sinh\eta + \cos\nu)^{-1}$.

Thus if A and B are constants, we must have

$$\frac{A}{\mathrm{i}\sinh\eta + \cos\nu} + \frac{B}{\mathrm{i}\sinh\eta - \cos\nu} = \frac{f(\eta) + g(\nu)}{\sinh^2\eta + \cos^2\nu},$$

that is

$$\begin{aligned}(A + B)\,\mathrm{i}\sinh\eta &= f(\eta)\\ (A - B)\cos\nu &= g(\nu)\,.\end{aligned}$$

If consideration is restricted to a potential that is symmetrical about the equator, A must equal B and so

$$f(\eta) = A\sinh\eta$$

and the potential is

$$\mathrm{i}A\sum_{n\,\text{even}} (2n + 1)\,Q_n(\mathrm{i}\sinh\eta)\,P_n(\cos\nu)\,.$$

The HAMILTON-JACOBI equation now separates into an η equation and a ν equation:

$$S_1'^2 - \frac{\alpha_3^2}{\cosh^2\eta} + 2c^2A\sinh\eta - 2c^2\alpha_1\cosh^2\eta = \alpha_2$$

$$S_2'^2 + \frac{\alpha_3^2}{\sin^2\nu} + 2c^2\alpha_1\sin^2\nu = -\alpha_2\,.$$

Then,

$$S_1 = \pm\int_{\eta_1}^{\eta}\left(\alpha_2 + 2c^2\alpha_1\cosh^2\eta - 2c^2A\sinh\eta + \frac{\alpha_3^2}{\cosh^2\eta}\right)\mathrm{d}\eta$$

and

$$S_2 = \pm\,\mathrm{i}\int_{\nu_1}^{\nu}\left(\alpha_2 + 2c^2\alpha_1\sin^2\nu + \frac{\alpha_3^2}{\sin^2\nu}\right)\mathrm{d}\nu$$

and the constants, β_1, β_2, β_3 are obtained by differentiating S:

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial S_1}{\partial \alpha_1} + \frac{\partial S_2}{\partial \alpha_1}$$

$$= \pm \int\limits_{\eta_1}^{\eta} \{\alpha_2 (1 + \sinh^2 \eta) + 2c^2 \alpha_1 (1 + \sinh^2 \eta)^2 + $$

$$+ 2c^2 A (1 + \sinh^2 \eta) + \alpha_3^2\}^{-\frac{1}{2}} \mathrm{d}(\sinh \eta)$$

$$\pm \mathrm{i} \int\limits_{\nu_1}^{\nu} \{\alpha_2 (1 - \cos^2 \nu) + 2c^2 \alpha_1 (1 - \cos^2 \nu)^2 +$$

$$+ \alpha_3^2\} \mathrm{d}(\cos \nu).$$

The integrals are both elliptic integrals of the first kind (WHITTAKER and WATSON, 1940, p. 515).

The corresponding integrals in VINTI's co-ordinate system have been evaluated by VINTI (1961a) and by ISZAK (1960) who have each related the canonical constants involved to the elements of the elliptical orbit. To interpret the results, the potential must be related to the form in spherical polar co-ordinates.

Write

$$V = V_0 + V_2 + V_4 + \ldots$$

where

$$V_0 = -\frac{\mu}{\mathrm{i}c} Q_0 (\mathrm{i} \sinh \eta)$$

and

$$V_2 = -\frac{5\mu}{\mathrm{i}c} Q_2 (\mathrm{i} \sinh \eta) P_2 (\cos \nu).$$

In spherical polar co-ordinates,

$$V_0 = \frac{\mu}{r} 1 \left\{ -\frac{1}{3}\left(\frac{c}{r}\right)^2 P_2 (\cos \theta) + \frac{1}{5}\left(\frac{c}{r}\right)^4 P_4 (\cos \theta) - \frac{1}{7}\left(\frac{c}{r}\right)^6 P_6 (\cos \theta) + \ldots \right\}$$

(A. H. COOK, 1959)

and

$$V_2 = -\frac{5\mu}{\mathrm{i}c} \cdot \frac{2}{15} \frac{1}{(\mathrm{i} \sinh \eta)^3} P_2 (\cos \theta) + \ldots$$

$$= -\frac{2}{3} \frac{\mu}{c} \left(\frac{c}{r}\right)^3 P_2 (\cos \theta) + \ldots.$$

Hence

$$V = V_0 + V_2 = \frac{\mu}{r}\left[1 - \left(\frac{c}{r}\right)^2 P_2(\cos\theta) + \ldots\right]$$

the terms omitted being of order

$$\left(\frac{c}{r}\right)^4$$

and less.

Accordingly

$$\left(\frac{c}{R}\right)^2 = J_2$$

which is VINTI's result.

ISZAK (1960) includes second order terms in his expressions for the motions of the node and perigee, with the following results ($s = \sin i$):

Speed of node:

$$-\frac{3}{2}\left(\frac{c}{a}\right)^2 \frac{\cos i}{(1-e^2)^2} + \frac{3}{16}\left(\frac{c}{a}\right)^4 \frac{\cos i}{(1-e^2)^4}\{18 - 13s^2 + 24s^2e^2\};$$

Speed of perigee:

$$\frac{3}{4}\left(\frac{c}{a}\right)^2 \frac{4-5s^2}{(1-e^2)^2} + \frac{1}{64}\left(\frac{c}{a}\right)^4 \frac{1}{(1-e^2)^4} \times$$

$$\times \{288 - 1296s^2 + 1035s^4 - (144 + 288s^2 - 510s^4)e^2\}.$$

Remembering that

$$J_2 = \left(\frac{c}{R}\right)^2,$$

these results are identical with those of other theories to first order but the J_2^2 terms do not agree with those of other theories and the relation of the elements used by ISZAK to osculating elements, for example, needs further study.

The result of VINTI's theory is that it is possible to obtain an exact solution to the equations of motion in terms of canonical *constants* for a potential having exactly the same J_2 term as the actual potential of the Earth. It is now of interest to see how far higher harmonics in this potential correspond to those of the actual Earth or to those in the potential outside a rotating spheroidal equipotential surface for which, as for VINTI's potential, the higher harmonics are fixed once the J_2 term is chosen.

The comparison is as follows, J_2being $(C-A)/MR^2$,

	$10^6 J_2$	$10^6 J_3$	$10^6 J_4$	$10^6 J_5$	$10^6 J_6$
VINTI		0	-1.2	0	1.3×10^{-3}
Equipotential		0	-2.4	0	2.5×10^{-3}
observed	1083	-2.4	-1	0	0.73

It is not of course to be expected that the terms for the equipotential surface will be the same as those for VINTI's potential because those in the latter are uniquely determined by the relation,

$$J_{2n} = -(-J_2)^n$$

whereas those for the former are fixed by the spin angular velocity of the rotating body by the relations

$$J_4 = -\tfrac{9}{70}(J_2 + \tfrac{1}{2}m)(7J_2 - m), \qquad J_6 = \tfrac{1}{21}(3J_2 + m)^2(9J_2 - 2m), \ldots$$

where m is the ratio of centrifugal to gravitational force on the equator.

More fundamentally, VINTI's potential expressed in spheroidal harmonics, appears as an infinite series whereas the potential outside a rotating body bounded by a spheroidal equipotential surface is given exactly by just the first two harmonics,

$$V_0 = \frac{\mu}{ic} Q_0(i \sinh \eta) P_0(\cos v)$$

and

$$V_2 = -\tfrac{1}{3}\omega_r^2 R_e^2 \frac{Q_2(i \sinh \eta)}{Q_2(i \sinh \eta_0)} \cdot P_2(\cos v)$$

where ω_r is the spin angular velocity of the body and R_e is the equatorial radius.

TABLE 1

NOTATIONS FOR THE POTENTIAL OF THE EARTH

Author	μ	J_2	J_3	J_4	J_5
KING-HELE, 1958	gR^2	$+\frac{2}{3}J$		$-\frac{8}{35}D$	
KOZAI, 1959	GM	$\frac{2}{3}\frac{A_2}{R^2}$	$-\frac{A_3}{R^3}$	$-\frac{8}{35}\frac{A}{R^4}$	
VINTI, 1959		J_2		J_4	
BROUWER, 1959	μ	$\frac{2k_2}{R^2}$	$-\frac{A_{3.0}}{R^3}$	$-\frac{8}{3}\frac{k_4}{R_m{}^4}$	$-\frac{A_{5.0}}{RI}$
GARFINKEL, 1959		$\frac{2k}{R^2}$		$-\frac{k'}{R^4}$	
BRENNER and LATTA, 1960	$-gR^2$	$\frac{2}{3}J$		$-\frac{8}{35}D$	
PETTY and BREAKWELL, 1960	μ	$\frac{2}{3}J_2$	J_3	$-\frac{8}{35}J_4$	
GROVES, 1960	μ	$-\frac{R_0A_2{}^\circ}{\mu}$		$-\frac{R_0A_2{}^\circ}{\mu}$	
ŽONGOLOVIČ, 1960, 1962	fM	$-c_{20}$	$-c_{30}$	$-c_{40}$	
MERSON, 1961	μ	J_2	J_3	J_4	J_5
STRUBLE, 1961	gR^2	$+\frac{2}{3}J$	$-$	$-\frac{8}{35}D$	

VINTI's potential matches the observed J_4 term very closely, more closely in fact than seemed the case when VINTI proposed it. All other known coefficients up to J_{12} are of order 10^{-6} and so are not represented by the potential.

3.5 SUMMARY AND COMPARISON OF RESULTS

The results of the theories outlined above are given in a number of different forms and it is difficult to appreciate the relations between them at first sight. The purpose of this section is to present the various results so far as possible in a common notation, to bring out the most significant results so far as practical applications are concerned and to see how far the results are in agreement between one treatment and another.

In the first place, although all theories except that of VINTI take the potential in the form of a series of spherical harmonics, there are many different notations for the coefficients. In this paper, the recommendation of the I.A.U. (Commission 7) is followed and the potential is taken in the form

$$V = -\frac{\mu}{r}\left\{1 - \sum_n J_n\left(\frac{R}{r}\right)^n P_n(\cos\theta)\right\}.$$

Other notations that have been employed are summarised in Table 1.

In the second place, the variations of the elements may be presented in different ways. The rate of change with respect to time may be given, or the change in one anomalistic or draconic (nodal) period, or, for periodic terms, the value as a function of the argument ω. Such differences do not affect the first-order secular changes, which are given in Tables 2a, 2b, 2c, in the J_2 notation, but they do affect the first-order periodic changes with argument ω.

BROUWER (1959) and KOZAI (1959) for example, give these long periodic terms as the actual values at a particular value of ω and for an element E they are in the form

$$\delta E = \frac{J_n}{J_2}\cdot f_E(e, i)\left(\begin{matrix}\sin\\ \cos\end{matrix}\, q\omega\right),$$

J_2 coming in as a denominator because the speed of ω is determined by the value of J_2.

MERSON (1961) and ŽONGOLOVIČ (1960a), on the other hand present the changes in one nodal period. Then if $E(\omega_1)$ is the value of E when $\omega = \omega_1$, the change of E in one period,

$$\begin{aligned}\Delta(E) &= E(\omega_1 + \dot{\omega}T) - E(\omega_1)\\ &\propto \left\{\begin{matrix}\sin\\ \cos\end{matrix}(q\omega_1 + q\dot{\omega}T) - \begin{matrix}\sin\\ \cos\end{matrix}\, q\omega_1\right\}\\ &= \left(\begin{matrix}\cos\\ -\sin\end{matrix}\, q\omega_1\right) q\dot{\omega}T\end{aligned}$$

since $\dot{\omega}T$ is small.

But

$$\dot{\omega} = 3n\left(1 - \tfrac{5}{4}\sin^2 i\right)J_2\left(\frac{R}{p}\right)^2$$

and so

$$\Delta E = 2\pi \times 3q(1 - \tfrac{5}{4}\sin^2 i)J_n\left(\frac{R}{p}\right)^{-2} f_E\left(-\frac{\cos}{\sin} q\omega_1\right).$$

These long-period results, also, are summarised in Table 2, in the form of the changes in one draconic period.

There are some discrepancies between the theories. The results for e and i (which are not independent) are in good agreement, but there are more serious divergences in the results for the node and for perigee, some of which arise from the fact that BROUWER's results are in terms of mean elements whereas MERSON (1961) and ŽONGOLOVIČ (1960a) use the values of the osculating elements at the node. Only those terms for which there is no difficulty are included in the Tables but as noted there, more extensive results will be found in the original papers. The discrepancies do not, however affect the general nature of the conclusions. All elements are subject to long-periodic variations, with speeds $2p\omega$ for harmonics of even order, and speeds $(2q + 1)\omega$ for harmonics of odd order, the highest speeds being in each case $(n - 2)\omega$ where n is the order of the harmonic. e and i are not subject to secular changes but the node and

TABLE 2

CHANGES OF THE ELEMENTS IN ONE NODAL PERIOD

NOTATION

$$V = -\frac{\mu}{r}\left\{1 - \sum J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta)\right\}$$

R : radius constant of the Earth at sea level, usually the equatorial radius
p : semi-latus rectum of orbit
e : eccentricity
i : inclination at ascending node
S : $\sin^2 i$

REFERENCES

B : BROUWER, 1959
G : GARFINKEL, 1959
Gr : GROVES, 1960
K–H : KING-HELE, 1958
K : KOZAI, 1959
M : MERSON, 1961 (formulae in terms of osculating elements at ascending node)
Ž1 : ŽONGOLOVIČ, 1960
Ž2 : ŽONGOLOVIČ, 1960a
ŽP : ŽONGOLOVIČ and PELLINEN, 1962
s : secular terms
l–p : long-periodic terms

TABLE 2 *(Continued)*

a. Eccentricity and Inclination: First order long periodic terms

$$\Delta e = 2\pi J_n \left(\frac{R}{p}\right)^n i_n$$

$$i_n = -e \cdot e_n (1-e^2)^{-1} \cot \mathrm{i}$$

n	e_n	Reference
2		
3	$-\frac{3}{2}(1-e^2)\left(1-\frac{5}{4}\right)\sin \mathrm{i} \cos\omega$	B, K, M, Ž2
4	$-\frac{45}{16}(1-e^2)\left(1-\frac{7}{6}S\right) Se \sin 2\omega$	B, K, M, Ž2
5	$\frac{15}{4}(1-e^2)\sin i\left(1+\frac{3}{4}e^2\right)\left(1-\frac{7}{2}+\frac{21}{8}S^2\right)\cos\omega$	B*, M, Ž2
	$+\frac{105}{32}e^2(1-e^2)\sin i\left(1-\frac{9}{8}S\right)\mathrm{S}\cos 3\omega$	B, M, Ž2
6	$\frac{525}{32}(1-e^2)S\left[\left(1-35+\frac{33}{16}S^2\right)\left(1+\frac{1}{2}e^2\right)e\sin 2\omega + \right.$	M, Ž2
	$\left.+\frac{3}{16}\left(1-\frac{11}{10}S\right)\mathrm{S}e^3\sin 4\omega\right]$	

* B has $(1-e^2)^{-1}$, evidently a misprint

b. Node: First order secular and long periodic terms

$$\frac{\Delta \Omega}{2\pi} \quad J_n\left(\frac{R}{p}\right)^n \Omega_n$$

n	Ω_n		Reference
2	$-\frac{3}{2}\cos i$	(3)	B, G, Gr, K–H, K, M, Ž1, Ž2, ŽP
3	$\frac{3}{2}\left(1-\frac{15}{4}S\right)e\sin\omega\cot i$		M, Ž2, B, K (1)
4	$\frac{15}{4}\cos i\left(1-\frac{7}{4}S\right)\left(1+\frac{3}{2}e^2\right)$		
	$-\frac{45}{16}\cos i\left(1-\frac{7}{3}S\right)e^2\cos 2\omega$		s : B, K, M, Ž1, 2 1 − p: M, Ž2 (2)
5	$-\frac{15}{4}\cot i\left(1-\frac{21}{2}S+\frac{105}{8}S^2\right)\left(1+\frac{3}{4}e^2\right)e\sin\omega$		M, Ž2
	$-\frac{105}{32}\cos i\left(1-\frac{15}{8}S\right)e^3\sin 3\omega$		
6	$-\frac{105}{16}\cos i\left(1-\frac{9}{2}S+\frac{33}{8}S^2\right)\left(1+5e^2+\frac{15}{8}e^4\right)$		M, Ž2

TABLE 2 *(Continued)*

$$- \frac{525}{32} \cos i \left(1 - 6\,S + \frac{99}{16} S^2 \right)\left(1 + \frac{1}{2} e^2 \right) e^2 \cos 2\omega$$

$$- \frac{1575}{512} \cos i \left(1 - \frac{33}{20} S \right) S\, e^4 \cos 4\omega$$

(1) B, K have $\left(1 - \frac{5}{4} S\right)$

(2) B's form differs slightly in the function of S

(3) Additional l–p terms proportional to e^2 in B

c. Longitude of Perigee: First order secular and long-periodic terms

$$\Delta\omega = 2\,\pi\, J_n \left(\frac{R}{p} \right)^n$$

n	ω_n	*Reference*
2	$3\left(1 - \frac{5}{4} S \right)$	
3	$\frac{3}{2e} \sin i \left\{ \left(1 - \frac{5}{4} S \right) - \left(\operatorname{cosec}^2 i + \left(\frac{5}{4} S - \frac{9}{4} \right) \right) e^2 \right\} \sin \omega$	B, K, M
4	$- \frac{15}{32} \left\{ (16 - 62\,S + 49\,S^2) + \left(18 - 63\,S + \frac{189}{4} S^2 \right) e^2 \right\}$	B, K, M
	$+ \frac{15}{32} \left(+ 6 - 35\,S + \frac{63}{5} S^2 \right) e^2 \cos 2\omega$	
5	$\frac{105}{16} e^{-1} \frac{\sin \omega}{\sin i} \left[\left(- \frac{4}{7} + 2\,S - \frac{3}{2} S^2 \right) S + \right.$	M
	$+ \left(\frac{4}{7} - \frac{87}{7} S + \frac{67}{2} S^2 - \frac{357}{16} S^3 \right) e^2 +$	
	$\left. + \left(- 1 + \frac{9}{8} S \right) S^2\, e^2 \cos \omega + \,.\,.\, 0\,(e^4) \right]$	
6	$\frac{525}{6} \left[\frac{8}{5} \left(1 - 8S + \frac{129}{8} S^2 - \frac{297}{32} S^3 \right) + \right.$	M
	$+ \left(2 - 6S + \frac{33}{8} S^2 \right) S \cos 2\omega +$	
	$\left. + 6e^2 \left(1 - \frac{43}{6} S + \frac{109}{8} S^2 - \frac{121}{8} S^3 \right) + \ldots \right]$	

KING-HELE, COOK and REES (1963) have extended some of these results to harmonics of order 14.

perigee show such changes for harmonics of even order. In the variations of any element, the terms of successively greater speeds are multiplied by successively higher powers of e, for example, in the eccentricity the terms are

$$\cos\omega\,, \qquad e\sin 2\omega\,, \qquad e^2\cos 3\omega\,, \qquad e^3\sin 4\omega\,.$$

It follows that in general the only terms of any importance are those with speeds ω and 2ω.

The variation of certain of the coefficients with inclination is shown in Figure 5.

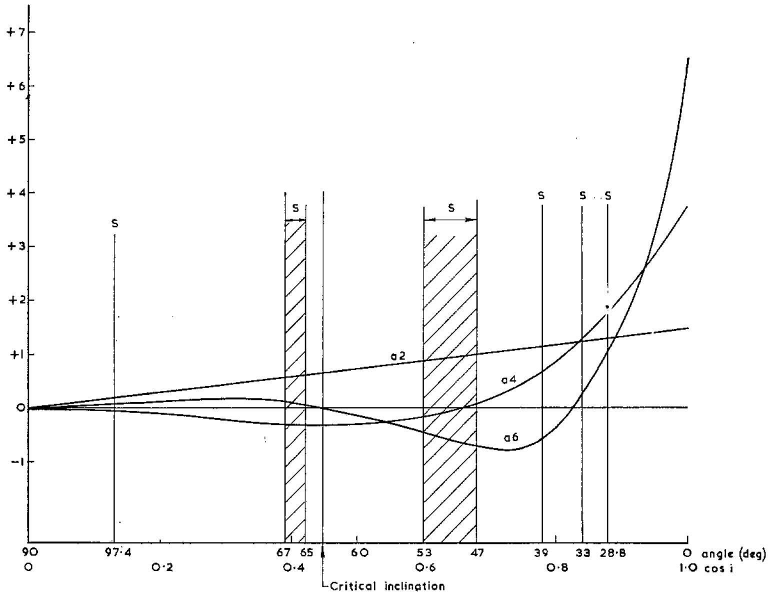


Fig. 5. Coefficients a_n of j_n in secular part of motion of node. S-bands in which satellite inclinations lie.

At first sight, the secular terms proportional to J_2^2 look very different but for the node at least, the differences arise from the definitions of the elements and the way of presenting the change of element. The time rate of change (BROUWER, 1959; KOZAI, 1959) or the change in one draconic period (MERSON, 1961; ŽONGOLOVIČ and PELLINEN, 1962; KING-HELE, 1958) may be given and the semi-major axis constant may be either the value it would have if all the J_n were zero (BROUWER, MERSON, ŽONGOLOVIČ and PELLINEN) or the mean value actually observed (KOZAI, KING-HELE). The results of the four principal theories are given in Table 3. The equivalence of the results will now be demonstrated.

Take first the transformation from a rate of change with respect to time to the change in one draconic period. The former expression must be multiplied by $n_0 T_N/2\pi$ to obtain the latter, where n_0 is the mean motion for $J_2 = 0$ and T_N is the draconic period. Now, in BROUWER's notation $l'' + g''$ increases by 2π in T_N and from BROUWER's first order results,

$$l'' + g'' = n_0 t\{1 + \tfrac{3}{4}j_2'[-1 - \eta + (5 + 3\eta)C]\}$$

TABLE 3

TERMS PROPORTIONAL TO $(J_2)^2$ IN THE SECULAR MOTIONS OF NODE AND PERIGEE

NOTATION

$$\eta = (1 - e^2)^{\frac{1}{2}}, \qquad C = \cos^2 i, \qquad S = \sin^2 i$$

$$j_2' = J_2 \left(\frac{R}{a}\right)^2 \eta^{-4}$$

1. *Mean elements, a and n for $J_2 = 0$, time rates of change* (BROUWER, 1959; KOVALEVSKY, 1960).

$$\frac{h''}{n_0 t} = -\frac{3}{2} j_2' \cos i \left\{ 1 - \frac{j_2'}{16} [-5 + 12\eta + 9\eta^2 - (35 + 36\eta + 5\eta^2)\, C] \right\}$$

$$\frac{g''}{n_0 t} = -\frac{3}{4} j_2' (-1 + 5C) + \frac{3}{128} j_2'^2 [-35 + 24\eta + 25\eta^2 + (90 - 192\eta - 126\eta^2)\, C + $$
$$+ (385 + 360\eta + 45\eta^2)\, C^2]$$

2. *Osculating elements at node, semi-latus rectum for $J_2 = 0$, change in draconic period* (MERSON, 1961; ŽONGOLOVIČ, 1960a).

$$\frac{\Delta \Omega}{2\pi} = -\frac{3}{2} j_2' \cos i \left\{ 1 - \frac{j_2'}{16} [12 - 80S - (4 + 5S)\, e^2] \right\}$$

$$\frac{\Delta \omega}{2\pi} = -\frac{3}{4} j_2' (4 - 5S) + \frac{3}{128} j_2'^2 [(760 - 890S)\, S + (56 - 36S - 45\, S^2)\, e^2]$$

With mean osculating elements (ŽONGOLOVIČ and PELLINEN, 1962)

$$\Delta \Omega = -\frac{3}{2} j_2' \cos i \left\{ 1 - \frac{j_2'}{16} [12 - 20S - (4 + 5S)\, e^2] \right\}$$

$$\Delta \omega = -\frac{3}{4} j_2' (4 - 5S) + \frac{3}{128} j_2'^2 [(136 - 170S)\, S + (56 - 36S - 45S^2)\, e^2]$$

3. *Mean osculating elements, n and a for $J_2 \neq 0$, time rate of change* (KOZAI, 1959).

$$\dot{\Omega} = -\frac{3}{2} n j_2' \cos i \left\{ 1 + \frac{3}{2} j_2' \left[\frac{5}{3} - 2\eta - \frac{1}{6}\eta^2 - S\left(\frac{35}{24} - 3\eta + \frac{5}{24}\eta^2\right) \right] \right\}$$

$$\dot{\omega} = -\frac{3}{4} n j_2' (4 - 5S) \left\{ 1 + \frac{3}{2} j_2' \left[\frac{5}{2} - 2\eta - \frac{1}{2}\eta^2 - S\left(\frac{85}{48} + 3\eta + \frac{1}{48}\eta^2\right) \right] \right\}$$
$$- \frac{15}{16} j_2'^2 (1 - \eta^2) \cdot C^2.$$

4. *Mean osculating elements, mean radius vector for $J_2 \neq 0$, change in draconic period* (KING-HELE, 1958; MESSAGE, 1960).

$$\frac{\Delta \Omega}{2\pi} = -\frac{3}{2} j_2' \cos i \left\{ 1 + \frac{3}{2} j_2' \cos i \left(\frac{23}{6} S - \frac{5}{2} \right) \right\}.$$

5. *From* VINTI's *Theory* (ISZAK, 1960).

Motion of node:

$$-\frac{3}{2} j_2' \cos i + \frac{3}{16} j_2'^2 \cos i\, [18 - 13S + 24Se^2]$$

Motion of perigee:

$$\frac{3}{4} j_2 (4 - 5S) + \frac{j_2'^2}{64} [288 - 1296S + 1035S^2 - (144 + 288S - 510S^2)\, e^2].$$

where, as in the Table, $C = \cos^2 i$, $\eta = (1 - e)^{\frac{1}{2}}$,
and

$$j_2' = J_2\left(\frac{R}{a}\right)^2 \eta^{-4}.$$

Hence

$$n_0 T_N = 2\pi \{1 - \tfrac{3}{4} j_2' [-1 - \eta + (5 + 3\eta) C]\}.$$

(see also ŽONGOLOVIČ and PELLINEN, 1962)

BROUWER's formula for the node is

$$\frac{h''}{n_0 t} = -\tfrac{3}{2} j_2' \cos i_0 \{1 - \tfrac{1}{16} j_2' [-5 + 12\eta + 9\eta^2 - (35 + 36\eta + 5\eta^2) C]\}$$

so that multiplying by $n_0 T_N / 2\pi$

$$\frac{\Delta \Omega}{2\pi} = -\tfrac{3}{2} j_2' \left\{1 + \frac{j_2'}{16}[-12 + 20S + 4e^2(4 + 5S)]\right\} \quad (S = \sin^2 i_0)$$

in agreement with the result of ŽONGOLOVIČ and PELLINEN (1962) expressed in mean elements.

Likewise, multiplying BROUWER's expression

$$\frac{g''}{n_0 t} = -\tfrac{3}{4} j_2' (-1 + 5C) + \tfrac{3}{128} j_2'^2 [-35 + 24\eta + 25\eta^2 + \\ + (90 - 192\eta - 126\eta^2) C + (385 + 360\eta + 45\eta^2) C^2]$$

by $n_0 T_N / 2\pi$, it is found that

$$\frac{\Delta \omega}{2\pi} = -\tfrac{3}{4} j_2' (-1 + 5C) + \tfrac{3}{128} j_2'^2 [(136 - 170S) S + (56 - 36S - 45S^2) e^2],$$

that is, the result of ŽONGOLOVIČ and PELLINEN in terms of mean osculating elements.

A further variant arises here because MERSON (1961) and ŽONGOLOVIČ (1960, 1960a) give results in terms of the osculating elements at the *ascending node* instead of mean elements. ŽONGOLOVIČ and PELLINEN (1962) carry out the transformation to mean elements which, from the correspondence of the formulae, seem to be equivalent to the elements used by BROUWER and KOZAI.

Whereas BROUWER uses the values, a_0 and n_0 for $J_2 = 0$, KOZAI takes the actual values a and n and KOZAI's results must be multiplied by

$$\frac{n}{n_0} \cdot \left(\frac{a_0}{a}\right)^2$$

to obtain those of BROUWER.

From KOZAI's first order results,

$$\frac{n}{n_0} \cdot \left(\frac{a_0}{a}\right)^2 = 1 + \tfrac{9}{2} j_2' (1 - \tfrac{3}{2} S) \eta$$

and so

$$\frac{h''}{n_0 t} = -\tfrac{3}{2}j_2' \cos i\{1 + \tfrac{9}{2}j_2'(1 - \tfrac{3}{2}S)\eta\}$$
$$- \tfrac{9}{4}j_2'^2 \cos i\{\tfrac{5}{3} - 2\eta - \tfrac{1}{6}\eta^2 - (\tfrac{35}{24} - 3\eta - \tfrac{5}{24}\eta^2)S\}$$
$$= -\tfrac{3}{2}j_2' \cos i - \tfrac{9}{4}j_2' \cos i\{\tfrac{5}{24} - \tfrac{1}{2}\eta - \tfrac{9}{24}\eta^2 + (\tfrac{35}{24} + \tfrac{3}{2}\eta + \tfrac{5}{24}\eta^2)C\}$$

in agreement with BROUWER's results.

Similarly, taking KOZAI's expression for $\dot{\omega}$, BROUWER's result for g'' is given by

$$\frac{g''}{n_0 t} = -\frac{3}{4}j_2'(4 - 5S)\left[1 + \frac{9}{2}j_2'\left(1 - \frac{3}{2}S\right)\eta + \right.$$
$$\left. + \frac{3}{2}j_2'\left\{\frac{3}{2} - 2\eta - \frac{1}{2}\eta^2 - \left(\frac{85}{48} - 3\eta + \frac{\eta^2}{48}\right)S\right\}\right] - \frac{15}{16}j_2'^2(1 - \eta^2)C^2$$
$$= \tfrac{3}{4}j_2'(-1 + 5C) + \tfrac{3}{128}j_2'^2[-35 + 24\eta + 25\eta^2 +$$
$$+ C(90 - 192\eta - 126\eta^2) + C^2(385 + 360\eta + 457\eta^2)]$$

that is, the same as BROUWER's expression.

The equivalence of KING-HELE's theory to others is important. In terms of the draconic period, the perturbed mean radius vector and mean elements, it is

$$\frac{\Delta \Omega}{2\pi} = -\tfrac{3}{2}j_2 \cos i_0[1 + \tfrac{3}{2}j_2(\tfrac{23}{6}S - \tfrac{5}{2})]$$

where j_2 has been written instead of j_2' because e is neglected.

To obtain the result in terms of the unperturbed a_0, it must be multiplied by

$$\left(\frac{a_0}{a}\right)^2,$$

that is by

$$1 + 3j_2(1 - \tfrac{3}{2}S)$$

leading to

$$\frac{\Delta \Omega}{2\pi} = -\tfrac{3}{2}j_2 \cos i - \tfrac{3}{2}j_2^2 \cos i(\tfrac{3}{4} - \tfrac{5}{4}S)$$

which is the result of ŽONGOLOVIČ and PELLINEN for $e = 0$.

Lastly, to relate BROUWER's result and MESSAGE's form of KING-HELE's, the former must be multiplied by

$$\frac{n_0 T_N}{2\pi} \times \left(\frac{a}{a_0}\right)^2,$$

that is, by

$$1 - \tfrac{3}{2} j_2' (5 - 7S)$$

leading to

$$\frac{\Delta \Omega}{2\pi} = -\tfrac{3}{2}\, j \cos i \,\{1 + \tfrac{3}{2} j_2 (\tfrac{23}{6} S - \tfrac{5}{2})\}$$

which is MESSAGE's result.

To summarize, it has been shown that, apart from MERSON's results in terms of smoothed elements, the four theories considered give equivalent results for the second order secular motions of node and perigee.

In addition, LYDDANE and COHEN (1962) have shown that there is good agreement between BROUWER's theory and numerical integration by COWELL's method.

4. Special Problems

The theories which are of by far the greatest importance in the determination of the external potential of the Earth have been discussed in Section 3 but as indicated there, they cannot be applied in all circumstances and may break down for very small inclinations or eccentricities or when the inclination is $\cos^{-1} 5^{-\frac{1}{2}}$ for which the J_2 term in the secular motion of the perigee vanishes. They also take no account of variation of the potential with longitude, that is of tesseral harmonics in the potential. Although these special problems are of considerable interest, they are not of great practical importance in the estimation of the potential of the Earth, except that as the accuracy of observations increases, it should be possible to obtain more information about the tesseral harmonics. In general however, the special circumstances in which the theories of the effects of zonal harmonics break down can be avoided although the Russian satellites have inclinations close to the critical one.

It is quite straightforward to include tesseral harmonics in the disturbing function and to obtain the corresponding short-periodic variations of the elements of the orbit to the first order (GROVES, 1960; KAULA, 1961; KOZAI, 1961; ISZAK, 1960; O'KEEFE and BACHELOR, 1957; ROBE, 1959; MUSEN, 1960; SEHNAL, 1960). It is found that if the potential is taken in the form

$$V_{pq} = -J_{pq}\left(\frac{R}{r}\right)^{p+1} P_p^q(\cos\theta)\cdot\cos q\,(\lambda - \beta_{pq})$$

where $P_p^q(\cos\theta)$ is an Associated LEGENDRE function, λ is the longitude referred to axes fixed in the Earth, and β_{pq} is a constant, then most elements show variations proportional to

$$J_{pq} \begin{matrix}\sin\\ \cos\end{matrix}\, q\,(l_{pq} - \Omega)$$

where

$$l_{pq} = \lambda_{pq} + \text{Greenwich sidereal time}.$$

4.1 SPECIAL VALUES OF ECCENTRICITY AND INCLINATION

It can be seen from Tables 2b and 2c that the long periodic terms in the motions of node and perigee for odd harmonics appear to become very large when the eccentricity or inclination are small. Again the term proportional to J_2^2 in the motion of motion of perigee has a component $e^{-1}\cos\omega$ when expressed in osculating elements at the node. The last is not a real singularity for it vanishes when the results are expressed in smoothed elements (MERSON, 1961) or in mean elements (BROUWER, 1959) but the first order terms do have real singularities, again to some extent a feature of the way in which the motion is described, for they do not imply that the perturbations of the satellite, regarded as changes in rectangular co-ordinates, are large. Rapid changes in the longitude of the node when the inclination is small, for instance, are just a geometrical consequence of the fact that the angle between the orbital plane and the equator is in fact small. The theoretical problem is therefore to choose elements that will be better adapted to the situation. It is well known that the way to do that is to take quantities

$$u = M + \omega , \qquad \xi = e\cos\omega , \qquad \eta = -\,e\sin\omega$$

when the eccentricity is small and

$$h_2 = \sin i \sin ☊ , \qquad k_2 = \sin i \cos ☊$$

when the inclination is small (PLUMMER, 1960, p. 148). It is a straightforward matter to write down the equations of variation for these pairs of elements and KOZAI has used the first set to solve the problem of the artificial satellite with small eccentricity (KOZAI, 1960). He then shows the variations in the radius vector and the argument of latitude, $u = v + \omega$ are

$$\mathrm{d}r = a\left(1 + \tfrac{1}{6}\sin^2 i\cos^2 u\right)$$
$$\mathrm{d}u = -\left(\tfrac{1}{2} - \tfrac{7}{12}\sin^2 i\right)\sin 2\lambda$$

which are the same as are deduced from the theory in which e is not small, showing that the variations in the position of the *satellite* are not singular.

KOZAI points out that it is possible for there to be two peri-positions in a nearly circular orbit.

VINTI's theory as originally developed, could not be applied too close to zero inclination but he has subsequently shown (1962) how to overcome that limitation.

The preceding problems are ones of defining the parameters in which to describe the bounded motion of a satellite but the difficulties connected with the critical inclination of cos i reflect a real physical situation. As can be seen from Table 2b the secular motion of perigee vanishes at this inclination and certain terms which contain $(1-5\cos^2 i)$ as a divisor become infinite. The problem is to some extent akin to that of small divisors arising from nearly commensurable motions in planetary theory and as in that context has been studied by developing the disturbing function not in terms of J_2 but in terms of $(J_2)^{\frac{1}{2}}$. (GARFINKEL, 1960; HORI, 1960). HORI's treatment is in terms of a

VON ZEIPEL type of transformation of the canonical variables but with the determining function and the Hamiltonian expanded in powers of $(J_2)^{\frac{1}{2}}$. The results for the secular motions in the neighbourhood of the critical inclination are

$$\frac{\mathrm{d}\dot{g}}{\mathrm{d}t} = -\tfrac{3}{2}\frac{\mu^2 k_2}{L^3 G'^4} + \tfrac{3}{10}\frac{\mu^6 k_2^2}{L^5 G'^6}\left\{-7 + \tfrac{50}{3}\lambda + \frac{L^2}{G'^2}\left(9 - \tfrac{82}{3}\lambda\right)\right\}$$

and

$$\frac{\mathrm{d}\dot{h}}{\mathrm{d}t} = \frac{3}{5^{\frac{1}{2}}}\frac{\mu^4 k_2}{L^3 G'^4}\left(1 - \frac{t}{2} - \frac{t^2}{8}\right) + \frac{3}{5^{\frac{1}{2}}}\frac{\mu^6 k_2^2}{L^5 G'^6}\left[-9 + \tfrac{50}{3}\lambda - \tfrac{12}{5}\frac{L}{G'} + \right.$$

$$\left. + \frac{L^2}{G'^2}(11 - 22\lambda)\right]$$

where

$$k_2 = \tfrac{1}{2}J_2R^2, \qquad k_4 = -\tfrac{3}{8}J_4R^4, \qquad \lambda = -\tfrac{3}{2}J_4/J_2^2$$

and

$$t = 1 - 5\cos^2 i_0 .$$

HORI shows that there are no singularities in the behaviour near the critical inclination if $J_4/(J_2^2)$ is -1. This is just the value of the ratio for VINTI's potential and there is in fact no singularity in the solution that VINTI obtains.

The general feature of the solution derived by GARFINKEL, HORI and STRUBLE (1961) may be understood by reference to Figure 6. In this the rate of rotation of ω as a function of u is plotted against the value of ω. When the trajectories are straight lines parallel to the axis of ω, the value of ω increases continually and there is a secular motion of perigee but when the trajectories are closed curves, the value of ω oscillates

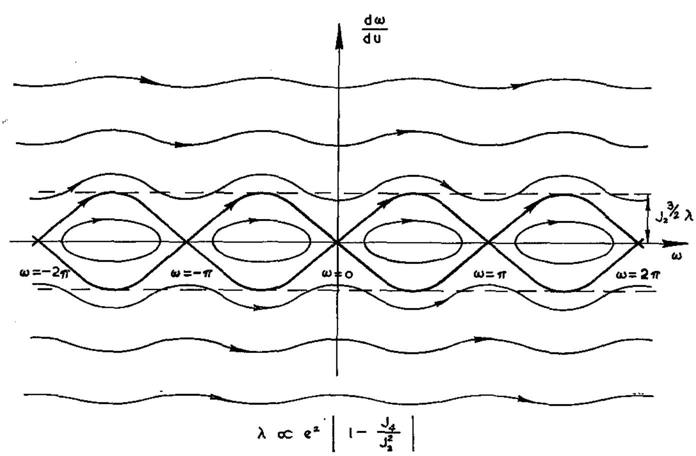


Fig. 6. Oscillations of Perigee.

about the points of maximum latitude if $J_4/J_2^2 > -1$ and about the nodes if $J_4/J_2^2 < -1$. In the former case the nodes are unstable points and in the latter case, the points of maximum latitude are unstable. The amplitude of the oscillatory part of the curves is proportional to

$$-e\left(1+\frac{J_4}{J_2^2}\right)^{\frac{1}{2}} J_2^{\frac{3}{2}}.$$

As with the cases of zero eccentricity and inclination, it has been suspected that the behaviour at the critical inclination is a feature of the parameters used to describe the motion and not of the motion itself but MESSAGE, HORI and GARFINKEL (1962) have shown that the radius vector and the sine of the latitude as derived from BROUWER's theory both contain the factor $(1\text{–}5\cos^2 i)$ as a small divisor and therefore argue that the singularity is a physical one.

ISZAK (1962), however, maintains that the development in powers of $J_2^{\frac{1}{2}}$ itself breaks down when the terms of order J_2^2 are included and he has undertaken an analysis of the problem in terms of VINTI's theory.

4.2 COMMENSURABLE MOTIONS

If the potential of the Earth is truly axi-symmetrical the orbit depends in no way on the period of rotation of the Earth but if the potential depends on longitude then it is possible that special effects may arise when the period of the satellite is commensurable with the rotation of the Earth. The possibility, first examined by GROVES (1960) and A. H. COOK (1961), is of especial interest in connexion with satellites of period 24h and MUSEN (1962), MUSEN and BAILIE (1962) and BLITZER *et al.* (1962) have studied this particular problem.

In the general case, take a term in the potential

$$V_{pq} = J_{pq}\cdot\frac{R_m^p}{{}_r pM} P_p^q(\cos\theta)\sin q\,(l+\beta_{pq})$$

where l is the longitude measured from axes fixed in the Earth.

If λ is the longitude of the satellite and if λ_g is the longitude of Greenwich, referred to sidereal axes, then

$$l = \lambda - \lambda_G = \lambda - \omega_r t$$

where ω_r is the spin angular velocity of the Earth about her polar axis.

Now if

$$\psi = \lambda - \Omega$$

then

$$\tan\psi = \tan(\omega + v)\cos i$$

or for small inclinations,

$$\psi \approx \omega + v - \tfrac{1}{4}\gamma^2 \sin 2(\omega + v)\ldots$$

where

$$\gamma = \tan i.$$

Hence

$$l = \omega + \Omega + v - \omega_r t - \tfrac{1}{4}\gamma^2 \sin 2(\omega + v) \ldots$$

When V_{pq} is differentiated for substitution in the right side of the LAGRANGE equations, terms with factors such as

$$\sin 2(\omega + v) \cos 2(l + \beta_{22})$$

are obtained. Such a term will give rise to a secular or very long periodic term in an element if

$$\omega + v = \omega + v + \Omega - \omega_r t - \tfrac{1}{4}\gamma^2 \sin 2(\omega + v).$$

That relation cannot be satisfied because $\dot{\Omega}$ is about 4 deg/day, but for a term like

$$\sin^2(\omega + v) \sin 2(l + \beta_{22}) \cos M,$$

the relation is

$$M = 2(l + \beta_{22}).$$

In general it is not possible to satisfy this relation exactly because M contains long-periodic parts, but if

$$M = 2(l + \beta_{22}) - \eta$$

there will be a term proportional to sin η in the variation of the element. In this way terms proportional to cos η or sin η can be picked out and are shown in Tables 4 and 5.

Now.

$$\omega = \omega_0 + \dot{\omega} t + \delta\omega_{\text{per}},$$

$$\Omega = \Omega_0 + \dot{\Omega} t + \delta \Omega_{\text{per}}$$

and

$$M = nt + \chi + \delta M_{\text{per}}$$

and therefore if

$$n = 2\omega_r - 2(\dot{\omega} + \dot{\Omega}),$$

$$\eta = 2(\omega_0 + \Omega_0 + \beta_{22}) + \lambda + \text{per. terms}.$$

It will be noted that η depends on χ, the position of the satellite at zero time and it follows that if the amplitude and phase angle of an harmonic term are both unknown, they cannot be separated from data on a single orbit.

In general if $(p - q)$ is even,

$$q(l + \beta_{pq}) = M + \eta,$$

$$(q - 1)n + q(\dot{\omega} + \dot{\Omega}) = q\omega_r$$

and

$$\eta = q(\omega_0 + \Omega_0 + \beta_{pq}) + (q - 1)\chi + \ldots$$

while if $(p - q)$ is odd,

$$q(l + \beta_{pq}) = \omega + v + \zeta$$

$$(q - 1)(n + \dot{\omega}) = q(\omega_r -)\dot{\Omega}$$

and

$$\zeta = (q - 1)(\omega_0 + \chi) + q(\Omega_0 + \beta_{pq}) + \dots$$

NORTON, in an appendix to COOK's paper, has given a complete set of the conditions for commensurability and the speeds of the motions that arise when they are satisfied.

The above conditions apply when e is small but if that is not so, the expansion of the disturbing function contains terms such as $e^{\nu} \sin \nu M$ which give rise to conditions like

$$q(l + \beta_{pq}) = \nu(M + \eta_{\nu})$$

which will be satisfied approximately whenever

$$n = \frac{q\omega_r}{q - \nu}.$$

But there is an infinite set of pairs (q, ν) that give the same value of n and so an infinite set of harmonic terms contributes to a variation of an element at any particular speed. Again, when $\nu = 1$ and q is large, the mean motions that satisfy the condition approach the angular velocity of the Earth. However, the situation is much simpler

TABLE 4

FACTORS OF TERMS PROPORTIONAL TO SIN η OR COS η IN DERIVATIVES OF V_{22}

	$\eta = 2(l + \beta_{22}) - M$			
	Order of term			
Derivative with respect to	**1**	e	γ^2	γe
a	—	$-\frac{27}{2a} e \sin \eta$	—	
e	$-\frac{3}{2}\sin \eta$ —	—	$\frac{3}{4}\sin^2 i \sin \eta$	
S	—	$\frac{21}{2} e \cos \eta$	—	
Ω	—	$9e \cos \eta$	—	
ω	—	$9e \cos \eta$	—	
i	—	—	—	$-\frac{9}{2} e \sin i \cos i \sin \eta$

All terms to be multiplied by $\frac{J_{22}}{a_e}\left(\frac{a_e}{a}\right)^3$

TABLE 5

FACTORS OF TERMS PROPORTIONAL TO SIN η OR COS η IN RATES OF CHANGE OF ELEMENTS. $\eta = 2(l + \beta_{22}) - M$

Element	*Order of term*				
	e^{-1}	1	e	$\gamma^2 e^{-1}$	γe
$\dot{a}$	–	–	$\frac{21}{na} e \cos \eta$	–	–
$\dot{e}$	–	$\frac{3}{2\,na^2} \cos \eta$	–	–	–
$\dot{\chi}$	$\frac{3}{2\,na^2} \frac{\sin \eta}{e}$		$\frac{27}{na^2} e \sin \eta$	$-\frac{3}{4\,na^2} \frac{\sin^2 i}{e} \sin \eta$	–
$\dot{\Omega}$	–	–	$-\frac{9}{2\,na^2} e \cos i \sin \eta$	–	–
$\dot{\omega}$	$-\frac{3}{2\,na^2} \frac{\sin \eta}{e}$	–	$\frac{9}{2\,na^2} e \cos^2 i \sin \eta$	$\frac{3}{4\,na^2} \frac{\sin^2 i}{e} i \sin \eta$	–
$\frac{\mathrm{d}i}{\mathrm{d}t}$	–	–	–	–	$-\frac{9}{na^2} e \tan \frac{1}{2} i \cos \eta$

All terms to be multiplied by $\frac{J_{22}}{a} \left(\frac{a_e}{a}\right)^3$

for circular orbits. MUSEN (1962) has developed more satisfactory methods for dealing with appreciable eccentricity.

MORANDO (1962), using canonical variables, has derived very similar results but he also is more concerned with the stability and libration of the 24h satellite.

There does not seem much likelihood of satellites with commensurable motions being used to determine tesseral harmonics, mainly because even if satellites with the correct mean motions and small eccentricities were launched, the difficulties involved in separating the effects of individual harmonic terms seem to be too great. Should the observation of such satellites become a practical possibility, it would be necessary to consider the effect of the Sun and the Moon on the stability of the commensurability conditions.

5. Observations and Results

5.1 OBSERVATIONS

The most significant and, until recently, the only information about the gravitational potential of the Earth to come from observations of satellites is values of the zonal harmonics of the lower orders derived from the secular and long-periodic changes of the elements of orbits, in particular, values of the even harmonics from the secular changes of node and perigee, the only elements that show such changes, and values of the odd harmonics from the long-periodic changes in eccentricity for which they are particularly well defined. These topics will be the main matter of this section. It should be noticed that it is long-periodic changes with argument ω that are of greatest importance because changes with arguments that are multiples of ω are smaller by a factor of e at least and most orbits that are dealt with have eccentricities of 0.2 or less.

Satellites are observed visually or photographically or by radio interferometers or by means of Doppler radio measurements. Doppler measurements give the velocity of the satellite relative to the observing station and are proving to be a very powerful means of tracking satellites accurately as has been shown by the performance of the Transit system. But only special satellites are equipped with the necessary radio transmitters emitting very accurately controlled frequencies and the ground equipment is not generally available, so that most of the data from which the Earth's potential has been deduced have been obtained from observations of the directions of satellites made optically or with radio interferometers. Details of different systems will be found in the report of the second COSPAR symposium published as Space Research II, 1961 and particularly the paper of VEIS and WHIPPLE (1961) and MASSEVIČ (1961). The most extensive systems are the Minitrack radio network and the Smithsonian Astrophysical Observatory Baker-Nunn cameras, both of which are distributed over the whole Earth. At the same time very valuable data have been acquired from relatively simple instruments because the motion of the node in particular can be obtained from rather elementary observations at one site and most of the early and most important discoveries were made in this way (MERSON and KING-HELE, 1958; MERSON, 1959).

The observed directions must be converted to directions from the centre of the

Earth, which can in the first instance be done with an approximate distance deduced from the period, and corrections must be applied for zero errors of the equipment. Elliptic elements can then be fitted, usually by an iterative computer program, to runs of observations. Particulars of such processes are given by MERSON (1959), MERSON and KING-HELE (1958) and MASSEVIČ (1961a). The orbits so derived will show changes due to air drag and these must be removed before long-periodic changes, especially in the eccentricity, can be isolated. One way of doing this (O'KEEFE, ECKELS and SQUIRES, 1959), is to determine the change in a from the change in period and then to assume that the distance of perigee does not change, as is very closely true when most of the drag occurs near perigee, and hence derive the change in e. When that has been done, the changes in the other elements due to drag may be deduced. It may be possible to improve the correction for drag by taking advantage of the fact that the drag is correlated with solar activity. When an estimate of the change in a is available, corrections can be applied for the rotation of the atmosphere which gives rise to changes in node and perigee. The following formulae have been obtained by G. E. COOK (1961):

$$\frac{\mathrm{d}\Omega}{\mathrm{d}a} = \frac{I_2 + 2eI_1 + cI_4 \cos 2\omega}{I_0 + 2eI_1 + cI_2 \cos 2\omega} \cdot \frac{T_d \Lambda \sin 2\omega}{4a(1 - {}_dT\Lambda \sin i}\{1 + 0(e^2)\}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}a} = \cos i \cdot \frac{\mathrm{d}\Omega}{\mathrm{d}a} - \frac{2cI_2 - \frac{1}{2}ce(I_1 - 9I_3) + 2c^2 J_4 \cos 2\omega}{I_0 + 2eI_1 + cI_2 \cos 2\omega} \cdot \frac{H \sin 2\omega}{a^2 e^2} \times$$

$$\times \{1 + 0(e^2)\}$$

where T_d is the orbital period in days,

$$\Lambda = \frac{\text{angular velocity of atmosphere}}{\text{angular velocity of Earth}}$$

and I_0 I_1, ... are BESSEL functions of the first kind with imaginary argument.

Corrections must also be applied for the attraction of the Sun and the Moon which give rise to secular and long-periodic terms in the motion of node and perigee. KOZAI (1959) gave formulae for the secular parts and G. E. COOK (1962) has included the long-periodic parts in the following formula:

$$\dot{\Omega} = \frac{3KC}{4n(1 - e^2)\sin i}\left[5Ae^2 \sin 2\omega + B(2 + 3e^2 - 5e^2 \cos 2\omega)\right]$$

where $K = n_d^2 m_d$, n_d being the mean motion of the disturbing body and m_d its mass in terms of the Earth's mass.

$$A = \cos(\Omega - \Omega_d)\cos u_d + \cos i_d \sin u_d \sin(\Omega - \Omega_d)$$
$$B = \cos i\left[-\sin(\Omega - \Omega_d)\cos u_d + \cos i_d \sin \mu_d \cos(\Omega - \Omega_d)\right] + \sin i \sin i_d \sin \mu_d$$
$$C = \sin i\left[\cos \mu_d \sin(\Omega - \Omega_d) - \cos i_d \sin \mu_d \cos(\Omega - \Omega_d)\right] + \cos i \sin i_d \sin \mu_d .$$

d refers to the disturbing body, u is the argument of latitude.

There is a similar result for the contribution to the motion of perigee. The effect of the Sun is about half that of the Moon. It will be noticed that the periodic parts are very small for nearly circular orbits. KAULA (1962) and SMITH (1962a) have also published theories of the luni-solar effects.

Radiation pressure from the Sun can also affect the orbit of a satellite and BRYANT (1961) has given formulae to determine the effects by numerical integration. The perturbations are very small for satellites with a small ratio of area to mass and satellites with a large area/mass ratio would not in practice be used for gravitational studies.

The quantity actually determined from the data is not the coefficient J_2, say, but the combination

$$J_2\mu^{-2/3} R_m^2 = -\tfrac{2}{3}\,\dot{\Omega}_2\, T_N^{7/3} \sec i \left[1 - \tfrac{3}{8} J_2 \left(\frac{R}{p}\right)^2 (7\cos^2 i - 1)\right]^{-7/3}.$$

Thus for J_2 at least, it is necessary to use good values of the Earth's radius constant R, and of the constant μ, and when values of J_2 are to be combined with other data to determine consistent values of geodetic constants, it must be borne in mind that the observation equation is of this form.

It will be seen that the inclination has a dominant influence on the magnitudes of the various changes of the elements and it is therefore important to have a good value of this parameter. The value of J_2 is especially dependent on the value of i because J_2 is about 1000 times greater than any other coefficient, and because the change of the node can be found very precisely if the observations can be continued for long enough, it may happen that the uncertainty in J_2 is determined almost entirely by the uncertainty of the inclination.

5.2 EVEN HARMONICS

As soon as observations were obtained on Sputnik 2, it was found that the motion of the node was very definitely less than would be expected from the value of J_2 inferred from surface gravity measurements, (BUCHAR, 1958; Harvard Card, 1958; KING-HELE and MERSON, 1958; MERSON, 1959) and although J_2 and J_4 could not be separated by means of data from just one satellite, it seemed, by taking data from gravity measurements into account (A. H. COOK, 1958) that the value of J_2 was about 1083×10^{-6}. As soon as Vanguard 1 was launched, however, it was possible to find J_2 and J_4 separately from satellite observations alone (JACCHIA, 1958; KING-HELE, 1959a) using the observed motions of the nodes of Sputnik 2 and Vanguard 1 which have the very different inclinations of 65° and 34° respectively. There was also an indication that J_6 might be significant, and with the launching of Explorer 4 at yet a third inclination (50°) it was possible to obtain the following values: (KING-HELE and MERSON, 1959; KING-HELE, 1959b).

$$\begin{aligned} 10^6 J_2 &= 1083.0 \pm 0.2 \\ 10^6 J_4 &= -\ 1.3 \pm 0.2 \\ 10^6 J_6 &= -\ 0.1 \pm 1.5 \end{aligned}$$

TABLE 6

ESTIMATES OF HARMONICS OF EVEN ORDER COEFFICIENTS $10^6 J_n$ (IN TERMS OF EQUATORIAL RADIUS)

Author	*Satellites*	*Values of i*	*Method*	J_2	J_4	J_6	J_8
O'Keefe and others, 1959	1958 $\beta2$*	34°	Numerical fit to ω and ☊	1082.5 (0.1)**	−1.7 (0.1)		
Kozai, 1961	1959 η, 1958 $\beta2$, 1959 η	50°, 34°	Secular terms ω and ☊	1082.19 (0.03)	−2.1 (0.1)	−0.23 (0.02)	
Michielsen, 1961	1960 $\varepsilon1$, 1960 $\gamma1$, 1958 $\beta2$	65°, 51°, 34°	Secular terms in ☊	1082.66	−1.72	+0.73	
			Secular term in above and term in ω for 1958 $\beta2$	1082.66	−1.68	+0.73	+0.08
Smith, 1961	1958 $\delta2$, 1959 $\alpha1$	65°, 33°	Secular terms in ☊	1083.15 (0.2)	−1.4 (0.3)	0.7 (0.6)	
	1960 $\beta2$, 1960 $\gamma1$	50°		1082.8 (0.2)	−1.3 (0.3)	if J_6 assumed zero	
Žongolovič, 1960	1957 $\beta1$, 1958 $\delta1$, 1958 $\delta2$	65°	Secular terms and l–p ☊ and ω	1083.4 (0.7)	−4.1 (0.7)		
Buchar, 1958	1957 $\beta1$, 1958 $\delta2$	65°					
	1959 $\alpha1$, 1958 $\beta2$, 1958 α	30°		1083.0 (0.4)	−1.1		
Kozai, 1962	1957 $\beta1$, 1958 $\delta1$, $\delta2$, 1961 $o2$, 1960 $\eta1$	65°	Secular motions of ☊ and ω	1082.48 (0.04)	−1.84 (0.09)	0.39 (0.09)	−0.02 (0.07)
	1960 $\gamma1$ 1960 $\gamma2$, 1960 $\iota2$ 1960 $\beta2$, 1960 $\iota1$	47–51°					
	1958 $\beta1$, $\beta2$, 1959 $\alpha1$, 1959 η	33°					

TABLE 6 *(continued)*

Author	*Satellites*	*Values of i*	*Method*	J_2	J_4	J_6	J_8
KING-HELE, COOK	1961 α1	97.4					
and REES, 1963	1960 η1	66.7	Secular motions	1082.78	−0.78	0.70	0.24
	1962 o1	53.8	of ☊	(0.05)	(0.2)	(0.1)	(0.2)
	1960 ι2	47.2		together with	$10_6J_{10} = -0.50 \pm 0.2$		
	1961 δ1	38.8			$10_6J_{12} = -0.28 \pm 0.2$		
	1959 α1	32.9		1082.79	−1.09	0.73	three terms
	1961 ν	28.8		(0.05)	(0.20)	(0.20)	only

* KEY

1957 β1 Sputnik 2
1958 α Explorer 1
1958 β1 Vanguard 1 Rocket
β2 Vanguard 1
1958 δ1 Sputnik 3 Rocket
δ2 Sputnik 3
1959 α1 Vanguard 2
1959 η Vanguard 3
1959 ι Explorer 7
1960 β2 Tiros 1
1960 γ1 Transit 1B Rocket
1960 γ2 Transit 1B
1960 ε1 Sputnik 4
1960 η1 Transit 2A
1960 ι1 Echo 1
1960 ι2 Echo 1 Rocket
1960 ζ1 Explorer 8
1960 π1 Tiros 2
1961 α1 Samos 2
1961 δ1 Explorer 9
1961 ν Explorer 2
1961 o1 Transit 4A
1962 o1 Ariel 1

** Figures in brackets are authors' estimates of standard deviations.

(Note that the values of J_2 are given here and elsewhere in terms of the equatorial radius of the Earth and not the mean radius, for the sake of easy comparison with other papers, although in a wider geodetic context it is advantageous to use the mean radius).

The preceding work was based entirely on the motion of the node but the very accurate orbits that were obtained for Vanguard 1 on account of its being very little perturbed by drag, made it possible to derive J_2 and J_4 from the one satellite by itself. It was mentioned that for the purposes of determining orbits by numerical integration, theories had been developed for computer application in which numerical instead of literal values of the various parameters were used. It is necessary in these theories that the calculated and observed secular motions of the node and perigee should be in exact agreement and this condition can be fulfilled by iterative changes of the numerical parameters until the observed and calculated positions of the satellite agree. The first results obtained from Vanguard 1 alone were reported by LECAR, SORENSON and ECKELS (1959) and subsequently the following values were given by O'KEEFE, ECKELS and SQUIRES (1959a):

$$10^6 J_2 = 1082.5 \pm 0.1$$
$$10^6 J_4 = -1.7 \pm 0.1.$$

It will be seen that despite the quite different method of deriving these results, they agree well with those from the three satellites. It may be mentioned that it has been suggested that they do not take account of the J_2^2 contribution to the secular terms but in fact it seems that although this is not done explicitly it is taken account of by means of the numerical procedure.

With the many satellites now launched, a number of more detailed studies have recently been made, and the results for the harmonics of even order are summarised in Table 6. The harmonic coefficients are found by solving equations of the form

$$a_2 J_2 + a_4 J_4 + a_6 J_6 + \cdots = \delta_a$$
$$b_2 J_2 + b_4 J_4 + b_6 J_6 + \cdots = \delta_b$$

one such equation being obtained for each satellite. The right sides are the observed secular motions of the node or perigee and the coefficients on the left hand side are functions of the orbital elements and, since the ranges of a and e are not very great for most satellites that have been launched, the coefficients are most dependent on the inclination. Then if two satellites have the same inclination it is found that they give almost identical left sides in the equations, and the separation of the different harmonic coefficients J_n is not very satisfactory unless the satellites used have very different inclinations. Inclinations are fixed by such considerations as range instrumentation and safety and so most satellites have inclinations that fall in three rather narrow bands near 34°, 50° and 65° but recently it has been possible to obtain observations on Discoverer satellites with very high inclinations.

Where many observation equations are available it is found that quite large differences can arise in the calculated J's according to the data used and the numbers of harmonics determined. Žongolovič (1960) uses the changes of ω and ☊ and takes account of the long-periodic terms to solve for J_2, J_3 and J_4. According to the data he includes, he obtains values in the following ranges:

$$10^6 J_2 : 1080.6 \text{ to } 1084.1$$
$$10^6 J_3 : \quad 1.7 \text{ to } \quad 10$$
$$10^6 J_4 : -3.9 \text{ to } -5.0$$

Smith examined the effect of omitting J_6 from his solution and as can be seen from the Table 6, J_2 is appreciably altered but J_4 is almost unchanged. Kozai (1962), on the other hand, found a large change in J_4 as well as in J_2 when J_6 was included but the addition of J_8 made little difference:

Solution	$10^8\,\delta J_2$	$10^8\,\delta J_4$	$10^8\,J_6$	$10^8\,J_8$	*Standard error of Observation*
1	–	–	–	–	13.5
2	10.1	–	–	–	3.0
3	10.4	0.6	–	–	3.0
4	30.8	29.3	41.6	–	1.0
5	29.3	26.1	38.6	−2.4	1.0

The latest determination is by King-Hele, Cook and Rees (1963 and private communication) who have used data from seven satellites with inclinations covering the range from 28° to 97°, a considerable improvement on previous sets of data, for Kozai's most recent values really depend on only three mean satellites. They find it possible to derive the coefficients of $J_2 \ldots J_{12}$ but if the potential is to be represented by fewer harmonics, they find that it is best fitted by the first three, J_2, J_4 and J_6. In view of this finding, it is useful to compare the values obtained by the different authors who have fitted three terms:

	$10^6\,J_2$	$10^6\,J_4$	$10^6\,J_6$
Michielsen, 1961	1082.66	−1.72	0.73
Smith, 1961	1083.15	−1.4	0.7
Kozai, 1962	1082.48	−1.81	0.42
King-Hele *et al.*, 1963	1082.79	−1.09	0.73

The values of J_2 are not very sensitive to the number of terms retained provided at least J_6 is included and except for Kozai's result, they agree well. The analysis of King-Hele *et al.* shows that J_4 depends very much on the range of inclination employed, for their value is considerably less than that of all authors and they are the first who have used data with large inclinations. Again, apart from Kozai, the values of J_6 agree remarkably well considering the uncertainties assigned to them. On the face of it, the differences between Kozai's results and those of other workers seem to be connected with his use of data from perigee. A rough statistical analysis shows clearly how Kozai's results stand apart from the others. The following are the values of χ^2 obtained on dividing the departure of each result from the mean value by the

TABLE 7

ESTIMATES OF HARMONICS OF ODD ORDER

Author	*Satellites*	*Values of i*	*Data*	$10^6 J_3$	$10^6 J_5$	$10^6 J_7$	$10^6 J_9$
MICHIELSEN, 1961	1960 ε1*	65°	radius at perigee	{ −2.46	+0.33	−0.64	–
	1960 γ1	51°		{ −2.27	−0.12	–	−0.55
	1958 β2	34°	motion of perigee	{ −2.54	−0.03	−0.43	–
				{ −2.41	−0.33	–	−0.37
SMITH, 1961	1960 ξ1	50°		−2.37	−0.05		
	1960 π1	48.5	Eccentricity	(0.18)**	(0.15)		
	1961 ν	29°					
O'KEEFE *et al* 1959	1958 β2	34°		−2,4	−0.1		
				(0.3)	(0.1)		
KOZAI, 1961	1959 η	50°	e, i, ω, Ω	−2.29	−0.23		
	1958 β2 1959 ι1	34°		(0.02)	(0.02)		
NEWTON *et al.*, 1961	1960 η1	66°	perigee	−2.36	−0.19	−0.28	
	1960 γ2	51°	distance	(0.14	(0.10)	(0.11)	
	1958 β2	34°					
KOZAI, 1962	1960 η, 1961 o2	65°	e, i, ω, Ω	−2.562	−0.064	−0.470	0.117
	1960 β2, 1960 ι2	47–51°		(0.007)	(0.007)	(0.010)	(0.011)
	1958 β1, 1958 β2	33°					
	1959 α1, 1959 η						
	1959 ι						

*KEY

1958 β1 Vanguard 1 Rocket
β2 Vanguard 1
1959 α1 Vanguard 2
1959 η Vanguard 3
1960 β2 Tiros 1
1960 γ1 Transit 1B Rocket
1960 γ2 Transit 1B
1960 ε1 Sputnik 4
1960 η1 Transit 2A
1960 ι2 Echo 1 Rocket
1960 ξ1 Explorer 8
1960 π1 Tiros 2
1961 ν Explorer 2

** Standard deviation. D. E. SMITH (*Planet. Space Sci.* **11** (1963) 789) obtains:

$$10^6 J_3 = -2.44 \pm 0.07$$
$$10^6 J_5 = -0.18 \pm 0.03$$
$$10^6 J_7 = -0.30 \pm 0.03$$

assigned standard deviation, squaring the quotient and summing the squares for each harmonic. The expected value is the number of separate results, in this case 3 for each harmonic, since MICHIELSEN assigns no standard deviation. The values of χ^2 are

$$J_2 : 60$$
$$J_4 : 15$$
$$J_6 : \ 6$$

and in each case, the sum is almost entirely due to KOZAI's value. In part this is because KING-HELE's uncertainties are on a more liberal basis than KOZAI's which are derived strictly from the residuals of his observation equations whereas KING-HELE and his colleagues make some allowance for neglect of higher harmonics. This however only reinforces the conclusion that there is a real difference between KOZAI and other workers, that it is not due to statistical fluctuation but reflects some difference

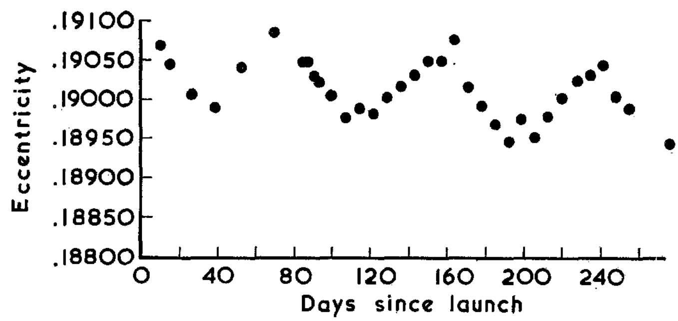


Fig. 7. Eccentricity of satellite 1958β2 (Vanguard). (From O'KEEFE, ECKELS, and SQUIRES, 1959)

apparently connected with the use of data from perigee, either in the theory employed or in systematic effects in the data available.

In the circumstances it is difficult to estimate a statistically optimum value and to assign it a standard deviation, for the discrepancies between the results of the different workers are clearly not of random origin. It seems that at the present time the most probable values are those of KING-HELE and his collaborators and they must certainly have greater weight than the other results because of the greater range of inclination of the satellites that were included.

5.3 ODD HARMONICS

Because the Vanguard 1 satellite is in an orbit for which air drag is very small, the secular change in the eccentricity is very small and it was possible to see long-periodic changes that for the Russian satellites were masked by the reduction and irregular variations due to drag. O'KEEFE, ECKELS and SQUIRES (1959) using data for the first 240 days (Figure 7) were thus able to establish that there must be odd harmonics in the potential of the Earth and subsequently (1959a) they were able to estimate J_2 and J_5, as well as even harmonics, from the motion of Vanguard 1 alone.

The estimates now available are listed in Table 7. KOZAI (1961) again shows the effect of including additional terms in the solution:

Solution	$10^8 J_3$	$10^8 J_5$	$10^8 J_7$	$10^8 J_9$	*Standard error of Observation*
1	0	–	–	–	84
2	–234.1	–	–	–	2.8
3	–230.0	–9.3	–	–	2.7
4	–255.3	–8.3	–41.5	–	1.9
5	–256.2	–6.4	–47.0	11.7	1.0

The standard error is the rms scatter of the residuals of the observations and from the way in which it changes with the terms included it will be seen that the third harmonic is much more definitely established than the others, that J_5 is not really significant, but that J_7 and J_9 do seem to be statistically significant. This behaviour is consistent with the scatter of estimates made by different authors. There is no serious doubt about the first order theory of the long-periodic terms and in particular the results for the eccentricity, which has considerable weight in the numerical work, are secure. The differences between results are therefore almost certainly due only to observational errors and to the different effects of neglecting higher harmonics at different inclinations. In these circumstances it is permissible to take the best values to be the mean of values from different studies; they are then

$$
\begin{aligned}
10^6 J_3 &: -2.41 \quad \pm 0.04 \\
10^6 J_5 &: -0.055 \pm 0.08 \\
10^6 J_7 &: -0.45 \quad \pm 0.09 \\
10^6 J_9 &: \quad 0.12
\end{aligned}
$$

5.3 TESSERAL HARMONICS

It was seen in Section 4 that tesseral harmonics give rise to terms with argument $m(l_{nm} - \Omega)$ in the variations of the elements. Since Greenwich sidereal time is involved, errors in the longitudes of observatories will lead to errors in the estimates of these terms and so it is not possible to derive them independently of station positions. For this reason the problem of the analysis of the data is much heavier for tesseral harmonics than for zonal ones and could not be undertaken until many accurate observations were available, preferably with accurate estimates of station positions. An accurate measure of time is essential for both the measurements of station longitude and the satellite observations themselves and in the Baker-Nunn camera system of the Smithsonian Astrophysical Observatory, the time system is based on atomic standards of frequency.

In addition to observational problems, there is the difficulty that it is not possible to separate different tesseral harmonics because components proportional to P_{n+2}^m, P_{n+4}^m, give rise to the same frequencies as P_n^m (KAULA, 1961a).

In the paper just referred to, KAULA has analysed observations of Vanguard 1 by

Minitrack over 385 days, determining the corrections to the geodetic datum systems of the Americas, South Africa and Austrialia as well as the coefficients of five tesseral harmonics (See KAULA, 1961). Two solutions were made using different runs of data and it appears as well from the sizes of the solutions in relation to the standard deviations, as from the differences in the solutions for the two sets of data, that most of the results are not statistically significant. The only harmonics that are established are P_2^2 and P_4^1 and only two or three datum shifts seem significant.

ISZAK (1961) has analysed the Baker-Nunn camera data but has assumed that the station positions are exact and has looked only for the P_2^2 term, corresponding to the ellipticity of the equator. A much more extensive and detailed study has been made by KOZAI (1961, 1962b) using data from three satellites observed with the Baker-Nunn cameras with an accuracy of 3–4 seconds of arc, as compared with about 2 min for the Minitrack data. He derives 8 tesseral harmonics. Most recently, KAULA (1963) has applied his method of analysis to the Baker-Nunn Camera data. He used data from

TABLE 8

ESTIMATES OF TESSERAL HARMONIC COEFFICIENTS IN THE EXTERNAL POTENTIAL

Coefficients are $10I\bar{C}_{nm}$, $10^6\bar{S}_{nm}$, where $\bar{C}_{nm}$, $\bar{S}_{nm}$ are as defined by KAULA (1963) (see Appendix).

n	m		ISZAK 1961	KAULA 1961	KOZAI 1961	NEWTON 1962	KAULA 1963
2	2	C	+ 6.9	+ 0.59	+ 0.93	3.34	1.84
		S	− 4.4	− 2.54	− 3.48	− 0.59	− 1.71
3	1	C			+ 2.78		1.77
		S			+ 1.10		− 0.21
3	3	C			− 12.1		
		S			+ 6.0		
4	1	C		+ 1.18	− 2.66	− 2.58	− 0.21
		S		− 0.27	+ 0.79	− 0.45	+ 0.46
4	3	C					0.50
		S					0.16

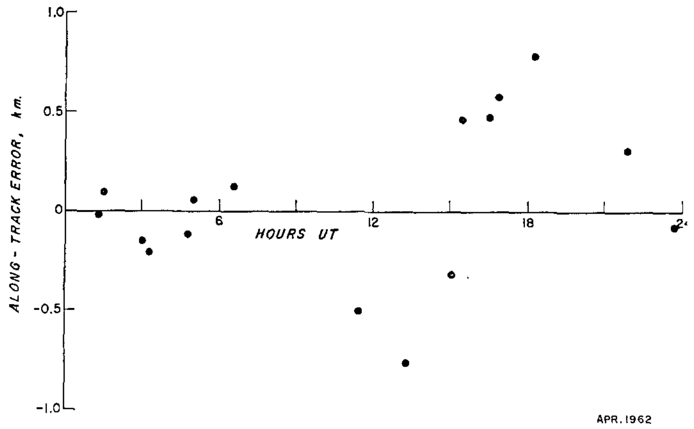
Satellites used

ISZAK,	1961	1959 α1, 1959 η
KAULA,	1961	1958 β2 (Minitrack observations)
KOZAI,	1961	1958 δ2, 1959 α1, 1959 η (Baker-Nunn observations)
NEWTON,	1962	1961 o1 (Transit observations)
KAULA,	1963	1960 ι2 (1959 α1 and 1959 η rejected) (Baker-Nunn observations)

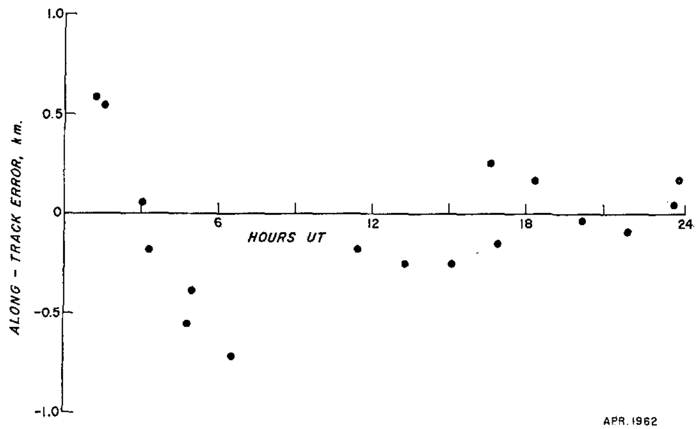
Remarks on Table. The numerical values are normalised to KAULA's (1963) form.

KAULA, 1961, uses P_n^m (cos θ) and a change of sign in the potential function.
ISZAK (1961), KOZAI (1961) and NEWTON (1962) use P_n^m (cos θ). A*s to Sign of Longitude;* NEWTON and KOZAI measure longitude position to East. It is assumed that other workers do so as well.

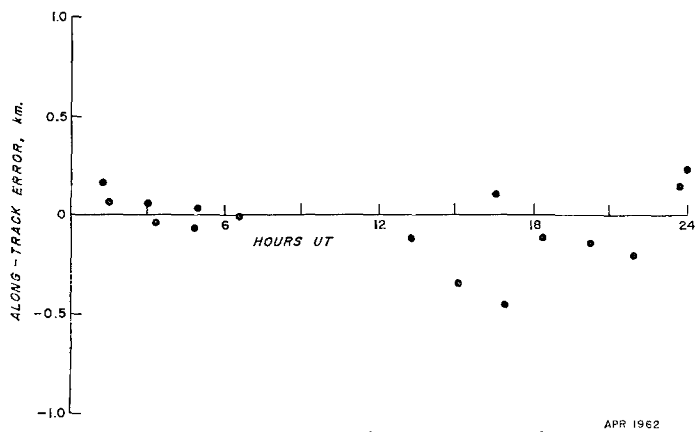
Numerical values are admitted in the Table if either the cosine or the sine term exceeds twice the author's estimate of its standard deviation.



a. $J_2^2 = 0, J_4' = 0$



b. $J_2^2 = 2.2 \times 10^{-6}, J_4' = 0$



c. $J_2^2 = 2.2 \times 10^{-6}, J_4 = 2.5 \times 10^{-6}$

Fig. 8. Along-track errors of Transit 4A as a function of time, 1961, October 23.

three satellites and found that the results showed considerable variation according to the orbits used. In particular, the results from 1959 α 1 and 1959 η give apparently significant values for the coefficient of $P_2^{\prime}$ which, it is known from dynamical considerations, must be infinitesimal; the final results were therefore based on 1960 ι 2 only. KAULA considers that the coefficients of four harmonics, P_2^2, P_3^1, P_4^1 and P_4^3, are significant and these are given in Table 8.

The preceding studies, like those from which zonal harmonics have been determined, start from variations in the elements of an elliptical orbit fitted to the observed positions of the satellite. It was pointed out at the beginning of this paper that the most direct way of determining the potential, if it were possible, would be to measure directly the variations in the acceleration vector of a satellite. This itself is not possible but the very high accuracy of the Doppler velocity measurements in the Transit navigation system has enabled the variations in the position of the satellite along the orbit, that is of the true longitude of the satellite, to be found and from them NEWTON (1962, and paper at IAG – COSPAR Symposium on the Geodetic Uses of Artificial Satellites, Washington, 1962) has estimated the tesseral harmonics proportional to P_2^2 and P_4^1. He shows that if the error along the track of the satellite as observed at a number of stations well spaced in longitude is called δ where $\delta = -\delta\omega - \delta M - \delta\Omega \cos i$ for small e then for Transit 4A, δ is found to have an amplitude of about 1.4×10^{-4} rad with a period of 12h (Figure 8) and the phase of the variation is found to change from day to day with the longitude of the node. The advantage of Transit 4A for such a study, apart from the accuracy of the observations, is that δ is proportional to $\sin^2 i$ and i is 66.8°, much greater than for any other satellite used to find tesseral harmonics. NEWTON first gave a result for P_2^2 only but subsequently on isolating a 24h component in δ, derived the P_4^1 term.

The results of these studies of tesseral harmonics are summarised in Table 8. The notation is

$$V = -\frac{\mu}{r}\left\{1 - \sum_{n\ m}\left(\frac{R}{r}\right)^n J_{nm} P_n^m(\cos\theta)\cos m(\lambda - \lambda_{mn})\right\}$$

or alternatively,

$$C_{nm} = J_{nm}\cos m\,\lambda_{mn}, \qquad S_{nm} = J_{nm}\sin m\,\lambda_{mn}.$$

There may be some doubt about the interpretation of some of the results as given in Table 8, for there may be ambiguity about the sign of the coefficients (e.g. KAULA, 1961, 1963) and there is also a number of ways of choosing the constant in the definition of P_n^m (see Appendix).

The data used by NEWTON seem to be the most reliable but, as he indicates, even there the interpretation is not clear cut because the geocentric positions of the observatories depend on the coefficients of the tesseral harmonics. It certainly seems from the comparison of the different estimates, that much more work needs to be done before reliable values of any tesseral harmonics are available.

6. Applications

Gravity values as measured at the surface of the Earth vary with the height of the site at which they are made but they may be all reduced to equivalent values on the equipotential surface at sea level by applying a correction equal to $\delta V/r$ where δV is the difference of potential between the site and sea level. δV is the quantity directly determined in spirit levelling surveys and when the results are expressed as difference of heights, allowance must be made for the known or assumed variation of gravity with height. The values of gravity so corrected are known as free-air values and if they are analysed into a series of surface harmonics, then according to the generally accepted theory, the relation between a spherical harmonic coefficient of the potential, J_n, say and the corresponding surface harmonic component of free-air gravity, $g\beta_n P_n(\sin\phi)$ say, is given by

$$\beta_n = (n-1)J_n.$$

For the zonal harmonic of order 2, the above result does not apply but

$$J_2 = \tfrac{2}{3}\{2m - \tfrac{3}{2}\beta_2\} + O(\beta_2^2)$$

where m is the ratio of centrifugal acceleration to gravitational acceleration at the equator, and ϕ is the geographical latitude.

A number of estimates have been made of the harmonics of lower order in free-air gravity, based on statistical analyses of the available gravity measurements, and the results are summarised in Table 9, which also contains the corresponding estimates from satellite data. The comparisons are not very satisfactory. There is a large range of values of β depending on the data and statistical methods used, and few of the

TABLE 9

VALUES OF HARMONICS IN THE POTENTIAL DERIVED FROM SURFACE GRAVITY

Coefficient ($\times 10^6$)	*Surface Gravity* JEFFREYS 1948	ŽONGOLOVIČ (*see* JEFFREYS) 1961	KAULA* 1959	*Satellite Estimate*
J_2	1072.5	1070.5	1083.5	1082.8
C_{22}	− 6.2	− 6.2		+ 1.6
S_{22}		− 0.3		− 2.1
J_3		− 7	+ 0.3	− 2.4
J_4			− 0.1	− 0.8
C_{41}			+ 0.32	− 1.1
S_{41}			+ 0.35	+ 0.1
J_5			+ 0.41	− 0.06
J_6			− 0.17	+ 0.7
J_7			0.00	− 0.45
J_8			− 0.04	+ 0.24

* KAULA's estimates of J_2, J_4 and J_6 involve a condition derived from satellite data.

estimates of other harmonics are statistically significant. There are two difficulties. About three-quarters of the surface of the Earth is covered by the seas and before satisfactory estimates of the surface harmonic components of free-air gravity can be made, the seas must be covered with a reasonably uniform network of gravity observations.

Technically it is much more difficult to measure gravity at sea than on land and until many more observations can be secured at sea it would be idle to expect better comparisons with satellite results.

In geodetic applications in the strict sense, the value of J_2 is used to obtain the form of the spheroid that best fits the sea-level equipotential surface by means of the relation

$$J_2 = \tfrac{2}{3}\{f(1 - \tfrac{1}{2}f) - \tfrac{1}{2}m(1 - \tfrac{2}{7}f)\},$$

where f, the ratio $(a - b)/a$, a and b being respectively the equatorial and polar radii of the spheroid. The actual form of the sea-level equipotential is obtained by adding to this spheroid harmonic components given by

$$N_n = aJ_n \cdot P_n(\cos\theta)$$

where N_n is the departure of the radius vector of the surface from the spheroidal value. For most geodetic purposes these other components may be neglected but the accuracy of the value of f obtained from satellite data is much greater than that hitherto available from terrestrial measurements and greatly simplifies the problem of determining the size of the Earth from geodetic surveys. A reference value of gravity may also be calculated from the corresponding value of f and with the best available value of mean gravity on the surface it is

$$978.0362\{1 + 5302.23 \times 10^{-6}\sin^2\phi - 6.40 \times 10^{-6}\sin^2 2\phi\}\,\text{cm s}^{-2},$$

where ϕ is the geographic latitude.

This formula, like that for f depends on the assumption that the sea-level surface is a spheroid.

According to MacCullagh's theorem, the moments of inertia of the Earth are related to the value of J_2 by the formula

$$J_2 = \frac{C - A}{Ma^2}$$

where a is the equatorial radius.

Furthermore, the precession of the Earth due to the attraction of the Sun and the Moon is proportional to $(C - A)/C$. The precessional constant is very accurately known and leads to the value

$$\frac{C - A}{C} = 3.27237 \times 10^{-3}$$

(see A. H. Cook, 1959a).

With the value of J_2 from satellite data,

$$\frac{C}{Ma^2} = 0.33089$$

$$\frac{A}{Ma^2} = 0.32981$$

(see also ARNOLD, 1960).

Although the precessional constant is very accurately known, the ratio $(C - A)/C$ is uncertain by about 1 part in 10^3 because it also depends on the mass of the Moon which is not too well determined. The values given here depend on JEFFREYS's (1948) value for the mass of the Moon since this seems to be in best agreement with other data (see A. H. COOK, 1963).*

Now if the Earth were in hydrostatic equilibrium, so that no shearing stresses could be supported, it was shown by G. H. DARWIN (1899) that the flattening f, related to J_2 by the formula given above, may be calculated from the polar moment of inertia as follows:

$$\frac{C}{Ma^2} = \frac{2}{3}\left[1 - \frac{2}{5}\left\{\frac{5}{2}\frac{m}{f}\left(1 - \frac{3}{2}m\right) - 1\right\}^{\frac{1}{2}}\right].$$

With the actual value of C given above (which is derived from observations alone and involves no assumption at all) f^{-1} would be 299.7 instead of the observed value, derived

TABLE 10

COMPARISON OF NON-HYDROSTATIC POTENTIAL WITH CONTRIBUTIONS FROM CONTINENTS

Order of harmonic		*Observed*	*Hydrostatic*	*Non-hydrostatic*	*Continents (1000 fathom line)*
n	m				
2		1082.8	1078.5	+ 4.3	− 0.42
3		− 2.4		− 2.4	− 0.17
4		− 0.8	− 2.3	+ 1.5	− 0.34
5		− 0.06		− 0.06	+ 0.56
6		+ 0.7	$\sim 10^{-3}$	+ 0.7	− 0.18
7		− 0.45		− 0.45	+ 0.34
8		+ 0.24	$\sim 10^{-6}$	+ 0.24	+ 0.03
2	2 C	+ 1.6		+ 1.6	+ 0.20
	S	− 2.1		− 2.1	+ 0.12
3	1 C	+ 2.3		+ 2.3	+ 0.02
	S	+ 0.46		+ 0.46	+ 0.01
4	1 C	− 1.1		− 1.1	+ 0.02
	S	+ 0.1		− 0.1	− 0.01
4	3 C	+ 0.50		+ 0.50	− 0.22
	S	+ 0.16		+ 0.16	—

Coefficients of Associated Legendre Polynomials correspond to

$$\int_s \left[P_n^m(\cos\theta)\right]^2 \frac{\sin^2}{\cos^2} m\lambda \, \mathrm{d}S = 4\pi$$

* An improved value is now available from space probe tracking.

from the observed value of J_2, of 298.2. Alternatively, if the Earth were in the hydrostatic state, the value of C/Ma^2 would be 0.33328 instead of 0.33089. There is no doubt of the reality of the departure of the observed values from the hydrostatic ones and therefore of the conclusion that the mass distribution in the Earth is in part supported by shear stresses. MUNK and MACDONALD (1960) have extended this comparison to other harmonics of low order and a revised version of their figures is given in Table 10. MUNK and MACDONALD compared the non-hydrostatic part of the potential with the contributions arising from the distribution of continents and oceans, allowing for the difference of structure of continents and oceans above the Mantle, and these, extended to compare with the more recent satellite data, are given in the Table. It is clear that MUNK and MACDONALD's conclusion, that the low order harmonics are in no way related to the distribution of continents and oceans, is still valid and the corresponding mass differences are presumably situated in the Mantle and supported by strengths of the order of 10^6b.* MUNK and MACDONALD also point out that the difference between the observed and hydrostatic moments of inertia corresponds to the Earth having had a larger rotation and therefore a larger bulge in the past. They estimated that the present bulge corresponds to the spin velocity 10 My ago; with the figures calculated here, it would be 3 My ago.

Lastly, the bearing of the satellite results on the motion and mechanical properties of the Moon are considered. The node and perigee of the Moon's orbit about the Earth show secular changes which arise for the most part from the attraction of the Sun but are in part due to the second harmonic in the potential of the Earth and in part due to second harmonics in the potential of the Moon. If the polar moment of inertia of the Moon is denoted by C_M and the moments about the other two principal axes by A_M and B_M, and if we write

$$J_{2M} = \frac{2C_M - (A_M + B_M)}{2M_M a_M^2},$$

$$K_M = (B_M - A_M)/M_M a_M^2,$$

$$L = J_{2M} + \tfrac{1}{2}K_M,$$

then the formulae for the non-solar parts of the motions of perigee and node are

$$\text{perigee} \quad 3897\,J_2 + 380\,L - 1192\,K_M = 4''.274 \text{ per year}$$

$$\text{node} \quad -3648\,J_2 - 460\,L = -4''.119 \text{ per year}$$

For comparison, the solar parts are $-46448''$ for the node and $97618''$ for perigee per year.

On substituting the value of J_2 from satellite data, it follows that

$$J_{2M} = 3.137 \times 10^{-4}$$

$$K_M = 1.074 \times 10^{-4}$$

* See also A. H. COOK, 1963a.

In addition a quantity β equal to $(C_M - A_M)/C_M$ may be derived from observations of the libration of the Moon and while there are difficulties with the observations and the reductions, it now seems that β is about 6.28×10^{-4} and therefore that

$$\frac{C_M}{M_M a_m^2} = 0.500.$$

The value for a uniform sphere is 0.4 and so it seems on the face of it that the Moon is lighter at the centre than at the surface. This improbable result is almost certainly due to errors in the calculation of the very large solar contributions to the observed motions of node and perigee and until that can be improved it cannot be expected that results from artificial satellites about the Earth will lead to better knowledge of the mechanical constitution of the Moon.

7. Conclusion

Within six years of the launching of the first artificial satellite, it is now possible to say that the theory of the orbits of such satellites, so far as it concerns the determination of the external gravitational potential of the Earth, is virtually complete and that the observational results have led to a body of information about the field that is unlikely to be changed in any significant way. It would be hard to think of a more striking example of a complete revolution in a well established traditional study as a result of new technical possibilities, especially when it is considered that almost all the results have been harvested from observations of a purely passive satellite, some of them, as has been emphasised, by the most elementary means. This is not to say that no problems remain. So far as the Earth is concerned, it seems likely that little will be known about the tesseral harmonics for some time to come, and it must be remembered that there is no physical reason why they should not be at least as important as the zonal harmonics other than J_2; they may indeed be more important for if there is convection in the mantle of the Earth, it would be expected to show up, if at all, in the tesseral rather than in the zonal harmonics. The results have also stimulated new theoretical studies of the gravitational field of the Earth, bringing to attention problems that were not so pressing when the external field could not be determined directly, and they have shown also how very necessary it is to intensify measurements of gravity at sea in order to obtain results from surface gravity measurements that shall be comparable in accuracy to the satellite results and so enable the two sets of results to be critically and profitably compared. At present it is premature to attempt to combine them statistically.

The main theoretical problems that remain relate to the orbits about the Moon, for since the equatorial ellipticity of the Moon is compatable with her polar flattening, the theoretical treatment that is adequate for the Earth will almost certainly not be so for the Moon. No doubt other critical cases will arise and commensurable motions will be more interesting theoretically, although perhaps not so important practically on account of the slow spin of the Moon, and it is to be hoped that it will be possible to extend an analysis of the type given by VINTI to the Moon. Further, more attention will have to be given to solar-terrestrial perturbations than is given to luni-solar perturbations for satellites of the Earth. Further the satellite results show up the inadequacy of existing results and emphasise the need for a new study of the theory of the motion of the Moon. Finally, little geophysical use has been made of the results but as more harmonics of higher order are determined it is to be expected that they will provide information about the variation of density in the Mantle.

Acknowledgements

I am indebted to Mr. KING-HELE for letting me have the results of his most recent studies prior to publication and to Dr. R. R. NEWTON for the material of Figure 8. I had some helpful correspondence with Mr. P. J. MESSAGE on the subject of the comparisons between different theories.

This paper has been prepared as part of the program of general research of the National Physical Laboratory and is published with the permission of the Director.

References

ARNOLD, K.: 1959, *Gerl. Beitr. Geophys.* **68,** 1.
ARNOLD, K.: 1960, *Gerl. Beitr. Geophys.* **69,** 191.
BLITZER, L., WEISFELD, M., and WHEELON, A. D.: 1956, *J. appl. Phys.* **27,** 1141.
BLITZER, L., BOUGHTON, E. M., KANY, G., and PAGE, R. M.: 1962, *J. Geophys. Res.* **67,** 329.
BRENNER, J. L. and LATTA, G. E.: 1960, *Proc. Roy. Soc. A.* **241,** 49.
BROUWER, D.: 1958, *Astron. J.* **63,** 433.
BROUWER, D.: 1959, *Astron. J.* **64,** 378.
BRYANT, R. W.: 1961, *Astron. J.* **66,** 430.
BUCHAR, E.: 1958, *Studia Geop. et Geod.* (Prague) **2,** 306.
COOK, A. H.: 1958, *Geophys. J.* **1,** 341.
COOK, A. H.: 1959, *Geophys. J.* **2,** 199.
COOK, A. H.: 1959a, *Geophys. J.* **2,** 222.
COOK, A. H.: 1961, *Geophys. J.* **4,** 53.
COOK, A. H.: 1963, Paper to be presented at IAU Symposium No. 21 on Fundamental constants of astronomy.
COOK, A. H.: 1963a, *Nature* (London), **198,** 1186.
COOK, G. E.: 1961, *Proc. Roy. Soc. A.* **261,** 246.
COOK, G. E.: 1962, *Geophys. J.* **6,** 271.
COOK, G. E.: 1962a, *Rep. Prog. Phys.* **25,** 63.
DARWIN, Sir G. H.: 1899, *Mon. Not. R. Astron. Soc.* **60,** 82.
GARFINKEL, B.: 1958, *Astron. J.* **63,** 88.
GARFINKEL, B.: 1959, *Astron J.* **64,** 353.
GARFINKEL, B.: 1960, *Astron. J.* **65,** 624.
GROVES, G. V.: 1960, *Proc. Roy. Soc.* A. **254,** 48.
HALL, N. S. and GAWLOWITZ, H. F.: 1961, *Space Res.* **II** 292.
HARVARD: 1958, Harvard Announcement Card 1408, 1958 June 24.
HERGENHAHN, G.: 1960, *Z. Vermessungswesen* **85,** 342.
HERGET and MUSEN, P.: 1958, *Astron. J.* **63,** 430.
HOBSON, E. W.: 1931, *Spherical and Ellipsoidal Harmonics.* Cambridge University Press.
HORI, G. I.: 1960, *Astron. J.* **65,** 291.
ISZAK, I. G.: 1960, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **52.**
ISZAK, I. G.: 1961, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **56.**

ISZAK, I. G.: 1962, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **90**.
JACCHIA, L. G.: 1958, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **19**, p. 1.
JEFFREYS, H.: 1948, *Mon. Not. R. Astron. Soc., Geophys. Suppl.* **5**, 219
JEFFREYS, H.: 1959, *The Earth*, 4th Ed. (Cambridge University Press).
JEFFREYS, H.: 1961, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **79**.
JEFFREYS, H. and JEFFREYS, B.: 1950, *Methods of Mathematical Physics* (Cambridge University Press).
KAULA, W. M.: 1959, *J. Geophys. Res.* **64**, 2401
KAULA, W. M.: 1961, *Geophys. J.* **5**, 104
KAULA, W. M.: 1961a, *Space Res.* **II**, 360, *Astron. J.* **66**, 355
KAULA, W. M.: 1962, *Astron. J.* **67**, 300.
KAULA, W. M.: 1963, *J. Geophys. Res.* **68**, 473.
KING-HELE, D. G.: 1958, *Proc. Roy. Soc. A.* **247**, 49.
KING-HELE, D. G.: 1959, *Proc. Roy. Soc. A.* **253**, 529.
KING-HELE, D. G.: 1959, *Proc. 10th Int. Astronaut Cong.*, pp. 1–20.
KING-HELE, D. G.: 1961, *Geophys. J.* **4**, 3.
KING-HELE, D. G.: 1962, *Geophys. J.* **6**, 270.
KING-HELE, D. G., COOK, G. E., and REES, J. M.: 1963, *Nature* (London) **197**, 785; *Geophys. J.* **8**,1.
KING-HELE, D. G. and MERSON, R. H.: 1958, *J. Brit. Interplan. Soc.* **16**, 446.
KING-HELE, D. G. and MERSON, R. H.: 1959, *Nature* (London) **183**, 881.
KOZAI, Y.: 1959, *Astron. J.* **64**, 367.
KOZAI, Y.: 1961, *Astron.* J. **66**, 8.
KOZAI, Y.: 1961a, *Astron. J.* **66**, 1288.
KOZAI, Y.: 1962, *Smithsonian Astrophys. Obs., Sp. Rep.* No. **101**.
KOZAI, Y.: 1962a, *Astron. J.* **67**, 446.
KOZAI, Y.: 1962b, *Proc. IUTAM Symp. Dynamics of Satellites,* Paris (Berlin, Springer Verlag).
KOVALEVSKY, J.: 1960, *Space Res.* I, 458.
KRAUSE: 1956, *Proc. 7th Int. Astronaut. Congr.*, Rome, 523.
LECAR, M., SORENSON, J., and ECKELS, A.: 1959, *J. Geophys. Res.* **64**, 209.
LEVALLOIS, J. J.: 1957, *Comptes Rendus Acad. Sci.* (Paris) **244**, 2479.
LYDDANE, R. H. and COHEN, C. J.: 1962, *Astron. J.* **67**, 176.
MASSEVIČ, A. G.: 1961, *Space Res.* II, 3.
MASSEVIČ, A. G.: 1961a, *Space Res.* II, 102.
MERSON, R. H.: 1959, *Astronautica Acta* **5**, 26.
MERSON, R. H.: 1961, *Geophys. J.* **4**, 17.
MESSAGE, P. J.: 1960, *Mon. Not. R. Astron. Soc.* **121**, 1.
MICHIELSEN, H. F.: 1961, *Paper presented at 7th Ann. Mtg. Amer. Astronaut Soc.*
MOLODENSKY, M. S.: 1948, *Izvest. Akad. Nauk (Ser. Geogr. Geophys.)* **12**, 113.
MORANDO, B.: 1962, *Comptes Rendus Acad. Sci.* (Paris) **254**, 635.
MUNK, W. H. and MACDONALD, J. G. F.: 1960, *The Rotation of the Earth* (Cambridge University Press).
MUNK, W. H. and MACDONALD, J. G. F.: 1960, *J. Geophys. Res.* **65**, 2169.
MUSEN, P.: 1959, *J. Geophys. Res.* **64**, 2271.
MUSEN, P.: 1960, *J. Geophys. Res.* **65**, 2783.
MUSEN, P.: 1961, *J. Geophys. Res.* **66**, 403.
MUSEN, P.: 1961a, *J. Geophys. Res.* **66**, 1659.
MUSEN, P.: 1962, *J. Geophys. Res.* **67**, 313.
MUSEN, P. and BAILIE, A. E.: 1962, *J. Geophys. Res.* **67**, 1123.
NEWTON, R. R.: 1962, *J. Geophys. Res.* **67**, 415.
NEWTON, R. R., HOPFIELD, M. S., and KLINE, R. C.: 1961, *Nature* (London), **190**, 617.
O'KEEFE, J. A. and BATCHELOR, C. D.: 1957, *Astron. J.* **62**, 183.
O'KEEFE, J. A., ECKELS, A., and SQUIRES, R. K.: 1959, *Science* **129**, 565.
O'KEEFE, J. A., ECKELS, A., and SQUIRES, R. K.: 1959a, *Astron. J.* **64**, 245.
PATTON, R. B. and RICHARD, V. W.: 1961, *Space Res.* **II**, 218.
PETTY, C. M. and BREAKWELL, J. V.: 1960, *J. Franklin Inst.* **270**, 259.
PIZETTI, P.: 1894, *R. C. Acad. Lincei* **3**, 166.
PLUMMER, H. C.: 1960, *An introductory treatise on dynamical astronomy* (Dover reprint, from Cambridge University Press, 1918).

PROSKURIW, V. F. and BATRAKOV, JU. V.: *Bull. Inst. Th. Astron.* **7**, 7.
ROBERSON, R. E.: 1957, *J. Franklin Inst.* **264**, 181 and 269.
SMART, W. M.: 1953, *Celestial Mechanics*. London, Longmans, Green & Co.
SMITH, D. E.: 1961, *Planet. Space Sci.* **8**, 43.
SMITH, D. E.: 1962, *Planet. Space Sci.* **9**, 93.
SMITH, D. E.: 1962a, *Planet Space Sci.* **9**, 659.
STERNE, T. E.: 1958, *Astron. J.* **63**, 28.
STOKES, Sir G. G.: 1849, *Trans. Camb. Phil. Soc.* **8**, 672.
STRUBLE, R. A.: 1960, *J. Math. Anal. Appl.* **1**, 300.
STRUBLE, R. A.: 1961, *Arch. Rational Mech. Anal.* **7**, 87.
VEIS, G. and WHIPPLE, F. L.: 1961, *Space Res.* **II**, 17.
VINTI, J. P.: 1959, *J. Res. Nat. Bur. Stand.* **63B**, 105.
VINTI, J. P.: 1961, *J. Res. Nat. Bur. Stand.* **65B**, 131.
VINTI, J. P.: 1961a, *Astron. J.* **66**, 514.
VINTI, J. P.: 1961b, *J. Res. Nat. Bur. Stand.* **655B**, 169.
VINTI, J. P.: 1962, *J. Res. Nat. Bur. Stand.* **66B**, 5.
WHITTAKER, E. T. and WATSON, G. N.: 1940, *Modern Analysis*, 4th Ed. (Cambridge University Press).
ŽONGOLOVIČ, I. D.: 1960, *Bull. Inst. Th. Astron.* **7**, 521.
ŽONGOLOVIČ, I. D.: 1960a, *Bull. Inst. Th. Astron.* **7**, 743.
ŽONGOLOVIČ, I. D.: 1960b, *Bull. Stations of Opt. Obs. of Sat. USSR* **2**, (12), pp. 1–24.
ŽONGOLOVIČ, I. D. and PELLINEN, L. P.: 1962, *Bull. Inst. Th. Astron.* **8**, 381.

NOTE

W. M. KAULA has discussed many of the topics considered here in his review article, *Celestial Geodesy* (*Adv. in Geophys.* **9**, (1961)) paying particular attention to the comparison of satellite results with other data.

Recent developments were discussed at the IAG–COSPAR Symposium on *Geodetic Uses of Artificial Satellites*, Washington, April 1962. (*The Use of Artificial Satellites for Geodesy*, ed. by G. Veis, Amsterdam, North-Holland Publishing Co., 1963).

APPENDIX

Tesseral Harmonics and Standard Forms of Associated Legendre Functions

Commission 7 of the International Astronomical Union

(Trans. I.A.U. Vol. XI B, (Proceedings, 1961) pp. 173–174) recommends that the force function (negative of the potential) of the Earth should be written in one of the alternative forms:

$$U = \frac{\mu}{r}\left[1 + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^n P_n^m(\cos\theta) \times \{C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\}\right]$$

or

$$U = \frac{\mu}{r}\left[1 + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^n p_{nm}(\cos\theta) \times \{A_{nm}\cos m\lambda + B_{nm}\sin m\lambda\}\right]$$

where

$$P_n^m(z) = (1 - z^2)^{\frac{1}{2}} \frac{d^m}{d_z^m} \cdot P_n(z)$$

and

$$p_{nm}(z) = \left[\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} P_n^m(z)$$

Now

$$\int_{-1}^{+1} \left[P_n^m(z)\right]^2 \mathrm{d}z = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

and therefore, on integrating over the surface of a unit sphere,

$$\int_S \left[P_n^m(z) \begin{matrix}\cos\\ \sin\end{matrix}\, m\lambda\right]^2 \mathrm{d}S = \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!},$$

while

$$\int \left[p_{nm}(z) \begin{matrix}\cos\\ \sin\end{matrix}\, m\lambda\right]^2 \mathrm{d}S = \frac{2}{2n+1}.$$

Accordingly

$$C_{nm}^2 = \frac{(n-m)!}{(n+m)!} A_{nm}^2 .$$

KAULA (1961, 1963) has chosen a third normalising factor such that if Y_{nm} is a surface harmonic, the integral over a unit sphere is 4π. He writes the coefficients in this convention as $\bar{C}_{nm}$, $\bar{S}_{nm}$;

$$\bar{C}_{nm}^2 = \frac{1}{2(2n+1)}\cdot\frac{(n+m)!}{(n-m)!} C_{nm}^2 = \frac{1}{2(2n+1)}\cdot A_{nm}^2 .$$

A fourth normalising factor arises if the $p_n^m(z)$ functions introduced by JEFFREYS and JEFFREYS (1950, chap. 24) are used.

$$p_n^m(z) = \frac{(n-m)!}{n!} P_n^m(z)$$

and therefore the integral over the unit sphere is

$$\frac{2\pi}{2n+1}\cdot\frac{(n-m)!(n+m)!}{(n!)^2}$$

and if the cosine and sine coefficients of an expansion in terms of these functions are a_{nm}, b_{nm},

$$a_{nm}^2 = C_{nm}^2\left[\frac{n!}{(n-m)!}\right]^2$$

and so on.

This is the convention used by MUNK and MACDONALD (1960).

The results of determinations of tesseral harmonics listed in Table 8 are normalised with KAULA's convention. The following conversion factors are required:

n	m	$\bar{C}_{nm}/C_{nm}$
2	2	1.55
3	1	0.93
	3	7.17
4	1	1.05

The comparison of the observed harmonics in the potential with those expected from MUNK and MACDONALD's continentality function is as given in Table 10 requires the following factors:

n	m	$\bar{C}_{nm}/a_{nm}$
2	2	0.77
3	1	0.30
	3	1.2
4	1	0.26
4	3	0.70