Approval voting, Condorcet's principle, and runoff elections

PETER C. FISHBURN and STEVEN J. BRAMS*

Abstract

Approval voting allows each voter to vote for as many candidates as he wishes in an election but not cast more than one vote for each candidate of whom he approves. If there is a strict Condorcet candidate – a candidate who defeats all others in pairwise contests – approval voting is shown to be the only nonranked voting system that is always able to elect the strict Condorcet candidate when voters use sincere admissible strategies. Moreover, if a strict Condorcet candidate must be elected under ordinary plurality voting when voters use admissible strategies, then he must also be elected under approval voting when voters use admissible strategies, but the converse does not hold.

The widely used plurality runoff method can also elect a strict Condorcet candidate when voters use admissible strategies on the first ballot, but some of these may have to be insincere to get the strict Condorcet candidate onto the runoff ballot. Furthermore, there is no case in which the strict Condorcet candidate is invariably elected under the plurality runoff method when voters use admissible first-ballot strategies. Thus, approval voting is superior to the plurality runoff method with respect to the Condorcet principle in its ability to elect the strict Condorcet candidate by sincere voting and in its ability to guarantee the election of the strict Condorcet candidate when voters use admissible strategies. In addition, approval voting is more efficient since it requires only one election and is probably less subject to strategic manipulation.

Approval voting in a multicandidate election allows each voter to vote for as many candidates as he wishes. A candidate receives one full vote from each person who votes for him regardless of how many other candidates that person votes for; the candidate with the most votes wins the election. We

* College of Business Administration, The Pennsylvania State University, and Department of Politics, New York University, respectively. Dr. Fishburn's present address is Bell Laboratories, Murray Hill, N.J. 07974.

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have noted elsewhere [1, 3] that approval voting is superior in several ways to all other single-ballot systems that do not ask voters to rank candidates. These other systems differ from approval voting by restricting the number of candidates that voters are allowed to vote for, the prime example being plurality (vote for one) voting.

A major reason for preferring approval voting to plurality and other single-ballot nonranked systems is its lack of any restriction on the number of candidates for whom an individual can vote. Thus, voters can express their 'approval' for all candidates they find relatively acceptable, and they do not have to worry about abandoning their favorite candidates when such candidates have slim chances of winning. Approval voting also promotes sincere voting and discourages strategic or manipulative voting more than other single-ballot systems [1, 3] and thereby enhances the likelihood that a candidate who is acceptable to a large proportion of the electorate will be elected. Furthermore, approval voting is more likely to elect a Condorcet or majority candidate – one who is preferred by a majority to each other candidate – when one exists for certain types of situations [1].

Our purpose in the present paper is two-fold: (1) to present new results that reinforce the apparent superiority of approval voting over other single-ballot nonranked systems; and (2) to introduce runoff election systems into our comparative analysis. The generic class of runoff systems consists of voting procedures whose first ballots are similar to the ballots of the single-ballot nonranked systems; the second or runoff ballot is a simple majority ballot between the two candidates who receive the most first-ballot votes.

In the next two sections we describe the voting systems we shall analyze in more detail and specify our assumptions about voters' preferences, both between individual candidates and between what we shall refer to as outcomes of elections. In section 3 we define the notion of admissible voting strategies on the only (or first) ballot; for each combination of a voting system and preference order, we identify the set of strategies that are admissible (feasible and undominated) for that combination. We note that for most preference orders the set of admissible voting strategies for any ordinary (single-ballot) system is a proper subset of the set of admissible strategies for the corresponding runoff system.

Next, in sections 4 and 5 we use the admissibility results to determine the abilities of various systems to elect a strict Condorcet candidate when one exists. In the first of these two sections, we investigate the existence of admissible, and sincere admissible, strategies that will elect the Condorcet candidate. In the latter section we consider cases in which the Condorcet candidate is invariably elected when all voters use admissible strategies.

To reinforce our conclusion that ordinary approval voting is superior to other single-ballot nonranked systems, we argue that ordinary approval voting has several advantages over runoff systems. However, our comparison between ordinary approval voting and the popular runoff plurality system leaves ample room for further analysis. A summary of findings concludes the main part of the paper.

Proofs of the theorems are given in the Appendix. Theorems 1 and 3 are not proved here since their proofs appear in [1].

1. Voting systems

We shall assume that there are $m \geqslant 3$ candidates in the set X of contending candidates, exactly one of whom is to be elected by a prespecified voting system. Two main types of voting systems will be considered. We shall refer to these as ordinary and runoff systems. A specific system of either type is identified by a nonempty subset s of $\{1, 2, \ldots, m-1\}$. Since there are $2^{m-1} - 1$ such subsets [1], there are $2^{m-1} - 1$ different systems of each type.

Each *ordinary system* s is a single-ballot system. Each voter either abstains or votes for exactly k candidates for some $k \in s$. Voters do not rank their chosen candidates, and a candidate receives a full vote from each person who votes for that candidate. The candidate with the most votes is elected. Thus $s = \{1\}$ is the usual plurality system, and $s = \{1, 2, \ldots, m-1\}$ is the approval voting system.

Each *runoff system* s is a two-ballot system. On the first ballot each voter either abstains or votes for exactly k candidates for some $k \in s$. The two candidates with the most first-ballot votes go onto the second, or runoff, ballot.[1] The runoff ballot is a simple-majority ballot; the candidate who receives the greater number of votes on this ballot is elected.

What about ties? For ordinary systems we assume that all candidates who tie with the largest vote total have some chance of winning and that all others have no chance of being elected, but we will not be specific about how ties are broken. Given the possibility of ties, we define an *ordinary outcome* as the *set* of candidates who have the greatest vote total in an ordinary election.

Ties can occur for runoff systems at two points. First, a tie can affect who goes onto the runoff ballot. If more than two candidates have the largest first-ballot total, then we assume that each of these and no other candidate has a chance of being in the runoff, which always involves exactly two candidates. On the other hand, if exactly one candidate has the largest first-ballot total, and two or more candidates have the next-largest first-ballot total, then we assume that the former candidate is assured of being in the runoff whereas each of the latter tied candidates (but no other) has a chance of being in the runoff.

The second point at which a tie can occur is on the runoff ballot. When this occurs, we assume that each candidate *not* on the runoff has no chance of being elected. As with ordinary systems, specific tie-breaking procedures will not be prescribed for runoff systems.

For later purposes we define a *runoff outcome* as a triple $(A, B, >)$ where:

- A is the set of candidates who must be on the runoff ballot on the basis of the first-ballot vote;
- B is the set of candidates who have some chance of being in the runoff but are not assured of being in the runoff on the basis of the first-ballot vote;
- $>$ specifies what would happen in the runoff for every pair of distinct candidates from $A \cup B$ who might be on that ballot; $x > y$ means that x would beat y.

Let $|A|$ denote the number of candidates in A and $\emptyset$ be the empty set so that $|A| = 0$ if and only if $A = \emptyset$. Preceding assumptions require $|A| \in \{0, 1, 2\}$, with $B = \emptyset$ if $|A| = 2$ and with $|A| + |B| \geqslant 3$ if $|A| < 2$. Within a runoff outcome, (A, B) may take the following forms: $(A, B) = (\{x, y\}, \emptyset)$ says that x and y have more first-ballot votes than all other candidates; $(A, B) = (\{x\}, \{y, z\})$ says that x has the most first-ballot votes, followed by y and z, who are tied; and $(A, B) = (\emptyset, \{x, y, z\})$ says that x, y and z have the same largest first-ballot total.

The fact that we consider only voting systems that do not ask voters to rank candidates is motivated by practical concerns. Apart from the unusual s sets, such systems are relatively easy for voters to understand and respond to. This is especially true of the ordinary plurality and approval voting systems and the familiar plurality ($s = \{1\}$) runoff system. In addition, these systems can be implemented on present voting machines, which is obviously important in primary and general elections at the local, state, and national [5, 6, 7] levels.

Several important differences between ordinary and runoff systems will be noted here since later sections focus on other things. First, runoff systems are more costly than ordinary systems since they required two elections.[2] Second, runoff systems provide voters more opportunities for strategic manipulation. In particular, runoff systems encourage people to vote for candidates on the first ballot only because they could be beaten by the voters' favorite(s) in a runoff. Since the degree of manipulability in this sense is most likely to be positively correlated with the number of candidates a person can vote for on the first ballot, the plurality runoff system would appear to be the least manipulable, while the approval voting runoff system is the most manipulable. Although we will not analyze the comparative manipulability of runoff systems – as we did for ordinary systems in [1] – we think that coupling approval voting to a runoff system produces a flawed combination [2]. This aversion, however, does not diminish our ardor for ordinary approval voting.

Runoff systems may also encourage the entry of more potential candidates into X, either because these candidates feel they have a better chance of winning or because they are induced by supporters of other candidates to enter as stalking horses to prevent certain contenders from reaching the runoff. The latter aspect is a form of agenda manipulation that deserves closer scrutiny.

2. Voter preferences

We shall let P denote a voter's strict preference relation on X, so that xPy if and only if the voter prefers x to y. The indifference relation I for P is defined by xIy if and only if neither xPy nor yPx, and xRy means that either xPy or xIy. We assume that P on X is an asymmetric weak order so that P partitions X into one or more indifference classes $X_1, \ldots, X_K$ such that xIy if and only if x and y are in the same X_k, with xPy whenever $x \in X_j$, $y \in X_k$ and $j < k$. The most-preferred (X_1) and least-preferred (X_K) indifference classes for P will be denoted respectively as $M(P)$ and $L(P)$. A voter is *unconcerned* (and P is unconcerned) if and only if $K = 1$. or $M(P) = L(P) = X$; otherwise the voter (or P) is *concerned*. If $K = 2$ with $X = M(P) \cup L(P)$, then P is *dichotomous*. In general, $A \subseteq X$ is *high for* P when $x \in A$ implies $y \in A$ for all yPx, and $A \subseteq X$ is *low for P* when $x \in A$ implies that $y \in A$ for all y for which xPy. If P is unconcerned, then all subsets of X are both high and low for P. In general, A is high for P if and only if its complement $X \setminus A$ is low for P. Both $\emptyset$ and X are high and low for every P.

For convenience in discussing preferences between outcomes we shall use P, I and R for preference, indifference, and preference-or-indifference between outcomes as well as for the corresponding relations between individual candidates. We assume without further mention that R is always reflexive and that both O_1RO_2 and O_2RO_1 cannot hold simultaneously for outcomes O_i of the same type. Furthermore, for ordinary outcomes $\{x\}$ and $\{y\}$, we assume that $\{x\}P\{y\}$ if and only if xPy, and that $\{x\}R\{y\}$ if and only if xRy.

The following assumptions (cf. Assumptions P and R in [1]) will be presumed to hold for ordinary elections for all $x, y \in X$ and all ordinary outcomes A, B and C:

*Axiom P*1. If xPy then $\{x\}P\{x,y\}$ and $\{x,y\}P\{y\}$;

*Axiom R*1. If $A \cup B$ and $B \cup C$ are nonempty, and if aRb, bRc and aRc for all $a \in A$, $b \in B$ and $c \in C$, then $(A \cup B) R (B \cup C)$.

For runoff outcome $\gamma = (A, B, >)$, we shall let Γ denote the set of all pairs of candidates that might constitute the runoff ballot, and let $\Gamma(>)$ be the set of all candidates who have some chance of winning the election when γ obtains. For example, if $\gamma = (\{x\}, \{y, z\}, \{x > y, z > x\})$, then $\Gamma = \{\{x, y\}, \{x, z\}\}$ and $\Gamma(>) = \{x, z\}$.

For preferences between runoff outcomes, let Γ_i correspond to γ_i, and let $\Gamma_i \backslash \Gamma_j$ denote the set of all pairs in Γ_i that are not in Γ_j. The following will be presumed to hold for runoff elections for all runoff outcomes $\gamma_1 = (A_1, B_1, >)$ and $\gamma_2 = (A_2, B_2, >)$:

Axiom P2. If $\Gamma_1(>) = \{x\}$ and $\Gamma_2(>) = \{x, y\}$, then $\gamma_1 P \gamma_2$ if xPy, and $\gamma_2 P \gamma_1$ if yPx;

Axiom R2. $\gamma_1 R \gamma_2$ if either $\cup\{\{x, y\}: \{x, y\} \in \Gamma_1\} \subseteq M(P)$ or $\cup\{\{x, y\}: \{x, y\} \in \Gamma_2\} \subseteq L(P)$ or $[\cup\{\{x, y\}: \{x, y\} \in \Gamma_1 \backslash \Gamma_2\} \subseteq M(P)$ and $\cup\{\{x, y\}: \{x, y\} \in \Gamma_2 \backslash \Gamma_1\} \subseteq L(P)]$.

Axiom $P2$ is straightforward and is similar to Axiom $P1$. Although $>$ does not appear explicitly in Axiom $R2$, the same $>$ is presumed to apply to both γ_1 and γ_2 so that if $\{x, y\} \in L_1 \cap L_2$ and $x > y$, then x will win in either case if x and y are on the runoff ballot. The first alternative in Axiom $R2$ implies that $\Gamma_1(>) \subseteq M(P)$ regardless of $>$; the second alternative implies that $\Gamma_2(>) \subseteq L(P)$ regardless of $>$; and the third alternative guarantees the election of a most-preferred candidate if the runoff pair is in Γ_1 but not Γ_2, or the election of a least-preferred candidate if the runoff pair is in Γ_2 but not Γ_1. Hence $R2$ seems quite reasonable from the viewpoint of an individual's preferences.

3. Voting strategies

In this section we characterize admissible voting strategies for runoff systems. We shall also note from [1] which strategies are admissible for ordinary systems. Ensuing sections assume that voters either abstain or use admissible strategies on the only (or first) ballot. We assume also that everyone who votes on a runoff ballot votes for his more preferred candidate on that ballot. Hence, apart from abstaining on the runoff, a voter's only choice in a runoff system is what to do on the first ballot.

Voting strategies for both ordinary and runoff elections are *proper* subsets of X. A voter uses strategy $S \subset X$ on the only (or first) ballot when he votes for all candidates in S and no others. The abstention strategy is denoted by $\emptyset$. Strategy S is *feasible* for system s if and only if either $S = \emptyset$ or $|S| = k$ for some $k \in s$. Strategy S is *admissible* for system s and preference order P if and only if S is feasible for s and no strategy T that is also feasible for s dominates S for P.

To explicate our notions of dominance for ordinary and runoff systems, we consider a focal voter with preference order P, allow any finite number of other voters, and consider all ways that the others might vote. In the runoff case, it is presumed that the focal voter votes on the runoff if he votes on the first ballot and is not indifferent between the runoff candidates. When considering strategies S and T for the focal voter, we presume that both strategies are allowed by the system at hand. The inclusion of feasibility in the foregoing definition of admissibility takes account of cases for specific s sets in which one or both of S and T are not feasible.

Others' votes on the only (or first) ballot, as well as what might happen on a runoff ballot, are accounted for by contingencies. An *ordinary contingency* lists the number of votes for each candidate by all voters other than the focal voter in an ordinary election. A *runoff contingency* specifies the vote totals for all other voters on the first ballot *and* specifies by $>$ on X what would happen on every runoff ballot that might arise from the first ballot. Ordinary outcomes and runoff outcomes as defined in section 1 are unambiguously specified by a corresponding contingency and the strategy used by the focal voter on the only (or first) ballot. For example, a runoff contingency plus a strategy S for the focal voter uniquely determines a runoff outcome $(A, B, >)$.

We shall say that strategy S dominates strategy T in an ordinary election with respect to the focal voter with preference order P, or $S\ dom_1\ T$ *for* P, if and only if the focal voter likes the ordinary outcome when he uses S as much as (R) the ordinary outcome when he uses T, for every possible ordinary contingency, and strictly prefers (P) the ordinary S-outcome to the ordinary T-outcome for at least one ordinary contingency.

Similarly, for a runoff election we write $S\ dom_2\ T$ *for* P if and only if the focal voter with preference order P likes the runoff outcome when he uses S as much as (R) the runoff outcome when he uses T, for every possible runoff contingency, and strictly prefers (P) the runoff S-outcome to the runoff T-outcome for at least one runoff contingency.

In general, let $A \setminus B$ be the set of all elements in A but not B. Because dominance is based on all contingencies and the focal voter votes for all candidates in $S \cap T$ on the only or first ballot when he uses either S or T, it follows for either type of system that S dominates T for P if, and only if, $S \setminus T$ dominates $T \setminus S$ for P. This fact is reflected in the following theorems, which show that ordinary dominance (dom_1) and runoff dominance (dom_2) are substantially different.

Theorem 1 (Ordinary dominance). Suppose P is concerned and Axioms $P1$ and $R1$ hold. Then, for all strategies S and T, $S\ dom_1\ T$ *for* P if and only if $S \neq T$, $S \setminus T$ is high for P and $T \setminus S$ is low for P.

Theorem 2 (Runoff dominance). Suppose P is concerned and Axioms $P2$

and $R2$ hold. Then, for all strategies S and T, S dom_2 T *for* P if and only if P is dichotomous, $S \neq T$, $S \setminus T$ is either $\emptyset$ or $M(P)$, and $T \setminus S$ is either $\emptyset$ or $L(P)$.

Clearly runoff dominance demands much more of S, T and P than does ordinary dominance, with S dom_1 T *for* P whenever S dom_2 T *for* P. In general, then, more strategies will be admissible for runoff system s and preference order P than for ordinary system s and preference order P.

Theorem 3 (ordinary admissibility). Under the hypothesis of Theorem 1, strategy S is admissible for ordinary system s and preference order P if and only if S is feasible for s and either $C1$ or $C2$ holds:

$C1$: $M(P) \subseteq S$, and S cannot be partitioned into nonempty S_1 and S_2 such that S_1 is feasible for s and S_2 is low for P;
$C2$: $L(P) \cap S = \emptyset$, and no nonempty $A \subseteq X$ is such that $A \cap S = \emptyset$, $A \cup S$ is feasible for s, and A is high for P.

Theorem 4 (Runoff admissibility). Under the hypotheses of Theorem 2, strategy S is admissible for runoff system s and preference order P if and only if S is feasible for s and either $C3$ or $C4$ holds:

$C3$: P is not dichotomous;
$C4$: P is dichotomous, it is false that $L(P) \subseteq S$, and $S \setminus L(P)$ is feasible for $s\}$, and it is false that $\{S \subset L(P)$, and $M(P) \cup S$ is feasible for $s\}$.

Theorem 3 with concerned P implies that the abstention strategy $\emptyset$ is never admissible for an ordinary system. As noted in [1], $\{x\}$ is admissible for ordinary plurality voting and concerned P if and only if $x \notin L(P)$, and S is admissible for ordinary approval voting and concerned P if and only if $M(P) \subseteq S$ and $L(P) \cap S = \emptyset$. Thus, if P is dichotomous, there is a unique admissible strategy for ordinary approval voting, namely $M(P)$.

Theorem 4 shows that if P is concerned and not dichotomous, then every feasible strategy, including $\emptyset$, is admissible for runoff system s and P. But what if P is dichotomous? Then all strategies feasible for the runoff plurality system are admissible, except as follows: if $|M(P)| = 1$ then $\emptyset$ is not admissible – it is better to vote for the unique most-preferred candidate than to abstain; if $|L(P)| = 1$, then $L(P)$ is not admissible – it is better to abstain than to vote for the unique least-preferred candidate. In addition, strategy S is admissible for runoff approval voting if and only if something in $M(P)$ is in S and something in $L(P)$ is not in S. Hence $\emptyset$ is never admissible for runoff approval voting when P is dichotomous.

As in [1], we say that *strategy S is sincere for P* if and only if S is high for P, and that *system s is sincere for P* if and only if all admissible strategies for s and P are sincere. A sincere strategy does not require a voter to 'lie' about his preferences, as would be true of an insincere strategy wherein he votes for a candidate whom he prefers less than some candidate for whom he does not vote. Thus, sincere strategies better reflect voters' preferences.

Sincerity for ordinary systems is discussed in [1], wherein we note that the ordinary approval voting system is the only ordinary system that is sincere for every *trichotomous* ($K = 3$) P. Theorems 2 and 4 show that a runoff system can be sincere for concerned P only if P is dichotomous. Given that P is dichotomous, runoff plurality voting is sincere for P if and only if $|L(P)| = 1$, and runoff approval voting is sincere for P if and only if either $|M(P)| = 1$ or $|L(P)| = 1$. Thus, when there are exactly three candidates, the runoff approval voting system is sincere for every dichtomous P.

However, as noted earlier, runoff approval voting is subject to severe manipulation effects. For example, if a plurality of voters prefer x to y to z and are fairly sure that x would beat z but lose to y in a runoff, they may well vote for x and z on the first ballot in an attempt to engineer a runoff between x and z. As a concrete example of this type of thinking, we would point to the 1976 election for House Majority Leader [2].

4. Condorcet possibility theorems

In this section and the next we shall consider whether a given candidate $x \in X$ either can be elected or must be elected when various voting systems are used. In both sections we assume that all voters use admissible strategies on the only (or first) ballot. In the present section we shall be mainly concerned with the existence of admissible, or sincere admissible, strategies which can elect a Condorcet candidate. Cases in which x must be elected are examined in the next section.

Two simple examples with candidate set $X = \{x, a, b\}$ and Condorcet candidate x illustrate the results given below. Suppose first that there are seven voters such that:

2 voters prefer x to a to b
2 voters prefer x to b to a
3 voters are indifferent between a and b and prefer a and b to x.

Then both the ordinary and runoff approval voting and plurality voting systems can elect the Condorcet candidate x when all voters use sincere admissible strategies. However, consider what happens when $s = \{2\}$, in which case each voter must vote for exactly two candidates on the only (or first) ballot if he does not abstain. For either the ordinary or runoff system $\{2\}$, the only admissible strategy for the last three voters is $\{a, b\}$. This is easily verified from Theorems 3 and 4. It follows that ordinary system $\{2\}$ *must* elect either a or b when all voters use admissible strategies since x will get exactly four votes while a and b together get 10 votes. In addition, if sincere admissible strategies are used in runoff system $\{2\}$, then x will not go onto the runoff ballot for, regardless of which of the first four voters abstain rather than vote for their first two choices on the first ballot, both a and b will get more votes than x. On the other hand, insincere admissible voting on the first ballot can get x elected with runoff system $\{2\}$. For example, if the first four voters vote for x and b on the first ballot – with the first two voting insincerely in this case – then x and b will be in the runoff where x will beat b by a simple majority. Thus, system $\{2\}$ can elect the Condorcet candidate x only if some voters vote insincerely – and then only when there is a runoff – whereas sincere admissible strategies suffice under approval and plurality voting systems, with or without a runoff.

Our second example has five voters such that:

1 voter prefers x to a to b
2 voters prefer a to x to b
2 voters prefer b to x to a.

In this case, ordinary approval voting but not ordinary plurality voting can elect x when all voters use *sincere* admissible strategies, although x can be elected by ordinary plurality when some voters use insincere admissible strategies. Furthermore, if no voter abstains on the first ballot under the plurality runoff system, then some voter must vote insincerely to ensure that x gets on the runoff ballot. Now even plurality voting fails to elect the Condorcet candidate x if voters vote sincerely, whereas ordinary approval voting succeeds.

We next generalize these observations. Although the following theorems do not cover all conceivable cases, they bring out the main points. We shall let V denote a voter preference profile so that V assigns a preference order P to every voter. In addition, $V[x]$ denotes the set of all V in which candidate x is a strict Condorcet candidate in the sense that more voters have xPy than yPx for each $y \in X \setminus \{x\}$.

Theorem 5 (Ordinary systems). The following hold for ordinary systems s:

1. For each $m \geqslant 3$, there is a $V \in V[x]$ and an s such that no combination of admissible strategies for the voters in V will elect x;
2. If $1 \in s$, then, for every $V \in V[x]$, there are admissible strategies for V that will elect x;
3. If s is the approval voting system, then for every $V \in V[x]$ there are sincere admissible strategies for V that will elect x;
4. If s is any system other than the approval voting system, then there is a

$V \in V[x]$ such that no combination of sincere admissible strategies will elect x.

Parts 2 through 4 of the theorem apply to each $m \geqslant 3$. Together, parts 3 and 4 say that approval voting is the only ordinary system that is invariably able to elect a strict Condorcet candidate x when voters use sincere admissible strategies. This does *not* say that approval voting *must* elect x when $V \in V[x]$ and voters use sincere admissible strategies – only that it is possible to assign sincere admissible strategies to the voters under which x will be elected. (However, as shown in [1], if every P is dichotomous, then x must indeed be elected when all voters use admissible strategies under approval voting.)

Part 2 of Theorem 5 notes that some systems beside approval voting have the capability of electing x whenever $V \in V[x]$ and voters use (not necessarily sincere) admissible strategies, whereas part 1 notes that there are other ordinary systems that cannot make this claim. The first example of this section shows a case in which $s = \{2\}$ can never elect x when $m = 3$, V is as specified in the example, and voters use admissible strategies.

Theorem 6 (Runoff systems). The following hold for runoff systems s when each voter in V votes on the runoff ballot if and only if he is not indifferent between the candidates on that ballot:

1. For every s and every $V \in V[x]$, there are admissible strategies for V that give x as many first-ballot votes as every other candidate;
2. If $s \cap \{1, 2\} \neq \emptyset$, then for every $V \in V[x]$ there are admissible strategies for V that will elect x;
3. If $\{1, 2, \ldots, m-2\} \subseteq s$, then for every $V \in V[x]$ there are sincere admissible strategies for V that will elect x;
4. If $s = \{k\}$ for some $k \in \{1, \ldots, m-1\}$, then there is a $V \in V[x]$ such that no combination of sincere admissible strategies will elect x, except when $k = 1$ and $m = 3$. In the latter case there is a $V \in V[x]$ such that no combination of nonempty sincere admissible strategies will elect x.

Parts 2, 3 and 4 of Theorem 6 in conjunction with Theorem 5.3 imply the following for runoff systems $\{1\}$, $\{2\}$ and $\{1, 2\}$ when X contains exactly three candidates:

- $\{1, 2\}$ (approval voting) can always elect Condorcet candidate x through nonempty sincere admissible strategies for $V \in V[x]$;
- $\{1\}$ (plurality voting) can always elect x through sincere admissible strategies for $V \in V[x]$, but it may be unable to elect x using nonempty sincere admissible strategies;
- $\{2\}$ (vote for exactly two) can always elect x through admissible

strategies for $V \in V[x]$, but some of these strategies may have to be insincere.

Although part 1 of Theorem 6 says that Condorcet candidate x can always get as many first-ballot votes as every other candidate when $V \in V[x]$ and runoff system s is used, it does not say that admissible strategies can always ensure the election of x. For example, if V has exactly one voter with linear order $xyzw$ on four candidates, and if there is no other voter and $s = \{3\}$, then either the voter abstains or he votes for three of the four candidates. In either case, more than two candidates are tied after the first ballot and, by previous assumption, each of these has a chance of being in the runoff. Hence x might not be on the runoff ballot.

All four parts of Theorem 6 in comparison with their counterparts in Theorem 5 reflect the greater diversity of admissible strategies for runoff system s than for ordinary system s. For example, the '$1 \in s$' hypothesis of Theorem 5.2 is replaced in Theorem 6.2 by '$1 \in s$ or $2 \in s$'. And Theorem 6.3 notes that runoff system $\{1, 2, \ldots, m-2\}$ as well as the runoff approval voting system can always elect the strict Condorcet candidate x with the use of sincere admissible strategies.

Because the plurality runoff system is so widely used, it is appropriate to note from Theorem 6.4 that, when $m \geqslant 4$, there may fail to exist sincere admissible plurality voting strategies for $V \in V[x]$ that will elect x. On the other hand, Theorem 5.3 notes that ordinary approval voting always has sincere (and nonempty) admissible strategies for $V \in V[x]$ that will elect x.

We should also point out that if the general hypothesis of Theorem 6, which says that a voter votes on the runoff ballot if and only if he is not indifferent between the candidates on that ballot, is not assumed to hold, then the conclusions of the theorem remain true when 'elect x' is replaced by 'ensure that x is in the runoff' in parts 2 and 3, and 'elect x' is replaced by 'give x any chance of being in the runoff' in part 4. Indeed, our later proofs of Theorem 6 only consider whether x can, must or cannot be in the runoff. Effects of abstentions on the chances of the majority candidate on a runoff ballot being elected are examined in [4].

Along with Theorems 5 and 6 one might also consider abilities of voting systems to elect 'inferior' candidates. Because runoff systems allow a great diversity of admissible strategies, runoff ballots may contain candidates who almost surely would not be elected by an ordinary system such as approval voting. On the other hand, runoff systems guard against the election of a 'worst' candidate since such a candidate would be beaten in the runoff if he got on the runoff ballot.

5. Election guarantees

We shall conclude our technical analysis by considering the possibility that candidate x must win an election when all voters use admissible stategies. To simplify matters, only four systems will be analyzed, namely ordinary and runoff plurality voting and ordinary and runoff approval voting.

We begin by illustrating our analysis in this section with $X = \{x, y, z\}$ and the following four-voter profile V:

2 voters are indifferent between x and y and prefer x and y to z
1 voter prefers x to z to y
1 voter prefers z to x to y.

Under ordinary approval voting, the only admissible strategy for the first two voters is $\{x, y\}$. The other two voters have two admissible strategies apiece: vote for their most-preferred, or two most-preferred, candidates. Whichever strategies the latter voters use, x wins the election. Hence, ordinary approval voting must elect x. On the other hand, none of the other three systems noted above can guarantee the election of x when voters use admissible strategies.

Candidate x is a strict Condorcet candidate in the preceding example. This is not a coincidence since, as shown by the following theorem, x must be a Condorcet candidate when some system guarantees its election under the use of admissible strategies.

Theorem 7. Suppose that, regardless of which system is used, all voters use admissible strategies on the only or first ballot and, in the case of a runoff system, each voter votes on the runoff ballot if and only if he is not indifferent between the candidates on that ballot. Then:

1. There is no V for which x must be elected under runoff plurality voting;
2. There are V for which x must be elected under runoff approval voting if and only if $m = 3$; for any such V, x must also be elected under ordinary plurality voting;
3. For every $m \geqslant 3$, there are V for which x must be elected under ordinary plurality voting and, for any such V, x must also be elected under ordinary approval voting;
4. If x must be elected under ordinary approval voting, then x is a strict Condorcet candidate, i.e., $V \in V[x]$.

Theorem 7 yields the following chain of implications for any $m \geqslant 3$:

[x must be elected under runoff plurality voting]
$\Rightarrow$ [x must be elected under runoff approval voting]

$\Rightarrow$ [x must be elected under ordinary plurality voting]
$\Rightarrow$ [x must be elected under ordinary approval voting]
$\Rightarrow$ [x must be a strict Condorcet candidate] .

Since examples can be constructed to show that the converse implications are false for some m, the ability of a system to guarantee the election of a strict Condorcet candidate under the hypotheses of Theorem 7 is highest for ordinary approval voting, next highest for ordinary plurality voting, third highest for runoff approval voting, and lowest – in fact nonexistent – for runoff plurality voting.

6. Summary and conclusions

This paper extends our prior analyses [1, 3] of ordinary approval voting vis-à-vis other single-ballot nonranked voting systems – including ordinary plurality voting – and also brings runoff voting into the picture. We began by giving a complete characterization of admissible voting strategies for all ordinary and runoff systems. We then considered when various systems either can or must elect a strict Condorcet candidate.

In our earlier work on ordinary systems, we showed that approval voting is both more sincere and more strategyproof than other ordinary systems and argued that it is more likely to reflect voters' true preferences. We noted also that, when all voters have dichotomous preferences, ordinary approval voting is the only ordinary system that invariably elects a Condorcet candidate when all voters use admissible strategies.

In the present paper we have extended these results to cover cases in which there exists a strict Condorcet candidate x for a configuration of (not necessarily dichotomous) voter preference orders. In particular, we have proved that some ordinary systems may be unable to elect x regardless of which admissible strategies are used by the voters, that ordinary plurality can always elect x when voters use admissible strategies, and that approval voting is the only ordinary system that is always able to elect x when voters use sincere admissible strategies. Moreover, if candidate x must be elected under ordinary plurality with admissible strategies, then x must also be elected under ordinary approval voting with admissible strategies (but not conversely), and in this case x is a strict Condorcet candidate.

Although many questions about approval voting remain to be answered, the results obtained thus far strongly support its superiority over other single-ballot nonranked voting systems, including ordinary plurality. However, since runoff procedures are often used in multicandidate elections, ordinary approval voting must be compared also to runoff systems and, most especially, to the very popular plurality runoff system. Although our analysis has not yet produced unequivocal evidence on the merits of ordinary approval voting versus runoff plurality voting, we have obtained some interesting findings.

A main feature of runoff systems in comparison to ordinary systems is their wider array of admissible strategies. This is caused in part by the strategic objective of getting not only a favorite candidate into the runoff but of pairing this candidate with another that can be beaten by the favorite in a runoff. Because opportunities to strategically manipulate the runoff ballot appear to increase as voters are allowed more options on the first ballot, runoff plurality voting is probably better able than other runoff systems to minimize such opportunities. On the other hand, runoff approval voting is better able than runoff plurality voting to elect a strict Condorcet candidate x with sincere admissible strategies, and there are three-candidate cases in which x must be elected under admissible runoff approval voting but not under admissible runoff plurality voting.

In comparing runoff plurality voting (RPV) to ordinary approval voting (OAV) we have seen that, when there is a strict Condorcet candidate x, OAV can always elect x with sincere admissible strategies while RPV may be unable to elect x with sincere (and nonempty for $m = 3$) admissible strategies. Furthermore, OAV is better than RPV in being able to guarantee the election of a strict Condorcet candidate x when all voters use admissible strategies. In addition, RPV is more likely to encourage voters to vote insincerely, especially if their favorites have little chance of being in the runoff, and is usually more expensive to implement than is OAV.

Thus we feel that OAV has distinct advantages over the familiar RPV system. One potential redeeming feature of RPV is that its runoff ballot provision prevents the election of an obviously 'inferior' candidate. Although it seems highly unlikely that OAV would elect such a candidate, further analysis is needed before this issue can be resolved.

NOTES

1. A plurality runoff system often has a provision that does not require a second ballot when some candidate is supported by a sufficiently large proportion of the voters on the first ballot, in which case the candidate with the most first-ballot votes is elected. Such a system is a hybrid of an ordinary system and a runoff system as defined in this paper.
2. A modified runoff system as described in the preceding note may of course require only one ballot.

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APPENDIX

We shall assume throughout this Appendix that Axioms $P1, R1, P2$ and $R2$ hold.

Theorem 2: If P is concerned then $S\ dom_2\ T$ *for* P if and only if P is dichotomous, $S \neq T$, $S \setminus T \in \{\emptyset, M(P)\}$ and $T \setminus S \in \{\emptyset, L(P)\}$.

Proof: We shall assume that $S \neq T$ since otherwise the reflexivity of R prevents S from dominating T. In addition, since $S\ dom_2 T$ *for* P if and only if $S \setminus T\ dom_2\ T \setminus S$ *for* P, we shall suppose that $S \cap T = \emptyset$. (Then if $S\ dom_2 T$ *for* P and $A \subseteq X$ is disjoint from $S \cup T$, $S \cup A\ dom_2 T \cup A$ *for* P provided that neither $S \cup A$ nor $T \cup A$ is X.)

Thus, with $S \neq T$ and $S \cap T = \emptyset$, we are to prove that $S\ dom_2 T$ *for* P if and only if P is dichotomous, $S \in \{\emptyset, M(P)\}$ and $T \in \{\emptyset, L(P)\}$. In doing this we shall write $S(\alpha)$ to denote the runoff outcome that obtains when α is the runoff contingency and the focal voter uses strategy S on the first ballot. Then $S(\alpha)\ P\ T(\alpha)$ if and only if he prefers runoff outcome $S(\alpha)$ to runoff outcome $T(\alpha)$, and $S(\alpha)\ R\ T(\alpha)$ when he likes runoff outcome $S(\alpha)$ as much as runoff outcome $T(\alpha)$. When runoff outcome $S(\alpha)$ gives a chance of winning to every candidate in A and no other, we shall write $S(\alpha) \to A$. Thus $S(\alpha) \to A$ is similar in meaning to $\Gamma(>) = A$.

We begin with the following two hypotheses for T:

$H1$: there exist distinct x, y, z such that $xPy, y \notin T, z \in T$;
$H2$: there exist distinct x, y, z such that $xPy, x \in T, z \notin T$.

Candidates not mentioned in runoff contingencies α here and later are presumed to have no chance of being in the runoff. For $H1$ let α have x and y tied and one vote ahead of z, with $y > x$, $y > z$ and $x > z$. Then $S(\alpha) \to \{y\}$ and $T(\alpha) \to \{x, y\}$ so that $T(\alpha)\, P\, S(\alpha)$ by Axiom $P2$. For $H2$ let α have x and z tied and one vote behind y, with $x > y$, $x > z$ and $y > z$. Then $S(\alpha) \to \{x, y\}$ and $T(\alpha) \to \{x\}$ so that $T(\alpha)\, P\, S(\alpha)$ by Axiom $P2$.

Hence S cannot dominate T if either $H1$ or $H2$ holds. If both $H1$ and $H2$ are false then it is straightforward to show that either $T = \emptyset$ or P is dichotomous with $T = L(P)$.

We consider next two hypotheses for S:

$H3$: there exist distinct x, y, z such that $xPy, y \in S, z \notin S$;
$H4$: there exist distinct x, y, z such that $xPy, x \notin S, z \in S$.

Let α for $H3$ have x and z tied and one vote ahead of y, with $x > z$, $y > x$ and $y > z$. Then $S(\alpha) \to \{x, y\}$ and $T(\alpha) \to \{x\}$ so that $T(\alpha)\, P\, S(\alpha)$ by Axiom $P2$. Let α for $H4$ have x and y tied and one vote ahead of z, with $x > y$, $x > z$ and $y > z$. Then $S(\alpha) \to \{x, y\}$ and $T(\alpha) \to \{x\}$ so that $T(\alpha)\, P\, S(\alpha)$ by Axiom $P2$.

Hence S cannot dominate T if either $H3$ or $H4$ holds. If both $H3$ and $H4$ are false then either $S = \emptyset$ or P is dichotomous with $S = M(P)$. Therefore, since $S \neq T$, $S\ dom_2 T$ *for* P can only be true if P is dichotomous and (S, T) is $(\emptyset, L(P))$, $(M(P), \emptyset)$ or $(M(P), L(P))$. We now show that $S\ dom_2 T$ *for* P for each of these three cases. Since it is easily seen that $S(\alpha)\, P\, T(\alpha)$ for some α for each case, we need to prove that $S(\alpha)\, R\, T(\alpha)$ for all runoff contingencies α.

Suppose first that P is dichotomous and $(S, T) = (\emptyset, L(P))$. Then there are exactly ten distinct general ways that (A, B) in $S(\alpha) = (A, B, >)$ can differ from (A', B') in $T(\alpha) = (A', B', >)$. (If $(A, B) = (A', B')$ then $S(\alpha)\, R\, T(\alpha)$ by reflexivity.) These ten ways are shown in the first two columns of Table 1.

The reader can verify without undue difficulty that there are no other ways that $T(\alpha)$ and $S(\alpha)$ can differ. The next two columns of the table show what must be in $M(P)$ and what must be in $L(P)$ when the (A, B) and (A', B') patterns hold for each row. The final column then notes which alternative(s) in Axiom $R2$ apply to each row. For example, the first alternative in Axiom $R2$ applies to the second row since all candidates (namely x and y) for $S(\alpha)$ who could be on the runoff ballot are in $M(P)$. It then follows from Axiom $R2$ that $S(\alpha)\, R\, T(\alpha)$ for every row in the table. Since Table I exhausts all nonidentical possibilities for $S(\alpha)$ versus $T(\alpha)$ in the case at hand, it follows that $\emptyset\ dom_2 L(P)$ for P when P is dichotomous.

Table 1. The ten ways that $S(\alpha)$ can differ from $T(\alpha)$ when $S = \emptyset$ and $T = L(P)$ with P dichotomous

(A, B) for $S(\alpha)$ for $S = \emptyset$	(A', B') for $T(\alpha)$ for $T = L(P)$	Implications: $M(P)$	$L(P)$	Alternative in Axiom $R2$
$(\{x, y\}, \emptyset)$	$(\{x\}, \{y\} \cup B)$	$\{y\}$	$\{x\} \cup B$	third
$(\{x, y\}, \emptyset)$	$(\emptyset, \{x, y\} \cup B)$	$\{x, y\}$	B	first
$(\{x\}\ \{y\} \cup B)$	$(\{x, y\}, \emptyset)$	$\{x\} \cup B$	$\{y\}$	third
$(\{x\}, B)$	$(\{x\}, B \cup C)$	B	C	1st $(x \in M)$/ 3rd $(x \in L)$
$(\{x\}, B \cup C)$	$(\{x\}, B)$	C	B	2nd $(x \in L)$/ 3rd $(x \in M)$
$(\{x\}, B \cup C)$	$(\emptyset, \{x\} \cup B)$	$\{x\} \cup C$	B	third
$(\emptyset, \{x, y\} \cup B)$	$(\{x, y\}, \emptyset)$	B	$\{x, y\}$	second
$(\emptyset, \{x\} \cup B)$	$(\{x\}, B \cup C)$	B	$\{x\} \cup C$	third
$(\emptyset, B)$	$(\emptyset, B \cup C)$	B	C	first
$(\emptyset, B \cup C)$	$(\emptyset, B)$	C	B	second

Table 2. The ten ways that $S(\alpha)$ can differ from $T(\alpha)$ when $S = M(P)$ and $T = \emptyset$ with P dichotomous

(A', B') for $T(\alpha)$ for $T = \emptyset$	(A, B) for $S(\alpha)$ for $S = M(P)$	Implications: $M(P)$	$L(P)$	Alternative in Axiom $R2$
$(\{x, y\}, \emptyset)$	$(\{x\}, \{y\} \cup B)$	$\{x\} \cup B$	$\{y\}$	third
$(\{x, y\}, \emptyset)$	$(\emptyset, \{x, y\} \cup B)$	B	$\{x, y\}$	second
$(\{x\}, \{y\} \cup B)$	$(\{x, y\}, \emptyset)$	$\{y\}$	$\{x\} \cup B$	third
$(\{x\}, B)$	$(\{x\}, B \cup C)$	C	B	2nd $(x \in L)$/ 3rd $(x \in M)$
$(\{x\}, B \cup C)$	$(\{x\}, B)$	B	C	1st $(x \in M)$/ 3rd $(x \in L)$
$(\{x\}, B \cup C)$	$(\emptyset, \{x\} \cup B)$	B	$\{x\} \cup C$	third
$(\emptyset, \{x, y\} \cup B)$	$(\{x, y\}, \emptyset)$	$\{x, y\}$	B	first
$(\emptyset, \{x\} \cup B)$	$(\{x\}, B \cup C)$	$\{x\} \cup C$	B	third
$(\emptyset, B)$	$(\emptyset, B \cup C)$	C	B	second
$(\emptyset, B \cup C)$	$(\emptyset, B)$	B	C	first

Suppose next that P is dichotomous and $(S, T) = (M(P), \emptyset)$. The analysis in this case is similar to that just given: the details appear in Table 2 where the first two columns show (A', B') for $T(\alpha)$ and (A, B) for $S(\alpha)$ in that order to take advantage of a natural symmetry with Table 1. As before, Axiom $R2$ implies that $S(\alpha)\ R\ T(\alpha)$ for every row, so that $M(P)\ dom_2 \emptyset$ for P when P is dichotomous.

Finally, suppose P is dichotomous with $(S, T) = (M(P), L(P))$. For any generic runoff contingency α, let Γ_Z denote the set of potential runoff pairs when the focal voter uses strategy Z on the first ballot. We consider $Z \in \{M(P), \emptyset, L(P)\}$. The analysis with Table 2 shows that either

$$(1)\ \cup\,\Gamma_M \subseteq M(P)$$
or $$(2)\ \cup\,\Gamma_\emptyset \subseteq L(P)$$
or $$(3)\ \cup\,(\Gamma_M \setminus \Gamma_\emptyset) \subseteq M(P) \text{ and } \cup\,(\Gamma_\emptyset \setminus \Gamma_M) \subset L(P)\ ,$$

where $\cup\,\Gamma = \cup\{\{x, y\}: \{x, y\} \in \Gamma\}$. Similarly, the Table 1 analysis shows that either

$$(4)\ \cup\,\Gamma_\emptyset \subseteq M(P)$$
or $$(5)\ \cup\,\Gamma_L \subseteq L(P)$$
or $$(6)\ \cup\,(\Gamma_\emptyset \setminus \Gamma_L) \subseteq M(P) \text{ and } \cup\,(\Gamma_L \setminus \Gamma_\emptyset) \subseteq L(P)\ .$$

It then follows that either

$$(7)\ \cup\,\Gamma_M \subseteq M(P)$$
or $$(8)\ \cup\,\Gamma_L \subseteq L(P)$$
or $$(9)\ \cup\,(\Gamma_M \setminus \Gamma_L) \subseteq M(P) \text{ and } \cup\,(\Gamma_L \setminus \Gamma_M) \subseteq L(P)\ .$$

Since (1) implies (7) and (5) implies (8), suppose neither (1) nor (5) is true. Then, if (2) holds, (6) must hold, and in this case (2) and (6) imply that $\cup\,\Gamma_L \subseteq L(P)$, which is (5); likewise, if (4) holds then (3) holds and therefore $\cup\,\Gamma_M \subseteq M(P)$, which is (1). Hence if (1) and (5) fail then (3) and (6) must hold. Since it is true in general that $A \setminus B \subseteq (A \setminus C) \cup (C \setminus B)$, it follows from (3), (6), $\Gamma_M \setminus \Gamma_L \subseteq (\Gamma_M \setminus \Gamma_\emptyset) \cup (\Gamma_\emptyset \setminus \Gamma_L)$ and $\Gamma_L \setminus \Gamma_M \subseteq (\Gamma_L \setminus \Gamma_\emptyset) \cup (\Gamma_\emptyset \setminus \Gamma_M)$ that $\cup\,(\Gamma_M \setminus \Gamma_L) \subseteq M(P)$ and $\cup\,(\Gamma_L \setminus \Gamma_M) \subseteq L(P)$, which is (9).

Therefore, since (7), (8) or (9) holds when P is dichotomous and $(S, T) = (M(P), L(P))$, Axiom $R2$ implies that $S(\alpha)\ R\ T(\alpha)$ for all runoff contingencies α. Q.E.D.

Theorem 4: S is admissible for runoff system s and P if and only if S is feasible for s and either:

$C3$: P is not dichotomous; or

$C4$: P is dichotomous, not $\{L(P) \subseteq S$ and $|S \setminus L(P)| \in s \cup \{0\}\}$ and not $\{S \subset L(P)$ and $|M(P) \cup S| \in s\}$.

Proof: The first part of Theorem 4 is obvious from Theorem 2. Suppose then that S is feasible for s and P is dichotomous. Then, by Theorem 2, S is not admissible for s and P if and only if there is a $T \neq S$ that is feasible for s such that $(T \setminus S, S \setminus T)$ is either $(\emptyset, L(P))$, $(M(P), \emptyset)$, or $(M(P), L(P))$. The latter three possibilities for $(T \setminus S, S \setminus T)$ amount respectively to:

$$\begin{aligned} T \cap L(P) &= \emptyset \text{ and } & S &= L(P) \cup T\,; \\ S \cap M(P) &= \emptyset \text{ and } & T &= M(P) \cup S\,; \\ T &= M(P) \text{ and } & S &= L(P). \end{aligned}$$

We can ignore the last of these since if $S = L(P)$ then $T = \emptyset$ in the first expression shows that a feasible strategy, namely $\emptyset$, dominates S. The two bracketed expressions in $C4$ then follow from the first and second expressions just written. Q.E.D.

Theorem 5.1: *For each $m \geqslant 3$* there is an ordinary s and a $V \in V[x]$ such that admissible strategies for V will never elect x.

Proof: For $m = 3$ see the first example in Section 4. For $m \geqslant 4$ let $s = \{m-1\}$ with V a three-voter profile in which two voters prefer x to a_1 to a_2 to ... to a_{m-1} and the other is indifferent among a_1 through a_{m-1} and prefers these to x. When all voters use admissible strategies, Theorem 3 implies that x gets two votes and at least one of a_1 through a_{m-1} gets three votes. Q.E.D.

Theorem 5.2: If $1 \in s$ and $V \in V[x]$ then there are admissible strategies for V that will elect x when ordinary s is used.

Proof: Given $1 \in s$ and $V \in V[x]$ let P be a generic order in V. If $P = \emptyset$, assign admissible strategy $\emptyset$ or admissible strategy $\{x\}$ to P. Assume henceforth in this paragraph that $P \neq \emptyset$. If $x \in M(P)$ let $k_1 = \max\{k\colon k \in s$ and $k \leqslant |M(P)|\}$, and assign S to P for which $|S| = k_1$, $x \in S$ and $S \subseteq M(P)$. This S is admissible by Theorem 3. Next, if $x \notin M(P) \cup L(P)$ let $k_2 = \max\{k\colon k \in s$ and $k \leqslant |M(P)| + 1\}$, and assign S to P for which $|S| = k_2$, $x \in S$ and $S \setminus \{x\} \subseteq M(P)$. Again, by Theorem 3, S is admissible. Finally, if $x \in L(P)$, assign an admissible $S \subseteq M(P)$ – which exists by $1 \in s$ and Theorem 3 – to P.

Consider any $y \neq x$. By the preceding assignment, all voters with xPy vote for x and not y, and all voters with xIy vote for x if they vote for y. Since more voters have xPy than yPx, x gets more votes than y. Since this is true for every $y \neq x$, x wins the election. Q.E.D.

Theorem 5.3: If s is the ordinary approval voting system and $V \in V[x]$ then there are sincere admissible strategies for V that will elect x.

Proof: Given $V \in V[x]$ assign to each $P \neq \emptyset$ in V the following sincere admissible ordinary approval voting strategy: if $x \in M(P) \cup L(P)$ let $S = M(P)$; if $x \notin M(P) \cup L(P)$ let S contain x and all candidates strictly preferred to x. If $P = \emptyset$ let $S = \emptyset$. Then, as in the second paragraph of the preceding proof, x wins the election. Q.E.D.

Theorem 5.4: If s is an ordinary system different from the approval voting system then there exists $V \in V[x]$ such that no sincere admissible strategies for V will elect x.

Proof: Given that ordinary s is not the approval voting system let k be the smallest integer in $\{1, \ldots, m-1\}$ that is not in s. We consider two cases.

Case 1: $k = m - 1$, so $s = \{1, 2, \ldots, m-2\}$. Fix x and $y \neq x$, and let V be formed with $4m-7$ voters all of whom have linear orders $(K = m)$ as follows:

> 2 voters have $\ldots xy$ (x in next-to-last place; y in last place);
> 2 voters have $y \ldots xz$ for each $z \in X \setminus \{x, y\}$;
> $2m-5$ voters have x in first place.

Then, when all voters use sincere admissible strategies, the first two voters vote for neither x nor y, the next $2m-4$ (*re* z) vote for y but not x, and the last $2m-5$ vote for x and maybe for y. Hence x gets $2m-5$ votes and y gets at least $2m-4$, so y beats x. Now for each $w \in X \setminus \{x\}$, $2m - 5 + 2 = 2m - 3$ voters prefer x to w, and $2m - 4$ prefer w to x. Hence $V \in V[x]$ and x cannot win the election when all voters use sincere admissible strategies.

Case 2: $k < m-1$. Let V consist of $4m-9$ voters with the following linear orders:

> 1 voter has $y \ldots x$ (y first, x last);
> 2 voters have $\ldots xyz \ldots$ for each $z \in X \setminus \{x, y\}$ where x is in the kth place in each of these orders;
> $2m-6$ voters have $yx \ldots$

Then $V \in V[x]$ since x beats y by $2m - 4$ to $2m - 5$, and x beats $z \in X \setminus \{x, y\}$ by at least $2m - 4$ to $2m - 5$. However, y has more votes than x under system s when all voters use sincere admissible strategies since then the voter with $y \ldots x$ votes for y and not x, each voter with $\ldots xyz \ldots$

must vote for y if he votes for x (since $k \notin s$), and all voters with $yx \ldots$ vote for y if they vote for x. Q.E.D.

Theorem 6.1: If s is a runoff system and $V \in V[x]$ then there are admissible strategies for V that give x as many first-ballot votes as every other candidate.

Proof: Given runoff system s and $V \in V[x]$, we shall assume that s is not the approval voting system since otherwise the conclusion of Theorem 6.1 follows from Theorem 6.3, which will be proved shortly. We prove here that every voter in V either has an admissible strategy that contains x or else has $\emptyset$ as an admissible strategy. The use of such strategies then gives x as many first-ballot votes as every other candidate.

Suppose x is in no admissible strategy for s and P. Then, by Theorem 4, P must be dichotomous. If $x \notin L(P)$ then there will be a feasible S that contains x and has $L(P) \subseteq S$ and $S \subset L(P)$, in which case S is admissible by $C4$. Hence we require $x \in L(P)$. If $\{x\} \subset L(P)$ let S be such that $|S| =$ max s with $x \in S$ and with something from $M(P)$ in S along with $L(P) \subseteq S$ if $|S| > 1$. Since such an S is admissible by $C4$, we require $L(P) = \{x\}$. Given dichotomous P with $L(P) = \{x\}$, $C4$ implies that no admissible S contains x if and only if $s = \{1, 2, \ldots, k\}$ for some k. Suppose this is so. Then $k < m-1$ since s is not the approval voting system and, since $|M(P)| = m-1$, it follows from $C4$ that $\emptyset$ is admissible. Q.E.D.

Theorem 6.2: If runoff system s contains either 1 or 2 and $V \in V[x]$ then there exist admissible strategies for V that will elect x.

Proof: Given $s \cap \{1, 2\} \neq \emptyset$ and $V \in V[x]$ we are to prove that some admissible strategies for V ensure that x is in the runoff. This follows immediately from Theorem 5.2 if $1 \in s$ since any strategy that is admissible for ordinary s must also be admissible for runoff s. Suppose henceforth that $1 \notin s$ and $2 \in s$. The preceding proof, modified slightly, then shows that some S with $x \in S$ and $|S| = 2$ is admissible for s and P regardless of the nature of P. Consequently, when all voters use such strategies, x gets a vote from every voter and, since $|S| = 2$ for all voters, at most one other candidate can get a vote from every voter. This ensures that x will be in the runoff. Q.E.D.

Theorem 6.3: If s is the approval voting runoff system and $V \in V[x]$ then there exist sincere admissible strategies for V that will elect x. In addition, system $s = \{1, 2, \ldots, m-2\}$ also has this capability.

Proof: The first part of Theorem 6.3 follows immediately from Theorem 5.3. For the second part let $s = \{1, 2, \ldots, m-2\}$. If $P = \emptyset$ set $S = \{x\}$ or $S =$

$\emptyset$. Suppose then that $P \neq \emptyset$, and let $X = \{x, a_1, \ldots, a_{m-1}\}$. If $x \in M(P)$ let $S = \{x\}$, and if $x \notin M(P) \cup L(P)$ let S consist of x plus all candidates strictly preferred to x unless such an S has $|S| = m - 1$, in which case P must be nondichotomous and end in xa_j for some $a_j \in X$. In this latter case let $S = \emptyset$, which is sincere and admissible by Theorem 4. Then for each a_j, a voter votes for x and not a_j when xPa_j unless P is of the form $\ldots xa_j$ with nothing else indifferent to either x or a_j. Let n_j be the number of voters who have a P order of the form $\ldots xa_j$, for $j = 1, \ldots, m-1$, and let J denote the set of all such voters, with $|J| = n_1 + \ldots + n_{m-1}$. Within J, $\Sigma_{k \neq j} n_k$ voters prefer a_j to x and n_j prefer x to a_j. Since all voters in J abstain on the first ballot, no candidate in $\{a_1, \ldots, a_{m-1}\}$ except perhaps one such candidate can gain relative to x within J since at most one j can have $n_j > \Sigma_{k \neq j} n_k$. Consequently, since a voter not in J who has sPa_j votes for x and not a_j, there can be at most one a_j who has as many votes as x on the first ballot provided that it is never true that a voter votes for a_j and not x whenever a_jIx. The strategies defined above along with $S = \{a_j\}$ for some $a_j \in M(P)$ when $x \in L(P)$ and $P \neq \emptyset$ ensure the latter provision. Therefore, when $s = \{1, 2, \ldots, m-2\}$ and $V \in V[x]$, there are sincere admissible strategies for all voters which ensure that at most one other candidate has as many votes as x on the first ballot, so that x will be in the runoff. Q.E.D.

Theorem 6.4: If $|s| = 1$ for runoff system s then there exists a $V \in V[x]$ such that no combination of sincere admissible strategies will elect x, except when $s = \{1\}$ and $m = 3$, in which case there is a $V \in V[x]$ such that no combination of nonempty sincere admissible strategies will elect x.

Proof: The exception for $s = \{1\}$ and $m = 3$ is required by Theorem 6.3 for $m = 3$, and the final statement in Theorem 6.4 is proved by the second example in section 4. For the first part of the theorem, suppose first that $s = \{1\}$ with $m \geqslant 4$ and with a, b and c three distinct candidates in $X \setminus \{x\}$. Let V be a nine-voter profile as follows:

1 voter is dichotomous and prefers a to all in $X \setminus \{a\}$
1 voter is dichotomous and prefers b to all in $X \setminus \{b\}$
2 voters have $axbc \ldots$
2 voters have $bxac \ldots$
3 voters have $cxab \ldots$

where the final seven votes have linear preference orders. Clearly $V \in V[x]$. By Theorem 4, the first voter's only sincere admissible strategy is $\{a\}$ since $\emptyset$ is ruled out by the latter part of $C4$, and the second voter's only sincere admissible strategy is $\{b\}$. Moreover, $\{x\}$ is not a sincere strategy for any voter. Therefore, if all voters use sincere admissible strategies, x gets no votes and a and b each get at least one vote so that x will not be in the runoff.

Suppose next that $s = \{k\}$ with $2 \leqslant k < m$ for any $m \geqslant 3$. Let $X = \{x, a_1, \ldots, a_{m-1}\}$ and form V with seven voters as follows:

3 voters are dichotomous with $M(P) = \{a_1, \ldots, a_k\}$
2 voters have linear orders with $xa_1a_2 \ldots a_{m-1}$
2 voters have linear orders with $xa_2a_1 \ldots a_{m-1}$.

Then $V \in V[x]$ and, by Theorem 4, $\{a_1, \ldots, a_k\}$ is the only sincere admissible strategy for the first three voters. Hence, regardless of which of the final four voters abstain instead of voting for their k most-preferred candidates, a_1 and a_2 will get more first-ballot votes than x. Q.E.D.

Theorem 7.1: There is no V for which x must be elected under runoff plurality voting when all voters use admissible strategies.

Proof: The discussion of Theorem 4 in section 3 shows that $\{x\}$ is never the unique admissible runoff plurality voting strategy. Hence, regardless of V, x can get no votes on the first ballot and hence is never guaranteed of being in the runoff. Q.E.D.

Theorem 7.2: There are V for which x must be elected under runoff approval voting when voters use admissible strategies if and only if $m = 3$; and, for any such V, x must also be elected under ordinary plurality voting when voters use admissible strategies.

Proof: By Theorem 4, the only time a voter who uses an admissible strategy under runoff approval voting must vote for x occurs when P is dichotomous and $M(P) = \{x\}$. Let $X = \{x, a_1, \ldots, a_{m-1}\}$. Then, if $m \geqslant 4$, we can assign admissible strategy $S = \{x, a_1, \ldots, a_{m-2}\}$ to all dichotomous voters who have $M(P) = \{x\}$, and all other voters can be assigned admissible strategies that do not contain x. Then, under this assignment, x is not assured of being in the runoff.

Suppose henceforth in this proof that $m = 3$ with $X = \{x, y, z\}$. Then all dichotomous voters with $M(P) = \{x\}$ must vote for x and may vote for either y or z. Since we want to make it as difficult as possible for x to be assured of being in the runoff, assume that all such voters do in fact vote for either y or z as well as for x and let

$n_1(y)$ = number of dichotomous voters with $M(P) = \{x\}$ who vote for x and y,
$n_1(z)$ = number of dichotomous voters with $M(P) = \{x\}$ who vote for x and z.

We shall assume also that nobody else votes for x, and that they vote for both y and z if $\{y, z\}$ is admissible under runoff approval voting. Then, with

$n_2 =$ number of dichotomous voters with $M(P) = \{x, y\}$;
$n_3 =$ number of dichotomous voters with $M(P) = \{x, z\}$;

the n_2 voters vote only for y (since $\{y, z\}$ is not admissible for them), the n_3 voters vote only for z, and all remaining voters vote for both y and z since, by Theorem 4, $\{y, z\}$ is admissible except for the P noted above. Finally, let $n_1 = n_1(y) + n_1(z)$, let n_4 be the number of voters who vote for both y and z, and let $n = n_1 + n_2 + n_3 + n_4$, the total number of voters.

Then, under this worst-case-for-x construction,

x gets n_1 votes
y gets $n_1(y) + n_2 + n_4$ votes
z gets $n_1(z) + n_3 + n_4$ votes

on the first ballot. Hence, x is assured of being in the runoff iff and only if $n_1 > n_1(y) + n_2 + n_4$ or $n_1 > n_1(z) + n_3 + n_4$ for every $n_1(y) = n_1 - n_1(z)$ from zero to n_1. Suppose x is *not* assured of being in the runoff. Then addition of the two converse inequalities gives $n_2 + n_3 + 2n_4 \geqslant n_1$, or $2n \geqslant 3n_1 + n_2 + n_3$. Moreover, if $2n \geqslant 3n_1 + n_2 + n_3$, then n_1 can be divided into $n_1(y)$ and $n_1(z)$ [e.g., $n_1(y) = 2n_1 + n_3 - n$ and $n_1(z) = n - n_1 - n_3$] such that $n_1(y) + n_2 + n_4 \geqslant n_1$ and $n_1(z) + n_3 + n_4 \geqslant n_1$. Therefore x is assured of being in the runoff if and only if

$$3n_1 + n_2 + n_3 > 2n \ .$$

Since this inequality can be satisfied for feasible n_j, we have proved the first part of Theorem 7.2. In addition, y can beat or tie x when the ordinary plurality system is used only if $n \geqslant 2n_1 + n_3$ since in this case y can have no more than $n_2 + n_4 = n - n_1 - n_3$ votes while x must have at least n_1 votes. Since $3n_1 + n_2 + n_3 \geqslant 2n$ implies not $(n \geqslant 2n_1 + n_3)$, and since a similar implication holds for x versus z, ordinary plurality must elect x when runoff approval voting must elect x. Q.E.D.

Theorem 7.3: For every $m \geqslant 3$ there exist V for which x must be elected under ordinary plurality voting when voters use admissible strategies, and, for all such V, x must be elected under ordinary approval voting when voters use admissible strategies.

Proof: Theorem 3 implies that a voter must vote for x under ordinary plurality when he uses an admissible strategy if, and only if, P is dichotomous and $M(P) = \{x\}$. Let n_1 be the number of the total of n voters who have this dichotomous order, and for each $y \in X \setminus \{x\}$ let $n(y)$ be the number of the remaining $n - n_1$ voters who either have $P = \emptyset$ or have $P \neq \emptyset$ and do not have y in their $L(P)$. Since Theorem 3 for approval voting implies that y will not get more than $n(y)$ votes, i.e., that $\{y\}$ is admissible for a concerned voter if and only if $y \notin L(P)$, x must win under approval voting if and only if $n_1 > n(y)$ for all $y \in X \setminus \{x\}$. But in this case x must also win under ordinary approval voting since then $\{x\}$ is the only admissible strategy for the n_1 voters, and concerned voters with $y \in L(P)$ do not have an admissible approval voting strategy that contains y. Clearly, $n_1 > n(y)$ for all $y \in X \setminus \{x\}$ can hold for any $m \geqslant 3$, and the theorem is proved. Q.E.D.

Theorem 7.4: If x must be elected under ordinary approval voting when all voters use admissible strategies, then x is a strict Condorcet candidate.

Proof: For any V, the worst-case situation for x arises under admissible ordinary approval voting when voters vote as follows:

if $x \in M(P) \cup L(P)$, vote for all candidates not in $L(P)$;
if $x \notin M(P) \cup L(P)$, vote for all candidates not in $\{x\} \cup L(P)$.

Consequently, if x must win under admissible ordinary approval voting, then more voters must prefer x to y than prefer y to x for each $y \in X \setminus \{x\}$. This is because the foregoing assignment of strategies implies that voters with yPx always vote for y and not x, and voters with yIx always vote for y if they vote for x. Q.E.D.