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ON THE ONASSIS PROBLEM

ABSTRACT. A decision problem is by convention characterized by its outcome matrix and by a subsequent utility evaluation. In trying to set up an outcome matrix based on wealth values it may occur that due to ambiguities inherent in the decision problem it is not clear which standard of value (or numéraire) should be used in order to measure wealth. A typical example of this kind is Stützel's so called Onassis Paradox. We show that problems of this kind can be solved within the conventional framework of decision theory. The analysis proceeds in two steps. First, state-dependent utility functions are derived; second, a model for evaluating these utility functions is presented.

I. OUTCOME MATRIX AND BERNOULLI-CRITERION

Decision theory deals with the problem of optimal decision under uncertainty as to the *outcomes* of different *actions* (or strategies). The outcome of a certain action depends on exterior circumstances, which are beyond the influence of the decision maker, or, in terms of decision theory, on the *state of the world* that happens to be true, but is unknown to the decision maker. If the possible states of the world are $s_1, s_2, \ldots, s_m$ and can occur with probabilities $p_1, p_2, \ldots, p_m$ ($\sum p_i = 1$) and if the decision maker can choose between the actions $a_1, a_2, \ldots, a_n$ then his situation can be defined by means of the following outcome matrix:

$$
\begin{array}{c|lll}
 & p_1 & p_2 & \ldots p_m \\
\hline
 & s_1 & s_2 & \ldots s_m \\
\hline
a_1 & e_{11} & e_{12} & \ldots e_{1m} \\
a_2 & e_{21} & e_{22} & \ldots e_{2m} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
a_n & e_{n1} & e_{n2} & \ldots e_{nm}
\end{array}
$$

$e_{ij}(i=1,2, \ldots, n; j=1,2, \ldots, m)$ denotes the outcome which is safe to be obtained in case action a_i is chosen and state s_j is realized (e_{ij} thus is associated with the constellation (a_i, s_j)). We assume that the probabilities $p_1, \ldots, p_m$ are known to the decision maker.

In this case the *Bernoulli-criterion* (expected utility criterion) represents

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the most prominent decision criterion. It is based on extremely plausible and widely accepted axioms of rational behaviour. Applying this criterion we can find the optical course of action in the following way:

At first we determine the *utility function* u assigning (real) utility values $u(e_{ij})$ to the outcomes e_{ij}. Then we choose the optimal action, i.e. that action for which the expected value of the utility of all possible outcomes is maximized. Thus the criterion can be formulated as follows:

$$(1) \qquad \sum_{j=1}^{m} u(e_{ij})\, p_j \to \underset{i}{\text{Max}}!$$

$u(e_{ij})$ is determined in the following way[1]: From all possible outcomes we choose one most favourable and one most unfavourable result $\bar{e}$ and $\bar{\bar{e}}$, so that consequently all other results e_{ij} range between $\bar{e}$ and $\bar{\bar{e}}$ as far as the preference of the decision maker is concerned. If the decision maker (though perhaps only hypothetically) is offered a choice between the result e_{ij} and a lottery-ticket promising him the result $\bar{e}$ with a certain probability P and the result $\bar{\bar{e}}$ with probability $1-P$; in this case the utility value $u\ (e_{ij})$ is determined as that probability P for which the decision maker is indifferent whether he will choose e_{ij} or the lottery-ticket.

The Bernoulli-criterion is often applied in economic decision models in such a manner that'outcome' is defined as the monetary wealth available at the end of the planning period. If this '*terminal wealth*' depending on the chosen action a_i and the current state s_j is designated as $w(a_i, s_j)$ the Bernoulli-criterion adopts a more special form:

$$(2) \qquad \sum_{j=1}^{m} u(w(a_i, s_j))\, p_j \to \underset{i}{\text{Max}}!$$

We cannot emphasize often enough that (2) represents a quite restrictive specialization of (1). Thus it is near at hand to suspect that the general applicability of (2) is limited, even if on the other hand it is common practice to use criterion (2), i.e. to define terminal monetary wealth as the only relevant consequence of the chosen action[2]. Is this practice however justified? It seems in fact to be particularly problematic if different standards of value can be applied to the valuation of 'terminal wealth' and *a priori* none is distinguished before the others. In those cases even the categories 'certain' and 'uncertain' may dependend on the choice of the standard of value (the numéraire).

II. ON THE CHOICE OF THE NUMÉRAIRE

Stützel (1970)[3] has proposed a fine example, known as Onassis problem, which clearly elucidates this fact.

Onassis wants to invest £ 1.000 for one year at a fixed interest rate. He takes into consideration two possibilities:

a_1: to invest the money in sterling-loans bearing 8%;

a_2: as a_1; but simultaneously to sell £ 1.080 forward (in exchange of DM) at a discount of 2%;

a_2 is equivalent with a one year deposit in marks at a rate of 5.84%.

Onassis expects that with a certain probability p the rate of exchange will be maintained (state s_1), that on the other hand with a probability $1 - p$ the pound will be devaluated this year by 20% as compared with the mark (state s_2).

Now, according as Onassis expresses the sum of money being involved in the transactions in terms of marks or in terms of sterling, he will compute extremely different gains (in %)[4] (Table I).

TABLE I

		p	$1-p$
		s_1	s_2
a_1	DM	+8	−13.6
	£	+8	+8
a_2	DM	+5.84	+5.84
	£	+5.84	+32.3

If we consider the valuation in marks a_2 seems to be certain and a_1 risky. Just the opposite applies for the valuation in sterling. Even if a subsequent utility valuation of the DM- or £-percentage is carried through nothing would be changed about this. In the case of valuation in DM (resp. £-valuation) we would obtain:

DM	s_1	s_2
a_1	$u(8)$	$u(-13.6)$
a_2	$u(5.84)$	$u(5.84)$

£	s_1	s_2
a_1	$v(8)$	$v(8)$
a_2	$v(5.84)$	$v(32.3)$

As a precaution we have employed two different utility functions u and v. Again it is obvious that in the case of the DM-valuation a_2 is certain and a_1 uncertain, in the case of £-valuation a_1 is certain and a_2 uncertain.

Now, one might object that applying the Bernoulli-criterion (2) it is of no importance whether an action appears certain or uncertain, since only the expected value of utility is relevant which in any case is one single figure. One might continue to argue that the Bernoulli-criterion (2) will always indicate the right decision if only an adequate utility function is chosen – no matter if DM or £ is employed as a standard of value. In the appendix however we shall prove that this objection is not valid.

With good reason W. Stützel points out that the Onassis Problem in no case represents just an esoteric special case in decision theory. Numerous practical decision problems show a quite similar structure. Let us now introduce another example, which is – though very much simplified – an every-day problem (consumer problem):

Suppose the actions a_1 and a_2 are given. At the end of the planning period action a_1 will yield a monetary wealth amounting to DM 1.000 with certainty. If the decision maker chooses action a_2 then at this point of time he will have a monetary wealth of DM 800 in the presence of the state of the world s_1 and of DM 1.200 in the presence of s_2. It is assumed that $p_1 = p_2 = \frac{1}{2}$. If state s_1 occurs one unit of a certain bundle of consumer goods has a price of DM 8 at the end of the planning period, if however state s_2 occurs the price equals DM 12. Expressing the terminal wealth in terms of marks action a_1 seems to be certain and action a_2 uncertain. If however the terminal wealth is expressed in units of the bundle of consumer goods we obtain the opposite result: action a_1 appears to be

TABLE II

		s_1	s_2
price of bundle of consumer goods		8 DM/UC	12 DM/UC
a_1	DM	1000	1000
	UC	125	83.33
a_2	DM	800	1200
	UC	100	100

uncertain and action a_2 certain, which is shown in Table II. In this table the terminal wealth is once measured in marks, once in units of the consumer goods bundle (UC).

III. STATE-DEPENDENT UTILITY FUNCTIONS

Which standard of value should now be used as a numéraire in the above cases? There is but one answer to this question: in general it is not appropriate in problems of this kind to confine oneself on but one standard of value. Both standards – or in other problems even more than two standards of value – are necessary in order to charaterize the outcomes of the decision problem. Consequently criterion (2) can no more be applied unless it is modified in an appropriate way.

Let us return to the basic pattern of a decision problem, namely the outcome matrix. In this connection it nowhere is mentioned that the outcomes e_{ij} should be given numerically and that they should represent assets or incomes measured in some way or other. On the contrary, e_{ij} is to be regarded as a detailed description of *all* aspects being a result of action a_i given the state of the world s_j. In practice it is of course not necessary (and mostly not even possible) to enumerate virtually all details but only those *relevant* to the decision maker. By ignoring relevant details however even the introduction of a utility function can no longer guarantee that right decisions are taken.

In the examples discussed above the outcomes can sufficiently be described by two numbers each; that is in the Onassis problem by the per cent gains, once expressed in terms of marks, once in sterling, and in the consumer problem by the terminal wealth, once expressed in marks, once in units of the bundle of consumer goods, just as shown above in Table I and Table II.

The outcomes can as well be expressed by one profit or wealth figure each, if this figure is supplemented by a number charaterizing the state of world i.e. the rate of devaluation in the Onassis problem resp. the price of the bundle of consumer goods in the consumer problem. Thus we would have: (see Table I′, II′).

In either way of representation it is obvious that both actions, a_1 and a_2, will yield uncertain results. It now no longer depends on an arbitrary numéraire whether an action may be regarded as certain or uncertain. If

TABLE I′

Onassis problem

	s_1		s_2	
a_1	+8	0	+8	−20
a_2	+5.84	0	+32.3	−20

TABLE II′

Consumer problem

	s_1		s_2	
a_1	1000	8	1000	12
a_2	800	8	1200	12

this did not apply before, it can only be attributed to an incomplete description of the outcomes.

Generally, there seem to exist numerous decision problems the results of which can be expressed tentatively by a figure w_{ij} depending both on action and state of the world, yet, this figure has to be supplemented by a specification of the prevailing state of world. Thus:

$$(3) \qquad e_{ij} = (w_{ij}, s_j), \qquad w_{ij} = w(a_i, s_j).$$

w_{ij} represents the terminal wealth (or profit) measured in optional units, yielded as the result of action a_i in the presence of the state s_j. w may be interpreted as the wealth or profit function. By w_{ij} alone the result e_{ij} is not identified with sufficient precision, though. The additional description of the present state of the world s_j should be given, for this factor can exert great influence on the utility gained from the profit w_{ij}. In other words, two results showing the same profit can be valuated in very different manners, according to the existing state of the world.
The description of the state of the world s_j may of course be replaced by a figure characterizing s_j, as for example the rate of devaluation in the Onassis case or the price of the bundle of consumer goods in the consumer case.

Now the Bernoulli-criterion has the form:

$$(4) \qquad \sum_{j=1}^{m} u\,[w(a_i, s_j), s_j]\, p_j \to \underset{i}{\mathrm{Max}!}$$

This criterion obviously is more general than (2), but still it is a special case of (1). Another, yet completely equivalent formulation is possible:

$$(4') \qquad \sum_{j=1}^{m} u_j\,[w(a_i, s_j)]\, p_j \to \underset{i}{\mathrm{Max}!} \quad \text{with} \quad u_j(w) := u(w, s_j).$$

u_j can be viewed as a *state-dependent utility function*[5] of the terminal wealth w. Consequently for any state j we have to determine another utility function u_j. Basically this can be achieved in the same way as the evaluation of a single non-state-depending utility function even though one would have to invest a greater effort in order to do so. Using state-dependent utility functions we are able to measure the terminal wealth – as mentioned before – in units of any numéraire. Thus the problem of finding an adequate numéraire does not arise.

IV. OPPORTUNITIES FOR THE USE OF TERMINAL WEALTH

The problem of finding the utility function u_j can be discussed independently of the particular current decision problem, and as far as only the latter is under discussion u_j can be assumed as given. It is however possible to develop a model which more precisely analyzes the determination of state-dependent utility functions. It is the purpose of this paragraph to present such a model.

Evidently in the presence of the state s_j the utility valuation of the terminal wealth w depends on how the wealth w is intended to be used in future and which opportunities for doing so are available at the state s_j.

An *example*[6] will illustrate this: Suppose the decision maker is free to choose between two courses of action a_1 and a_2. The wealth available at the end of the planning period – in the following this point of time is denoted by t_1 – depends on which action is chosen and which state of the world does exist. Suppose only the states s_1 and s_2 are possible with probabilities $\frac{1}{2}$ each. If terminal wealth is measured in marks the following outcome matrix may arise (Table III).

According to criterion (2) the same expected value of utility $\frac{1}{2}\,u(100)+$ $+\frac{1}{2}\,u(200)$ is assigned to the two actions. They thus seem to be equivalent as far as this criterion is concerned. Still the decision maker needs not be indifferent with regard to the two actions, if for example he can choose

TABLE III

	$p_1=\frac{1}{2}$	$p_2=\frac{1}{2}$
	s_1	s_2
a_1	200 DM	100 DM
a_2	100 DM	200 DM

between the following opportunities of using his terminal wealth at time t_1:

The monetary wealth available at time t_1 is re-invested at an interest rate r up to the point of time t_2. At time t_0 when the decision maker has to decide whether to choose action a_1 or action a_2 the rate r is still unknown. With certainty r will equal 10% if state s_1 is prevalent. If however state s_2 is prevalent r will equal 10% only with a probability of 20%. With a probability of 80% the rate of interest will be lower, namely $r=5\%$.

By assumption the decision maker pursues the objective to obtain the greatest possible terminal wealth at time t_2, for example because he then wants to devote it to the purchase of consumer goods at prices which are already known.

If every amount of money shown in Table III is substituted by the terminal wealth realizable at time t_2, we arrive at the following outcome matrix (Table IV).

TABLE IV

	$p_1=\frac{1}{2}$	$p_2=\frac{1}{2}$
	s_1	s_2
a_1	220	105 or 110 with probability 0.8 or 0.2 resp.
a_2	110	210 or 220 with probability 0.8 or 0.2 resp.

Table IV replaces the outcome matrix of Table III, which has to be dismissed as incorrect as it does not consider all aspects relevant to the decision maker (most specifically: it does not take into account his state-dependent possibilities of future use of his wealth in time t_1). Note that not all outcomes in Table IV can be expressed by one figure only; e_{12} and e_{22} represent probability distributions over assets, and not simply assets. Even if we use another numéraire to measure wealth (as for example a unit of a bundle of consumer goods at time t_2), nothing will be changed about this unless we take utility units as a numéraire.

In this case Table IV is changed into Table V.

TABLE V

	$\frac{1}{2}$	$\frac{1}{2}$
	s_1	s_2
a_1	$u(220)$	$0.8\,u(105)+0.2\,u(110)$
a_2	$u(110)$	$0.8\,u(210)+0.2\,u(220)$

Applying a linear utility function we obtain:

TABLE V′

	$\frac{1}{2}$	$\frac{1}{2}$
	s_1	s_2
a_1	220	106
a_2	110	212

Now a_1 evidently is preferred to a_2, while from Table III one may get the impression that a_1 and a_2 are equivalent. The reason for this divergent judgment can be attributed to the fact that at date t_1 exactly the same amount of wealth has to be assessed in different ways depending on the state of the world which happens to prevail at t_1 and consequently on the opportunities for the use of money which then are open. Obviously it is preferable at time t_1 to possess DM 200.– in the presence of s_1 than to possess the same sum (DM 200.–) in the presence of s_2.

Before we are going to generalize this example we should like to mention another aquivalent way of presentation, which should make clear that it can be incorporated in the decision model outlined in the previous paragraph: The outcome matric as shown in Table IV can also be expressed this way (Table VI).

TABLE VI

	$\frac{1}{2}$	$\frac{1}{2}$
	s_1	s_2
a_1	200(r=10)	100(r=5 or r=10 with probability 0.8 or 0.2 resp.)
a_2	100(r=10)	200(r=5 or r=10 with probability 0.8 or 0.2 resp.)

The first figures represent the amount of wealth at time t_1, while in brackets one finds a description of the state-dependent opportunities for future use of this amount. It is this description which stands for the specification of the state itself as it was employed in our earlier analysis, especially in formula (3). It thus can be seen that our example is just a special case of the model considered in the last paragraph.

To *generalize* this example we assume that at the end of his first planning period, i.e. at time t_1, the decision maker wants to be positioned in the most favourable decision situation which is possible. His decision situation at this time firstly depends on the amount of money units that are available to him, i.e. the extent of his wealth at time t_1, and secondly on the alternatives of utilizing this money (e.g. reinvesting it or exchanging it for consumer goods etc.)

In the following the possible sets of alternatives available at time t_1 for utilizing money are briefly denoted as the possible *states of the usability world.* Assume that in total q states of the usability world $S_1, S_2, \ldots, S_q$ – excluding each other – are possible (in our example: the possible rates of interest r). The *decision situation* at time t_1 is now characterized by the constellation (w, S_k) $(k=1, 2, \ldots, q)$, where w represents the wealth available at time t_1.

The *conditional probability of the state S_k of the usability world* $(k=1, 2, \ldots, q)$ under the condition that the state of world s_j $(j=1, 2, \ldots, m)$ does exist, can be defined as

$$P(S_k \mid s_j) =: p(k \mid j).$$

If a particular action a_i is chosen and if the state s_j occurs, then firstly w is determined as $w=w(a_i, s_j)$ and secondly the probabilities $p(k \mid j)$ of the different states S_k of the usability world are fixed. Thus a_i and s_j determine a probability distribution for the possible decision situation at time t_1. This probability distribution has to be interpreted as the consequence of a_i in the presence of s_j:

$$(5) \qquad e_{ij} = [w(a_i, s_j), \{S_k, p(k \mid j)\}\ k = 1, \ldots, q].$$

A comparison of (5) and (3) demonstrates that (5) represents a particular specialization or modification of (3) which is implied by the more detailed features of our model: s_j is substituted by the probability distribution $\{S_k, p(k \mid j)\}\ k=1, \ldots, q$ which exclusively depends on s_j.

We now assume that a utility $U(w, S_k)$ can be obtained if all alternatives of applying the monetary wealth w, inherent in S_k, are utilized optimally. The techniques for determining this utility function U are principally the same as described in paragraph 1 for the utility function u. $U(w, S_k)$ represents the utility pertaining to the decision situation (w, S_k). In an abbreviated form it is denoted as

$$U_k(w) := U(w, S_k).$$

Considering all states of the usability world S_k which are possibly given in the presence of the state of the world s_j the expected utility of an outcome e_{ij} then yields

$$(6) \qquad u(e_{ij}) = \sum_{k=1}^{q} U_k [w(a_i, s_j)] \, p(k \mid j).$$

Finally the total expected utility realizable by action a_i is

$$(7) \qquad \sum_{j=1}^{m} u(e_{ij}) \, p_j = \sum_{j=1}^{m} \sum_{k=1}^{q} U_k [w(a_i, s_j)] \, p(k \mid j) \, p(j) \to \underset{i}{\mathrm{Max}}!$$

According to the Bernoulli-criterion the expected utility has now to be maximized with respect to a_i. Introducing the state-dependent utility functions u_j

$$(8) \qquad u_j(w) := \sum_{k=1}^{q} U_k(w) \, p(k \mid j)$$

(7) is changed into

$$(7') \qquad \sum_{j=1}^{m} u_j [w(a_i, s_j)] \, p_j \to \underset{i}{\mathrm{Max}}!$$

This however precisely coincides with the Bernoulli-criterion (4′) with a state-dependent utility function which was introduced earlier. Apparently this function can be derived in a natural way from a utility valuation of wealth in the presence of the different possibilities of utilization.

In our *example* the states of the usability world S_k are characterized by the current rate of interest r. A summary of the conditional probabilities $p(k \mid j)$ may be given in the following table:

k \ j	s_1	s_2
$S_1: r = 5$	0	0.8
$S_2: r = 10$	1	0.2

The utility resulting from the wealth w in the state of the usability world S_k is

$$U(w, S_k) = \begin{cases} u(w \cdot 1.05) & \text{in the case of} \quad S_1 \\ u(w \cdot 1.10) & \text{in the case of} \quad S_2 \end{cases}$$

The state-dependent utility functions for w therefore are

$$u_1(w) = u(w \cdot 1.10)$$
$$u_2(w) = 0.8\, u(w \cdot 1.05) + 0.2\, u(w \cdot 1.10)$$

With the help of these functions we can finally establish the utility matrix of Table V by inserting $w\ (a_i,\ s_j)$ for w according to Table III. Table VI represents the outcome matrix in agreement with (5).

Our example has a simple structure, firstly because only one single decision period exceeding t_1 is considered, secondly because from each of the two states of the usability world, S_1 and S_2, only one single alternative may be chosen (i.e. to invest the money at the current rate of interest), while in general we shall have to choose an optimal utilization from a multiplicity of alternatives. That means that in general an individual decision problem will have to be solved for each state S_k, before the utilities $U(w, S_k)$ can actually be determined. Nevertheless such problems of a more general nature can basically be solved since they represent special cases from the field of sequential decision models, for which solution methods have been developed.

Another generalization – though going beyond the limits of our decision model – would follow from the assumption that the probabilities of a particular state of the usability world S_k not only depend on the state of world s_j but also on the chosen action a_i. Then the conditional probability $P(S_k \mid s_j)$ would have to be replaced by $P(S_k \mid a_i,\ s_j) := p(k \mid i,j)$.

V. ONCE MORE ON THE CHOICE OF THE NUMÉRAIRE

The two first-named examples dealing with the problem of the numéraire, can be treated as special cases within the limits of our decision model.

As far as the *consumer problem* is concerned the states of the usability world are characterized by the price of the bundle of consumer goods. As only one price is associated with each state of the world, the latter may be identified with this price.

$$s_1 \Rightarrow S_1 : \text{price} = \ \ 8\ \text{DM/UC}$$
$$s_2 \Rightarrow S_2 : \text{price} = 12\ \text{DM/UC}$$

The conditional probabilities are

$$p(k \mid j) = \begin{cases} 1 & \text{for} \quad k = j \\ 0 & \text{for} \quad k \neq j \end{cases} \quad (k, j = 1, 2)$$

and $U_k(w) = u_k(w)$.

The utility $U_k(w)$ is yielded directly as the utility of the set of consumer goods obtainable at the price S_k. For this reason it is the set of consumer goods that has to be employed as a numéraire, not money – at least if we actually take interest on the application of criterion (2) (and not in that of criterion (4′) or (7′)) –. Hence, in the case of the consumer problem the question of the appropriate standard of value can clearly be answered. The outcome matrix of Table II which at first was precautiously expressed by number couples, can now be simplified (Table VII).

TABLE VII

	s_1	s_2
a_1	125	83.33
a_2	100	100

In the case of the *Onassis problem* we cannot simply consider one standard of value to be the only appropriate one. It all depends on Onassis' plans how to use the invested wealth later on and on the alternatives that are open to him. For example if in any case he intends to buy certain capital goods only in Great-Britain, the prices of which can be regarded as certain from the beginning, then sterling might be the only adequate numéraire. On the other hand, if prices are uncertain and Onassis is not determined to buy only in one particular country, and even more if Onassis is in a position to make his choice between different capital goods, which choice will depend on next year's prices (and thus on the corresponding rate of exchange), in this case it is no longer evident which measure could serve as a numéraire for the establishment of the outcome matrix, if an outcome is to be characterized by realized terminal wealth only and if a state-independent utility function (i.e. criterion (2)) is to be employed. In fact, in a case of this kind we should best give up searching for the 'right' numéraire. Instead, we could try to analyze Onassis' decision situation according to our model. We then shall have to enumerate the different potential states of the usability world Onassis might be confronted with one year later and to investigate them with regard to their realizable utility. As described above we thus arrive at state-dependent utility functions and can finally find the optimum. Using this method we are not dependent on a particular numéraire. It does not matter whether the computations are made in marks or in sterling. We cannot avoid to establish and to solve a sequential decision model, though.

Whatever may be the structure of this model it should be possible to trace it back to criterion (7) resp. (4) of our model and thus finally to criterion (1) of the basic model of decision theory, which here again proves to be highly flexible and adaptible[7]. It certainly will not be necessary to introduce further categories (in addition to 'probability', 'action', 'outcome', 'utility') such as the term 'unit of measurement' or 'standard of value' to the basic model. This of course does not exclude that by refining the model we shall have to introduce new terms.

VI. IDENTICAL UTILITY FUNCTIONS FOR DIFFERENT STATES OF THE WORLD

In this last paragraph we investigate under which circumstances some or all utility functions u_j are identical. According to (8) the states of the world s_j and s_i have identical utility functions if and only if the following (necessary and sufficient) condition is met:

$$\sum_{k=1}^{q} U_k(w)\, p(k \mid j) = \sum_{k=1}^{q} U_k(w)\, p(k \mid j') \quad \text{for all} \quad w\,.$$

This yields the condition

$$(10) \qquad \sum_{k=1}^{q} U_k(w)\, [p(k \mid j) - p(k \mid j')] = 0\,.$$

This condition is met in particular if the following applies:

$$(11) \qquad p(k \mid j) = p(k \mid j') \quad \text{for all} \quad k(k = 1, 2, \ldots, q)\,.$$

Hence, if the conditional probability of arriving at the state S_k of the usability world $(k = 1, 2, \ldots, q)$ is just as high in the presence of the state of the world s_j as in the presence of the state $s_{j'}$, then the state-dependent utility functions u_j and $u_{j'}$ are identical. Of theoritical interest is the case where the states of the usability world S_k and the states of the world s_j are stochastically independent, whence the condition $p(k \mid 1) = p(k \mid 2) = \ldots = = p(k \mid m)$ for each $k(k=1, 2, \ldots, q)$ is satisfied. Hence condition (11) is met for any two states s_j and s_j', and the same utility function u is applicable to each state of the world s_j $(j=1,2,\ldots,m)$.

A special case is given when at time t_0 only *one* state of the usability world is considered to be possible by the decision maker, thus $q=1$. Since then the state S_1 will be realized with certainty, we obtain $p(1 \mid 1) = = p(1 \mid 2) = \ldots = p(1 \mid m) = 1$. As we can see, condition (11) is met again and $u_1 = u_2 = \ldots = u_m$.

Provided that all states of world and of the usability world are stochastically independent of each other, money may thus be used as a numéraire without a state-dependent utility function having to be established (hence criterion (4′) coincides with criterion (2)). Yet, in this assuption there seems to be little realism. Generally the data characterizing the state of the world and data characterizing the usability world are stochastically dependent. If e.g. we invest money in shares and re-invest the money available at the end of the planning period in shares, then there exists a close stochastic dependency between the states of world on the one hand and the states of the usability world on the other hand. Between these states stochastic independence could be assumed if e.g. the decision maker spends the money available at the end of the planning period on consumer goods as long as the prices of the shares and those of the consumer goods are stochastically independent.

It is true, even if we employ a particular standard of value and the utility functions u_j are not identical, we can possibly find another standard, such that every u_j is made indepedent of the state of the world if this new standard of value is applied as a numéraire for the evaluation of wealth. This is achieved under the condition that when changing from ohne standard to the other the numéraire for the wealth has to undergo a (in general state-dependent) transformation:

$$w' = f_j(w),$$

in such a way that

$$u_j(w) = u(f_j(w)) = u(w')$$

with a new utility function u (now independent of s_j). And it is just the example of the consumer that complies with this condition (cf. the transition from Table II′ to Table VII).

APPENDIX

Applying the utility functions u and v to the Onassis problem we find the following expected values for the actions a_1 and a_2

	DM	£
a_1:	$pu(8)+(1-p)u(-13.6)$	$v(8)$
a_2:	$u(5.84)$	$pv(5.84)+(1-p)v(32.3)$

The utility functions may be chosen in such a way (and there are a great many) that for any probability p the Bernoulli-criterion definitely decides which of the two actions is preferable, i.e. no matter whether the utility function u or v is used. We arbitrarily set $u(-13.6)=u_0$, $u(8)=u_1$, $v(5.84)=v_0$, $v(32.3)=v_1$ with $u_0<u_1$, $v_0<v_1$. If now p^* is that probability (for the realization of the event 'devaluation') for which Onassis is indifferent with respect to the actions a_1 and a_2, then we set $u(5.84)=p^*u_1+(1-p^*)u_0$ and $v(8)=p^*v_0+(1-p^*)v_1$. Then according to the Bernoulli-Criterion – and besides in accordance with rational decision behaviour – a_1 is preferred if $p>p^*$ and a_2 is preferred if $p<p^*$. For in the first case $pu_1+(1-p)\,u_0>u(5.84)$ resp. $pv_0+(1-p)\,v_1<v(8)$ and in the second case the reversed relation applies. In any case, the Bernoulli-criterion (2) takes the right decision irrespective of the numéraire chosen (marks or sterling). Hence, by introducing the Bernoulli-criterion, with adequate utility functions, we seem to be able to solve the Onassis paradox.

Yet, appearances are deceptive. A strict application of the Bernoulli-criterion based on the alternatives of an assessment in marks or sterling entails inacceptable consequences. To simplify matters let us suppose $p/(1-p)=11.25$ (if another p is used the argumentation has to look somewhat different). In the following we at first take sterling as a numéraire and want to compare action a_1 (purchase of a sterling-loan bearing 8%) with action a'_1, where in the presence of s_1 profit is $(8-\Delta)\%$ and in the presence of s_2 profit is $(8+11.25\Delta)\%$. Δ is small, otherwise however an optional amount. Onassis can realize a'_1 investing some portion $b(0<b<1)$ of his initial amount of money in sterling-loans bearing 8% and by investing the remainder $1-b$ in a 5.84% DM-loan (or equivalently: in 8% sterling-loans combined with a forward selling of £ at a 2% discount), where $b=1-\Delta/2.16$. a and a' are promising the same expected profit of 8%. They consequently may be regarded as (approximately) equivalent (i.e. offering the same utility) if Δ is small, irrespective of the structure of the utility function v, provided it is differentiable at the point 8. This yields the following tables:

Sterling

	s_1	s_2
a_1	8	8
a'_1	$8-\Delta$	$8+11.25\,\Delta$

Marks

	s_1	s_2
a_1	8	-13.6
a'_1	$8-\Delta$	$-13.6+0.8{\cdot}11.25\,\Delta$

According to the Bernoulli-criterion, but now expressed by means of the utility function u, the equivalence between a_1 and a'_1 amounts to

$$pu(8)+(1-p)\,u(-13.6)=pu(8-\Delta)+(1-p)\times \times u(-13.6+0.8\cdot 11.25\Delta)$$

or (since $p/(1-p)=11.25$)

$$u(8)-u(8-\Delta)=0.8\,[u(-13.6+0.8\cdot 11.25\Delta)- - u(-13.6)]/(0.8\cdot 11.25)$$

or finally after going to the limit $\Delta\to 0$:

$$u'(8)=0.8\,u'(-13.6).$$

As we see the utility function cannot be chosen arbitrarily although on the other hand utility functions are to reflect the individual (or subjective) behaviour towards risk and should thus be unrestricted. The above restriction however is exclusively caused by our attempt to describe the same behaviour by means of different utility functions using different standards of value and is not due to certain ways of behaviour under risk. In fact this restriction will usually not be in accordance with individual behaviour towards risk. It is thus seen that our attempt to solve the Onassis paradox by introducing utility functions in the usual way leads to contradictions and must therefore be regarded as a failure.

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NOTES

[1] Cf. e.g. Luce and Raiffa (1957), pp. 23.

[2] Typical examples are portfolio theory and the theory of investment of the firm.

[3] See also Engels (1969).

[4] For the computations and further discussion of the problem see Stützel, (1970). Generally: $1+$ DM rate of interest $= (1 + £$ rate of interest$) \times (1 - £$ rate of devaluation).

[5] This concept has been introduced by Hirschleifer (1965, pp. 534), however without deriving it from a general decision model.

[6] The construction of this example shows some similarities with Stützel's (1970) second decision problem, which can be analyzed along the same line of arguments.

[7] Cf. also Schneeweiss (1966).