Rank-dominance in income distributions

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Rawls (1971) motivated his maximin principle for social choice through the artifact of the 'veil of ignorance': the assumption that individuals do not know their position in society. The Rawls maximin principle may be rationalized in terms of individuals identifying with the least will-off individual in each state. Analogously, we are concerned with the social choice criterion that is suggested, given the 'veil of ignorance' assumption, when individuals view themselves as potential occupants of each position in society. Identifying the 'position in society' of the individual with his or her rank in the income distribution, we arrive at the rank-dominance partial order on income distributions. We say that one income distribution rank-dominates another income distribution if and only if the income in each position in the first income distribution is at least as great as the income in the same position in the second income distribution. Just as the assumption on the part of the individual that he or she is going to occupy the lowest rank suggests the Rawls maximin principle, so does the assumption on the part of the individual that he or she is just as likely to occupy any rank as any other suggest the rank-dominance criterion.

This note parallels the work of Atkinson (1970) – subsequently extended by Dasgupta, Sen and Starrett (1973) and Rothschild and Stiglitz (1973) – on the measurement of inequality both in asking what social welfare function is implied by a particular partial ordering on income distributions and in applying work in the theory of choice under uncertainty to answer this question. To this end, we recall two results from the theory of choice under uncertainty. (The discrete versions of both theorems are stated.)

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Theorem A: (Hadar and Russell, 1969; Hanoch and Levy, 1969; Rothschild and Stiglitz, 1970)

Let F and G be (cumulative) distribution functions on income.

Then

$$\sum_{t=0}^{y} F(t) \leqslant \sum_{t=0}^{y} G(t)$$

for all y if and only if $E_F u \geqslant E_G u$ for all monotone-increasing concave function u: $R^1 \to R^1$, where $E_F(\cdot)$ denotes the expected value operator relative to F.

Theorem B: (Quirk and Saposnik, 1962)

Let F, G, E_F and E_G be defined as in Theorem A. Then $F(t) \leqslant G(t)$ for all t if and only if $E_F u \geqslant E_G u$ for all monotone (not necessarily concave) functions u: $R^1 \to R^1$.

Whereas Atkinson exploited Theorem A, this note exploits Theorem B.

For each income distribution x we write

$$x = (x(1), \ldots, x(n))$$

where $x(i)$ denotes the income of individual i in distribution x, x_i will denote the ith income in the distribution in ascending order. That is, x_i is the ith element in $(x_1, \ldots, x_n)$, where $x_1 \leqslant \ldots \leqslant x_n$. In the event that $x(i) = x(j)$, $i \neq j$, $x(i)$ will come before $x(j)$ in the ranking if $i < j$.

Given income distributions x and y we say $x \underset{R}{\geqslant} y$ (x rank-dominates y) if and only if $x_i \geqslant y_i$ for all i.

Let

$$f(t) = \frac{\#\{i | x(i) = t\}}{n}$$

and

$$g(t) = \frac{\#\{i | y(i) = t\}}{n} .$$

We say $x \underset{U}{\geqslant} y$ if and only if

$$\sum_{x(i)} u(t) f(t) \geqslant \sum_{y(i)} u(t) g(t)$$

for all monotone increasing u; Thus $\underset{U}{\geqslant}$ is the social preference relation associated with a social welfare function that is additively separable in monotone individual utility functions.

We say $x \underset{W}{\geqslant} y$ if, for all W: $R_+^n \to R^1$ satisfying the two conditions

$$W(x) \geqslant W(x') \text{ whenever } x \geqslant x' \tag{1}$$

and

$$W(\pi(x)) = W(x) \text{ for any permutation } \pi(x) \text{ of } x \tag{2}$$

we have $W(x) \geqslant W(y)$.

Thus $\underset{W}{\geqslant}$ is the social preference relation associated with a social welfare function that is monotone in individual incomes and has the symmetry (anonymity) property.

To the extent that the assumption of symmetry of the social welfare function is considered less objectionable than the assumption of additive separability, $\underset{W}{\geqslant}$ is preferable to $\underset{U}{\geqslant}$ as a partial order on income distributions. Accordingly, we have the following result.

Proposition 1: $\underset{W}{\geqslant}$, $\underset{U}{\geqslant}$ and $\underset{R}{\geqslant}$ are all equivalent.

Proof: We shall prove the proposition by showing that $\underset{W}{\geqslant} \Rightarrow \underset{U}{\geqslant} \Rightarrow \underset{R}{\geqslant} \Rightarrow \underset{W}{\geqslant}$.

1. $\underset{W}{\geqslant} \Rightarrow \underset{U}{\geqslant}$ since the class of additively separable social welfare functions is contained in the class of symmetric social welfare functions.

2. Suppose $x \underset{R}{\not\geqslant} y$. Let j be the first integer for which $x_j < y_j$. Then

$$F(x_j) = \sum_{t \neq x_j} f(t) = \frac{j}{n} > \frac{j-1}{n} = \sum_{t \leqslant x_j} g(t) = G(x_j) .$$

It follows from Theorem B that $x \underset{U}{\not\geqslant} y$. Thus $x \underset{U}{\geqslant} y \Rightarrow x \underset{R}{\geqslant} y$.

3. Suppose $x \underset{R}{\geqslant} y$. Then by monotonicity and symmetry of W, $W(x(1), \ldots, x(n)) = W(x_1, \ldots, x_n) \geqslant W(y_1, \ldots, y_n) = W(y(1), \ldots, y(n))$. Thus $x \underset{R}{\geqslant} y \Rightarrow x \underset{W}{\geqslant} y$. This completes the proof of the proposition.

We turn briefly to the relationship between the rank criterion and the Pareto criterion. It is obvious that rank-dominance does not imply Pareto-dominance. (To see this we need only consider the two-person income distributions $x(1) = 5, x(2) = 3$ and $y(1) = 2, y(2) = 4$. Here $x \underset{R}{\geqslant} y$ but x is not Pareto-superior to y.)

However, as the following shows, the converse is true.

Corollary: For income distributions x and y $x \underset{P}{\geqslant} y \Rightarrow x \underset{R}{\geqslant} y$, where $\underset{P}{\geqslant}$ denotes the (strong) Pareto criterion.

Proof: $x \underset{P}{\geqslant} y \Rightarrow x(i) \geqslant y(i)$ for all i. Therefore for any real number t, $\{i \mid x(i) \leqslant t\} \subseteq \{i \mid y(i) \leqslant t\}$. Obviously, then

$$F(t) = \frac{\#\{i \mid x(i) \leqslant t\}}{n} \leqslant \frac{\#\{i \mid y(i) \leqslant t\}}{n} = G(t) \text{ for all } t.$$

By Theorem B this means $x \underset{U}{\geqslant} y$, which by Proposition 1 is equivalent to $x \underset{R}{\geqslant} y$.

Finally, we show that the rank-dominance criterion is consistent with a version of the Suppes grading principle (see Sen, 1970: Ch. 9). We say that x is graded higher than y ($x \underset{S}{\geqslant} y$) if there is a permutation σ of individuals such that $x(i) \geqslant y(\sigma(i))$ for all i.

We then have:

Proposition 2: $x \underset{R}{\geqslant} y \Rightarrow x \underset{S}{\geqslant} y$.

Proof: Let $x = (x(1), \ldots, x(n))$ and $y = (y(1), \ldots, y(n))$ be such that $x \underset{R}{\geqslant} y$. $(x(1), \ldots, x(n)) \to (x_1, \ldots, x_n)$ and $(y(1), \ldots, y(n)) \to (y_1, \ldots, y_n)$ define permutations π and ρ on $\{1, \ldots, n\}$ such that for $i \neq j$,

$$\pi(i) < \pi(j) \text{ if } \left\{ \begin{array}{l} x(i) < x(j) \text{ or} \\ x(i) = x(j) \text{ and } i < j \end{array} \right. .$$

$$\rho(i) < g(j) \text{ if } \left\{ \begin{array}{l} y(i) < y(j) \text{ or} \\ y(i) = y(j) \text{ and } i < j \end{array} \right. .$$

That is, π uniquely assigns to each individual a rank in income distribution x and ρ does the same relative to income distribution y, so that π^{-1} and ρ^{-1} are well defined.

$$\begin{aligned} \text{Now } x \underset{R}{\geqslant} y &\Rightarrow x_{\pi(i)} \geqslant y_{\pi(i)} \text{ for all } i \\ &\Rightarrow x[\pi^{-1} \cdot \pi(i)] \geqslant y[\rho^{-1} \cdot \pi(i)] \\ &\Rightarrow x(i) \geqslant y[\rho^{-1} \cdot \pi(i)] \\ &\Rightarrow x \underset{S}{\geqslant} y \text{ where } \sigma \equiv \rho^{-1} \cdot \pi \; . \end{aligned}$$

This completes the proof of the proposition.

In summary, we have defined a partial order called rank-dominance on income distributions in terms of the position of incomes in the distribution. Specifically, one income distribution is said to rank-dominate a second distribution whenever the income in each position in the first distribution is at least as large as the income in the same position in the second distribition We showed that the partial ordering of income distributions defined by the rank-dominance relation is identical with that defined by a social welfare function which is additively separable in individual utility functions as well as with that defined by a social welfare function having the anonymity propert. Moreover, if one income distribution is Pareto-superior to a second, the first must also rank-dominate the second. Finally, we showed that one income distribution rank-dominates another only if the first distribution grades higher in the Suppes sense than the second distribution.

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