J. Math. Biol. (1992) 30: 755–763

Journal of
Mathematical
Biology

Research Announcement

Convergence results and a Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations*

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Received November 27, 1991; received in revised form January 16, 1992

Abstract. Conditions are presented under which the solutions of asymptotically autonomous differential equations have the same asymptotic behavior as the solutions of the associated limit equations. An example displays that this does not hold in general.

Key words: Dynamical systems – Asymptotic behavior – Butler–McGehee lemma – Chemostat – Epidemics

1 Introduction

Some 35 years ago, Markus (1956) published an often quoted (and sometimes misquoted) paper on asymptotically autonomous differential systems where he considers ordinary differential equations

$$\dot{x} = f(t, x), \tag{1.1}$$

$$\dot{y} = g(y), \tag{1.2}$$

in $\boldsymbol{R}^n$. Equation (1.1) is called *asymptotically autonomous* – with *limit equation* (1.2) – if

$$f(t, x) \to g(x), \quad t \to \infty, \quad \text{locally uniformly in } x \in \boldsymbol{R}^n,$$

i.e., for x in any compact subset of $\boldsymbol{R}^n$. For simplicity we assume that $f(t, x)$, $g(x)$ are continuous functions and locally Lipschitz in x. Further all solutions are supposed to exist for all forward times. The ω-limit set $\omega(t_0, x_0)$ of a forward bounded solution x to (1.1), satisfying $x(t_0) = x_0$, is defined in the usual way:

$$y \in \omega(t_0, x_0) \Leftrightarrow y = \lim_{j \to \infty} x(t_j) \quad \text{for some sequence } t_j \to \infty \ (j \to \infty).$$

Among other results Markus presents the following theorems:

Theorem 1.1 (Markus) *The ω-limit set ω of a forward bounded solution x to* (1.1) *is non-empty, compact, and connected. Moreover ω attracts x, i.e.,*

$$\operatorname{dist}(x(t), \omega) \to 0, \quad t \to \infty.$$

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* Research partially supported by NSF grant DMS 9101979

Finally ω is invariant under (1.2). *In particular any point in ω lies on a full orbit of* (1.2) *that is contained in ω.*

Theorem 1.1 (Markus, 1956, Theorem 1 and preceding remarks) has sometimes been misquoted in the form that ω-limit sets of (1.1) are unions of ω-limit sets of (1.2) or even subsets of ω-limit sets of (1.2). A counter-example will be presented below.

Theorem 1.2 (Markus) *Let e be a locally asymptotically stable equilibrium of* (1.2) *and ω the ω-limit set of a forward bounded solution x of* (1.1). *If ω contains a point y_0 such that the solution y of* (1.2), *with $y(0) = y_0$, converges to e for $t \to \infty$, then $\omega = \{e\}$, i.e., $x(t) \to e$, $t \to \infty$.*

Actually Markus proves more in his Theorem 2, but most applications use the formulation in Theorem 1.2 which is a consequence of Markus's (1956) Theorems 1 and 2. Markus's (1956) Theorem 2, in its original formulation, has been applied by Conway et al. (1978) in the proof of their Theorem 5.3. A generalization of Markus's Theorem 2 can be found in Hale (1980, III.2, Exercise 2.4).

Markus's (1956) Theorem 7 generalizes the Poincaré–Bendixson Theorem to asymptotically autonomous planar systems.

Theorem 1.3 (Markus) *Let $n = 2$ and ω be the ω-limit set of a forward bounded solution x of* (1.1). *Then ω either contains at least one equilibrium of* (1.2), *or ω is the union of periodic orbits of* (1.2).

Theorem 1.1 has heavily stimulated the development of the qualitative theory of non-autonomous differential equations and dynamical systems (see Miller 1965; Sell 1967, 1972; Dafermos 1971; as a small sample of references). It has been generalized to Volterra integral equations by Miller and Sell (1970), e.g. Theorems 1.2 and 1.3 are often applied to show that the solutions of population dynamic (notably chemostat) models converge to an equilibrium (Theorem 1.2: Hsu 1981, Hsu et al. 1981, Hsiao et al. 1987, Cushing 1989; Theorem 1.3: Busenberg and Iannelli 1985). Theorem 1.3 has also triggered some research on almost periodic solutions of asymptotically autonomous ODEs in the plane (Utz and Waltman 1967; Grimmer 1968).

Somehow Markus's paper has generated the feeling that the so-called *Inverse Limit Problem* has been fairly completely solved for asymptotically autonomous systems (Sell 1972), and little work on this topic has seemingly been done in the last ten or even twenty years except applying the known results. We phrase this problem as follows:

Question 1.4 *Assume that the equilibria of* (1.2) *are isolated and that any solution of* (1.2) *converges to one of them. Does every solution of* (1.1) *converge to an equilibrium of* (1.2) *as well?*

An answer to this question is helpful in analyzing certain chemostat/gradostat and epidemic models where one can show for a multi-dimensional system that some components converge to a limit independently of what the other components do (Jäger et al. 1987; Smith and Tang 1989; Waltman 1990; Smith et al. 1991; Blythe et al. preprint). It is also useful for determining the large-time behavior of solutions to non-linear reaction-diffusion systems with Neumann boundary conditions (Conway et al. 1978; Smoller 1983; Chap. 14, §D).

When considering convergence in an epidemic model (Blythe et al. preprint), Castillo-Chavez (personal communication, triggering this investigation) noticed that Theorems 1.1 to 1.3 were not sufficient to answer Question 1.4 for their special case. The following example illustrates that, even in the plane, Question 1.4 cannot be positively answered without further conditions.

Example 1.5 Consider the following system in cylindrical coordinates $r, \theta, x_3, x_1 = r\cos\theta, x_2 = r\sin\theta$:

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \beta r|\sin\theta| + x_3 \\ \dot{x}_3 &= -\gamma x_3 \end{aligned} \tag{1.3}$$

with initial data $0 < r(0) < 1$, $0 \leqslant \theta < 2\pi$, $x_3(0) \geqslant 0$ and positive constant parameters β, γ. This system has three equilibria: $(0, 0, 0)$, $(1, 0, 0)$, $(1, \pi, 0)$. Any solution $x(t)$ is attracted to the circle $r = 1$, $x_3 = 0$ for $t \to \infty$. If $x_3(0) = 0$, $r(0) > 0$, then $x(t)$ converges to $(1, 0, 0)$ or to $(1, \pi, 0)$ as $t \to \infty$. If $\beta > \gamma$, $x_3(0) > 0$, $r(0) > 0$, the ω-limit set is the whole circle $r = 1$, $x_3 = 0$, for θ in (1.3) is unbounded.

This can be seen by contradiction. Assume that $\theta(t)$ is bounded. As $\theta(t)$ is strictly increasing, $\theta(t)$ converges to some θ_∞ for $t \to \infty$, $\theta_\infty = 2k\pi$ or $\theta_\infty = (2k+1)\pi$ for some $k \in \mathbf{Z}$. On the one hand, for any $\varepsilon > 0$, one can find $M > 0$ such that

$$\theta_\infty - \theta(t) \leqslant M e^{-(\beta - \varepsilon)t}.$$

On the other hand, we have $\dot{\theta} \geqslant x_3(0) e^{-\gamma t}$ which leads to a contradiction for $\beta - \varepsilon > \gamma$. A detailed exposition and more examples (including some with C^∞ vector fields) can be found in Thieme (preprint b).

It is tempting to argue that, for determining the asymptotic behavior of (1.3), it is sufficient to study the limiting system $x_3 = 0$ because x_3 decreases to 0 exponentially. Example 1.5 shows that this shortcut (which can sometimes be found in the literature) is not allowed a priori. Fortunately it can often be a posteriori justified by Theorem 1.2 or the more general results presented in this note.

We give two answers to Question 1.4. The first is restricted to planar ODE systems and consists in extending Theorem 1.3 to a Poincaré–Bendixson type trichotomy.

Theorem 1.6 *Let $n = 2$ and ω the ω-limit set of a forward bounded solution x of* (1.1). *Assume that there exists a neighborhood of ω which contains at most finitely many equilibria of* (1.2). *Then the following trichotomy holds*:
(i) *ω consists of an equilibrium of* (1.2).
(ii) *ω is the union of periodic orbits of* (1.2) *and possibly of centers of* (1.2) *that are surrounded by periodic orbits of* (1.2) *lying in ω.*
(iii) *ω contains equilibria of* (1.2) *that are cylically chained to each other in ω by orbits of* (1.2).

In the third possibility the ω-limit set contains homoclinic orbits (phase unigons) connecting one equilibrium to itself and/or phase polygons with finitely many sides (connecting equilibria) all of which are traversed in the same direction.

Theorem 1.6 allows to use the Dulac criterion to show that all forward bounded solutions of (1.1) converge to an equilibrium of (1.2). See Hahn (1967), e.g. There remains the following

Open Problem. *Let $n = 2$ and ω be the ω-limit set of a forward bounded solution x of* (1.1). *Assume that the equilibria of* (1.2) *are isolated. Is ω the union of periodic orbits, equilibria, and orbits connecting equilibria associated with* (1.2)?

Our second answer to Question 1.4 is not restricted to planar systems (see Theorem 4.2 for details):

- *Assume the equilibria of* (1.2) *are isolated and not cyclically chained to each other.*

Both conditions are necessary in general: Cyclical chains have to be excluded as illustrated in Example 1.5, and certain connected sets of equilibria have to be ruled out as shown in an example by Smith (1991). Actually we assume more in Theorem 4.2, namely that the equilibria are isolated compact invariant sets of the limit equation. The assumptions of Theorem 4.2 are satisfied, e.g., in Corollary 2 by Smith (1991) who positively answers Question 1.4 for asymptotically autonomous tridiagonal competitive and cooperative ODE systems modeling neural nets.

Theorem 4.2 holds for general asymptotically autonomous semiflows and so applies to asymptotically autonomous (parabolic and hyperbolic) partial differential equations, functional differential equations, and to Volterra integral and integro-differential equations. We mention that persistence theory can be extended from autonomous to asymptotically autonomous semiflows, too. There are examples, however, for an equilibrium to be a uniform strong repeller for the autonomous limit-semiflow, but to be locally asymptotically stable (though not uniformly in time) for the asymptotically autonomous semiflow. (See Thieme, preprint b).

The key to generalizing or improving Markus's theorems lies in proving Butler–McGehee type and cyclicity results for asymptotically autonomous semiflows (Sect. 3). Rather than reproving such results it is more convenient to embed the asymptotically autonomous semiflow, operating on a metric space X, into an autonomous semiflow on a metric space $[t_0, \infty] \times X$, where on $\{\infty\} \times X$ the new autonomous semiflow is induced by the limit-semiflow (Sect. 2). This way, autonomous Butler–McGehee results translate easily into asymptotically autonomous ones, and Theorem 1.1 follows immediately from the corresponding results for ω-limit sets of autonomous semiflows. Combining these results one obtains Theorem 1.2 for general semiflows (Theorem 4.1). A more detailed presentation and further applications to epidemic models will appear in subsequent publications.

We wonder whether other Poincaré–Bendixson type results for autonomous semiflows (e.g. for competitive or cooperative three-dimensional ODE systems, Hirsch 1990; monotone cyclic feed-back systems, Mallet-Paret and Smith 1990; or reaction-diffusion equations on the circle, Fiedler and Mallet-Paret 1989) have extensions similar to Theorem 1.6.

2 Embedding of asymptotically autonomous into autonomous semiflows

Let X, d be a metric space. We consider a mapping $\Phi : \Delta \times X \to X$, $\Delta = \{(t, s); t_0 \leqslant s \leqslant t < \infty\}$. Φ is called a continuous (non-autonomous) semiflow if Φ

is continuous as a mapping from $\Delta \times X$ to X and

$$\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x), \quad t \geqslant s \geqslant r \geqslant t_0,$$
$$\Phi(s, s, x) = x, \quad s \geqslant t_0.$$

A semiflow is called autonomous if $\Phi(t+r, s+r, x) = \Phi(t, s, x)$. Setting $\Theta(t, x) = \Phi(t + t_0, t_0, x)$ one obtains $\Phi(t, s, x) = \Theta(t - s, x)$ with an autonomous continuous semiflow $\Theta : [0, \infty) \times X \to X$.

Definition 2.1 Let Φ be a (non-autonomous) continuous semiflow and Θ an autonomous continuous semiflow on X. Then Φ is called *asymptotically autonomous* – with *limit-semiflow* Θ – if and only if

$$\Phi(t_j + s_j, s_j, x_j) \to \Theta(t, x), \quad j \to \infty,$$

for any three sequences $t_j \to t$, $s_j \to \infty$, $x_j \to x$ $(j \to \infty)$, with elements $x, x_j \in X$, $0 \leqslant t, t_j < \infty$, and $s_j \geqslant t_0$.

We emphasize that we require Θ to be continuous rather than finding a definition or properties of Φ which would imply the continuity.

The relation to the ordinary differential equations (1.1) and (1.2) is the following: Let $\Phi(t, t_0, x_0)$ be the solution of $x(t)$ to (1.1) satisfying $x(t_0) = x_0$ and $\Theta(t, y_0)$ be the solution $y(t)$ to (1.2) satisfying $y(0) = y_0$. A Gronwall argument and the subsequent Lemma 2.3 show that Φ is asymptotically autonomous with limit system Θ in the sense of Definition 2.1, if (1.1) is asymptotically autonomous with limit equation (1.2).

In order to embed (Φ, Θ) into an autonomous semiflow we choose the state space $Z = [t_0, \infty] \times X$ where $[t_0, \infty]$ is compactified in the usual way. Z is then a metric space. We define an autonomous semiflow Ψ on Z by

$$\Psi(t, (s, x)) = \begin{cases} (t + s, \Phi(t + s, s, x)), & t_0 \leqslant s < \infty \\ (\infty, \Theta(t, x)), & s = \infty, \end{cases} \quad 0 \leqslant t < \infty. \tag{2.1}$$

Proposition 2.2 *If Φ is an asymptotically autonomous continuous semiflow with continuous autonomous limit-flow Θ, then Ψ is a continuous autonomous semiflow on Z.*

Sometimes the following version of Definition 2.1 will be easier to check in concrete cases:

Lemma 2.3 *Φ is asymptotically autonomous – with limit-semiflow Θ – if and only if*

$$d(\Phi(t_j + s_j, s_j, x_j), \Theta(t_j, x_j)) \to 0, \quad j \to \infty,$$

for any three sequences $t_j \to t$, $s_j \to \infty$, $x_j \to x$ $(j \to \infty)$, with elements $x, x_j \in X$, $0 \leqslant t, t_j < \infty$, and $s_j \geqslant t_0$.

The benefit of construction (2.1) lies in the relation between the ω-limit sets of Φ and Ψ.

Let a point $(s, x) \in Z$, $t_0 \leqslant s < \infty$, $x \in X$, have a pre-compact orbit $\{\Phi(t, s, x); t \geqslant s\}$. Then the ω-limit set of (s, x), $\omega_\Phi(s, x)$, is defined by

$$\omega_\Phi(s, x) = \bigcap_{\tau \geqslant s} \overline{\{\Phi(t, s, x); t \geqslant \tau\}}.$$

In other words, y is an element of $\omega_\Phi(s, x)$ if there is a sequence $s \leqslant t_j \to \infty, j \to \infty$, such that $\Phi(t_j, s, x) \to y, j \to \infty$.

The following statement is an obvious consequence of the definition of ω-limit sets, the definition of Ψ in (2.1), and Proposition 2.2.

Lemma 2.4 *y is an element of the ω-Φ-limit set of (s, x) if and only if (∞, y) is an element of the ω-Ψ-limit set of (s, x). If one of these two equivalent statements holds, then $\Psi(t, (\infty, y)) = (\infty, \Theta(t, y))$.*

Via Lemma 2.4 we can derive the following properties of ω-Φ-limit sets from the properties of ω-limit sets of autonomous semiflows and generalize Markus's Theorem 1.1. We say that a point y in X lies on an *entire* (*or full*) Θ-orbit if there exists a continuous function φ on $\boldsymbol{R}$ satisfying $\varphi(0) = y$ and $\varphi(t + s) = \Theta(t, \varphi(s))$.

Theorem 2.5 *ω-Φ-limit sets of points (s, x) with pre-compact (forward) orbits are non-empty, compact, and connected. Further they attract the orbits, i.e.,*

$$\operatorname{dist}(\Phi(t, s, x), \omega_\Phi(s, x)) \to 0, \quad t \to \infty.$$

Finally they are invariant under the limit-semiflow Θ, in particular any point y of $\omega_\Phi(s, x)$ lies on an entire Θ-orbit in $\omega_\Phi(s, x)$.

This proposition as well as others to come can be derived directly, of course, without using the semiflow Ψ defined in (2.1), see p.5. But it would be boring to repeat all the work done for autonomous semiflows in the case of asymptotically autonomous semiflows, the more so as the notation becomes much clumsier.

Finally we want to see how some other important dynamical systems concepts translate back and forth between Φ and Θ on one side and Ψ on the other. We use the notation

$$\Phi_s^t(x) = \Phi(t, s, x), \qquad \Theta_t(x) = \Theta(t, x).$$

Definition 2.6 A subset M of X is called *forward Θ-invariant* if and only if $\Theta_t(M) \subseteq M, t > 0$, and *$\Theta$-invariant* if and only if $\Theta_t(M) = M, t > 0$.

M is called *forward Φ-invariant* if and only if $\Phi_s^t(M) \subseteq M$ for all $t \geqslant s \geqslant t_0$, and *$\Phi$-invariant* if $\Phi_s^t(M) = M$ for all $t \geqslant s \geqslant t_0$.

Let Y be a forward Θ-invariant subset of X. A Θ-invariant subset M of Y is called an *isolated compact Θ-invariant subset* of Y, if and only if there is an open subset U of X such that there is no compact Θ-invariant set $\tilde{M}$ with $M \subseteq \tilde{M} \subseteq U \cap Y$ except M. U is called a *Θ-isolating neighborhood* of M in Y.

Lemma 2.7 *With the above definitions the following holds*:

(a) *A subset M of X is forward Φ-invariant if and only if $[t_0, \infty) \times M$ is forward Ψ-invariant.*

(b) *M is forward Θ-invariant if and only if $\{\infty\} \times M$ is forward Ψ-invariant.*

(c) *M is forward Φ-invariant and forward Θ-invariant if and only if $[t_0, \infty] \times M$ is forward Ψ-invariant.*

(d) *A subset $\tilde{M}$ of $[t_0, \infty] \times X$ is Ψ-invariant if and only if $\tilde{M} = \{\infty\} \times M$ with a Θ-invariant subset M of X.*

(e) *Let Y be a forward Θ-invariant subset of X. Then M is an isolated compact Θ-invariant subset of Y if and only if $\{\infty\} \times M$ is an isolated compact Ψ-invariant subset of $\{\infty\} \times Y$.*

3. Butler–McGehee type lemmas and cyclicity results

Let Φ be a continuous asymptotically autonomous semiflow on the metric space X and Θ its continuous limit-semiflow. Using the embedding of (Φ, Θ) into an autonomous semiflow outlined in the previous section, one can easily derive the following Butler–McGehee type result from the autonomous theory (Butler and Waltman 1986; Hale and Waltman 1989, Lemma 4.3; Thieme, preprint a, Sect. 4):

Lemma 3.1 *Assume that the point* (s, x), $s \geqslant t_0$, $x \in X$, *has a pre-compact* Φ*-orbit and that* $\omega = \omega_\Phi(s, x)$ *is its* ω-Φ*-limit set. Further let* M *be a* Θ*-invariant set such that* $M \cap \omega \neq \emptyset$, *but* $\omega \nsubseteq M$. *Finally assume that* $M \cap \omega$ *is an isolated compact* Θ*-invariant subset of* ω. *Then* M *has a non-empty stable and a non-empty unstable manifold in* ω *in the following sense*:

There exists an element $u \in \omega \backslash M$ *with* $\omega_\Theta(u) \subseteq M$ *and an element* $w \in \omega \backslash M$ *with a full* Θ*-orbit in* ω *whose* α-Θ*-limit set is contained in* M.

u *can be chosen such that its forward* Θ*-orbit is arbitrarily close to* M. w *can be chosen such that its backward* Θ*-orbit is arbitrarily close to* M.

We recall that the *α-Θ-limit* set of a full Θ-orbit $\varphi(t)$ is defined by

$$\alpha(\varphi) = \bigcap_{t \geqslant 0} \overline{\varphi((-\infty, -t])}.$$

We follow Hale and Waltman (1989) in the following definitions:

Definition 3.2 We call a union $M = \bigcup_{k=1}^m M_k$ a *Θ-invariant covering* of a set Ω if M contains Ω and if the sets M_k are pairwise disjoint Θ-invariant subsets of X.

A set $M \subseteq Y \subseteq X$ is said to be *Θ-chained* (in Y) to another (not necessarily different) set $N \subseteq Y$, symbolically $M \mapsto N$, if there is some $y \in Y$, $y \notin M \cup N$, and a full Θ-orbit through y in Y whose α-limit set is contained in M and whose ω-limit set is contained in N.

A finite number of sets $M_1, \ldots, M_k$, $k \geqslant 1$, is called *Θ-cylically chained* to each other (or a *Θ-cyclical chain*) in $Y \subseteq X$ if the following holds: In case that $k > 1$, M_j is chained to M_{j+1}, $j = 1, \ldots, k-1$, and M_k is chained to M_1, in Y. If $k = 1$, M_1 is chained to itself in Y.

A finite covering $M = \bigcup_{k=1}^m M_k$ of subsets in $Y \subseteq X$ is called *Θ-cyclic* in Y if, after possible renumbering, the sets $M_1, \ldots, M_k$ are cyclically chained to each other in Y for some $k \in \{1, \ldots, m\}$. M is called a *Θ-acyclic covering* in Y otherwise.

The following result can be easily derived from Lemma 3.1.

Proposition 3.3 *Let* ω *be the* ω*-limit set of a pre-compact* Φ*-orbit. Let* $M = \bigcup_{k=1}^m M_k$ *be a* Θ*-invariant covering of* Ω,

$$\Omega = \bigcup_{y \in \omega} \omega_\Theta(y),$$

such that, for all k, ω *is not contained in* M_k *and* $\omega \cap M_k$ *is an isolated compact* Θ*-invariant subset of* ω.

Then M *is* Θ*-cyclic in* ω.

Lemma 3.1 and Proposition 3.3 are the main tools (together with the autonomous Poincaré–Bendixson theorem) to prove Theorem 1.6. (See Thieme, preprint b.) We mention that, by the embedding in Sect. 2, one can also derive persistence results. Details will be explained elsewhere.

4 Convergence of pre-compact orbits

Let Φ be an asymptotically autonomous continuous semiflow on the metric space X and Θ its continuous limit-semiflow.

We recall that a Θ*-equilibrium* (or *fixed point*) is an element $e \in X$ such that $\Theta(t, e) = e$ for all $t \geqslant 0$. We first generalize Markus's Theorem 1.2.

Theorem 4.1 *Let e be a locally asymptotically stable equilibrium of Θ and $W_s(e) = \{x \in X; \Theta(t, x) \to e, t \to \infty\}$ its basin of attraction (or stable set). Then every pre-compact Φ-orbit whose ω-Φ-limit set intersects $W_s(e)$ converges to e.*

Proof. Let ω be an ω-Φ-limit set which has a point x in common with $W_s(e)$. By Theorem 2.5, $\omega_\Theta(x)$ is contained in ω. On the other hand $\omega_\Theta(x)$ just consists of e. Hence $e \in \omega$. As e is locally asymptotically stable, $\{e\}$ is an isolated compact Θ-invariant set. If ω also contains elements different from e, by Lemma 3.1, ω contains a full orbit through a point different from e whose α-Θ-limit set is $\{e\}$. This contradicts the local stability of e.

In order to answer Question 1.4 we make the following assumption (recall the definition of an isolated compact invariant set in Definition 2.6):

(E) *The equilibria of Θ are isolated compact Θ-invariant subsets of X. Further the ω-Θ-limit set of any pre-compact Θ-orbit contains a Θ-equilibrium.*

Theorem 4.2 *Let* (E) *hold and the point (s, x), $s \geqslant t_0$, $x \in X$, have a pre-compact Φ-orbit. Then the following alternative holds:*

- *$\Phi(t, s, x) \to e$, $t \to \infty$, for some Θ-equilibrium e.*
- *The ω-Φ-limit set of (s, x) contains finitely many Θ-equilibria which are chained to each other in a cyclic way.*

Proof. As the ω-Φ-limit set of (s, x), let us call it ω, is Θ-invariant, it contains an equilibrium. Assume that ω is no singleton. Let $e_1, \ldots, e_m$, $m \geqslant 1$, be the Θ-equilibria contained in ω. Set $M_j = \{e_j\}$ and apply Proposition 3.3. Then the M_j form a cyclic covering.

Example 1.5 warns us that the second possibility can occur even for planar ODE systems and has to be excluded in order to guarantee convergence towards an equilibrium.

Corollary 4.3 *Let* (E) *hold and assume that there is no Θ-cyclical chain of Θ-equilibria. Then any pre-compact forward Φ-orbit converges towards a Θ-equilibrium for $t \to \infty$.*

Remark 4.4 The results of this section remain true if equilibria are replaced by compact Θ-invariant subsets of X.

Acknowledgements. Carlos Castillo-Chavez triggered this investigation by pointing out to me that Markus's results may not be sufficient to guarantee convergence in certain epidemic models (Blythe et al., preprint). I also enjoyed fruitful discussions with my colleagues Matthias Kawski, Yang Kuang, Hal L. Smith, and Betty Tang at ASU, and with Karl P. Hadeler. I thank Jose Palacios and Jack VanWieren for helping me find references to Markus's theorems.

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