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Effect of domain-shape on coexistence problems in a competition-diffusion system

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Abstract. We discuss a competition-diffusion system to study coexistence problems of two competing species in a homogeneous environment. In particular, by using invariant manifold theory, effects of domain-shape are considered on this problem.

Key words: Coexistence of competing species – Competition-diffusion system – Invariant manifold theory

1. Introduction

One of the fundamental problems in population ecology is how and why a large number of competing species are able to coexist in a basically homogeneous environment. In studying this problem, numerous mathematical models have been proposed so far. One of the simplest models for the interaction of n competing species is the well-known Gause–Lotka–Volterra system:

$$\frac{dx_i}{dt} = \left(R_i - a_i x_i - \sum_{j \neq i}^{n} b_{ij} x_j\right) x_i \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

where $x_i(t)$ is the population density of species i at time t, R_i is the intrinsic growth rate, a_i is the intraspecific competition rate and b_{ij} $(i \neq j)$ is the interspecific competition rate for $i, j = 1, 2, \ldots, n$. All of the parameters are positive constants. Especially, when $n = 2$, (1.1) is reduced to

$$\begin{cases} \dfrac{dx_1}{dt} = (R_1 - a_1 x_1 - b_1 x_2) x_1, \\ \dfrac{dx_2}{dt} = (R_2 - b_2 x_1 - a_2 x_2) x_2, \end{cases} \tag{1.2}$$

where we simply write $b_{12} = b_1$ and $b_{21} = b_2$. The asymptotic behavior of solutions $(x_1(t), x_2(t))$ of (1.2) is completely classified into the following cases when

$(x_1(0), x_2(0))$ is positive:

$$\begin{cases} \text{when } \dfrac{b_1}{a_2}, \dfrac{a_1}{b_2} < \dfrac{R_1}{R_2}, & \lim_{t\to\infty} (x_1(t), x_2(t)) = \left(\dfrac{R_1}{a_1}, 0\right); \\ \text{when } \dfrac{b_1}{a_2}, \dfrac{a_1}{b_2} > \dfrac{R_1}{R_2}, & \lim_{t\to\infty} (x_1(t), x_2(t)) = \left(0, \dfrac{R_2}{a_2}\right); \\ \text{when } \dfrac{b_1}{a_2} < \dfrac{R_1}{R_2} < \dfrac{a_1}{a_2}, & \lim_{t\to\infty} (x_1(t), x_2(t)) = (x_1^*, x_2^*); \\ \text{when } \dfrac{a_1}{b_2} < \dfrac{R_1}{R_2} < \dfrac{b_1}{a_2}, & (x_1^*, x_2^*) \text{ is unstable and } (x_1(t), x_2(t)) \\ \text{generically approaches to either } \left(\dfrac{R_1}{a_1}, 0\right) \text{ or } \left(0, \dfrac{R_2}{a_2}\right) \text{ as time goes on,} \end{cases} \tag{1.3}$$

where

$$x_1^* = \frac{a^2 R_1 - b_1 R_2}{a_1 a_2 - b_1 b_2} \quad \text{and} \quad x_2^* = \frac{a_1 R_2 - b_2 R_1}{a_1 a_2 - b_1 b_2}.$$

That is, two species coexist only for the case when $b_1/a_2 < R_1/R_2 < a_1/b_2$ holds.

Horn and MacArthur [11] and Levin [15] consider the situation where the habitat is subdivided into two patches and propose the following model:

$$\begin{cases} \dfrac{dx_1^{(i)}}{dt} = (R_1 - a_1 x_1^{(i)} - b_1 x_2^{(i)}) x_1^{(i)} + D_1 (x_1^{(j)} - x_1^{(i)}), \\ \dfrac{dx_2^{(i)}}{dt} = (R_2 - b_2 x_1^{(i)} - a_2 x_2^{(i)}) x_2^{(i)} + D_2 (x_2^{(j)} - x_2^{(i)}), \end{cases} \quad (i, j = 1, 2; i \neq j) \tag{1.4}$$

where $(x_1^{(i)}, x_2^{(i)})$ are the population densities of two species in patch i $(i = 1, 2)$, and D_1, D_2 are the species-specific migration rates between patch 1 and patch 2. When $D_1 = D_2 = 0$, that is, each species can never migrate between two patches, $(x_1^{(1)}, x_2^{(1)})$ and $(x_1^{(2)}, x_2^{(2)})$ independently satisfy (1.2). On the other hand, if D_1 and D_2 are both very large, one could expect that $x_i^{(1)} - x_i^{(2)}$ $(i = 1, 2)$ tend to zero and as the result, the behavior of solutions of (1.4) is qualitatively the same as that of (1.2). If D_1 and D_2 are not necessarily large, Levin [15] showed that (1.4) admits stable positive equilibria for the case when $a_1/b_2 < R_1/R_2 < b_1/a_2$, that is, two competing species can coexist by suitably migrating from one patch to the other even if the interspecific competition is stronger than the intraspecific one. This implies that coexistence of competing species is possible in a patchy environment. The discussion above can be extended to n-competing systems of the form (1.4).

The situation can be also considered where two competing species move by diffusion in a bounded domain. It is described by the usual reaction-diffusion system which is the continuum version of (1.4):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + (R_1 - a_1 u_1 - b_1 u_2) u_1, \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + (R_2 - b_2 u_1 - a_2 u_2) u_2, \end{cases} \quad (t, x) \in (0, \infty) \times \Omega \tag{1.5}$$

with the zero-flux boundary conditions

$$\frac{\partial u_1}{\partial n} = 0, \qquad \frac{\partial u_2}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times \partial\Omega. \tag{1.6}$$

Here $u_1(t, x)$ and $u_2(t, x)$ are the population densities at time $t > 0$ and position $x \in \Omega$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, Δ is the Laplacian, d_1 and d_2 are the diffusion coefficients, and $\partial/\partial n$ is the outward normal derivative at $\partial\Omega$.

For the study of the coexistence problem of two competing species which move by diffusion, there has been much work which deals with existence or non-existence of stable spatially inhomogeneous positive equilibrium solutions of the problem (1.5), (1.6). First of all, we should note that if the diffusion coefficients d_1 and d_2 are both very large, any nonnegative solutions of (1.5), (1.6) with an arbitrarily fixed domain Ω tend to be spatially homogeneous as $t \to \infty$ (Conway et al. [4], S.-I. Ei [5]). This result implies that the asymptotic behavior of solutions of (1.5), (1.6) is qualitatively the same as that of (1.2). Moreover, even if d_1 and d_2 are not necessarily large, the assertions in (1.3) also hold for (1.5), (1.6) except for the case $a_1/b_2 < R_1/R_2 < b_1/a_2$, when $u_i(0, x) \geqslant 0$ and $u_i(0, x) \not\equiv 0$ $(i = 1, 2)$ (Ahmad and Lazer [1], for instance). For the case $a_1/b_2 < R_1/R_2 < b_1/a_2$, the situation is not so simple. When Ω is *convex*, Kishimoto and Weinberger [14] proved that for any d_1 and d_2, spatially inhomogeneous equilibrium solutions of (1.5), (1.6) are not stable even if these solutions exist. On the other hand, Matano and Mimura [16] and also Jimbo [13] showed that there is a nonconvex domain $\Omega \subset \mathbb{R}^2$ such that (1.5), (1.6) has *stable* spatially inhomogeneous nonnegative equilibrium solutions. We can thus understand that existence and stability of spatially inhomogeneous equilibrium solutions of (1.5), (1.6) crucially depend on the shape of Ω as well as the magnitude of d_1 and d_2. In ecological terms, coexistence of two competing species depends upon the shape of habitat-domain as well as the diffusion rates.

More general n-competing species models

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + \left(R_i - a_i u_i - \sum_{j \neq i}^{n} b_{ij} u_j \right) u_i \quad (i = 1, 2, \ldots, n) \tag{1.7}$$

have been investigated. In fact, it is shown in Mimura and Fife [19] and Mimura [17] that even if the domain Ω is convex, coexistence of n-competing species $(n \geqslant 3)$ is possible for suitable values of d_i, R_i, a_i and b_{ij} $(i, j = 1, 2, \ldots, n)$.

In this paper, we will discuss the possibility of coexistence of two competing species in two-dimensional habitat domains. Mathematically, we consider the dependency of domain-shape on existence and stability of equilibrium solutions of (1.5), (1.6). To do it, we impose the following two assumptions on our problem:

(A-1) Ω is a symmetric and connecting dumbbell-shaped domain with a small parameter ε, say Ω_ε, which consists of three disjoint unions $\Omega_\varepsilon = \Omega_0^L \cup \Omega_0^R \cup R_\varepsilon$. Here Ω_0^L and Ω_0^R are convex domains and R_ε, which is a handle connecting Ω_0^L and Ω_0^R, is given by $R_\varepsilon = \{(x_1, x_2) \mid |x_1| \leqslant 1, |x_2| \leqslant r(x_1, \varepsilon)\}$, where $r(x_1, \varepsilon)$ is a smooth and positive function with $|r(x_1, \varepsilon)| = O(\varepsilon)$ as $\varepsilon \to 0$. Moreover, as shown in Fig. 1, R_ε^L and R_ε^R are respectively symmetric domains of R_ε with respect to the axes $x_1 = -1$ and $x_1 = 1$, satisfying $R_\varepsilon^L \subset \Omega_0^L$ and $R_\varepsilon^R \subset \Omega_0^R$.

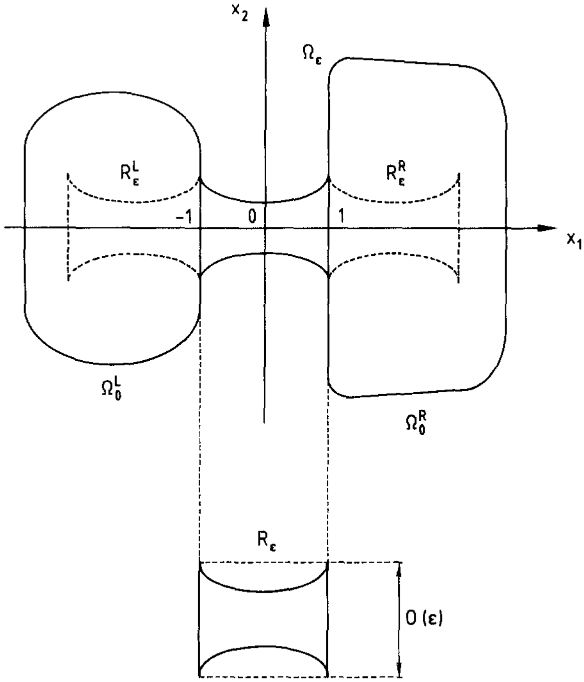


Fig. 1. The shape of domain Ω_ε, where $\Omega_\varepsilon = \Omega_0^L \cup \Omega_0^R \cup R_\varepsilon$

(A-2) d_i takes the form $d_i = \bar{d}_i/\varepsilon^\theta$ with positive constants θ and $\bar{d}_i$ $(i = 1, 2)$

To study a domain-shape problem for (1.5), (1.6), we introduce a parameter $\alpha = |\Omega_0^R|/(|\Omega_0^R| + |\Omega_0^L|)$ which controls the shape of Ω_ε. In fact, the limiting case when $\alpha \downarrow 0$ (resp. $\alpha \uparrow 1$) corresponds to the situation that Ω_ε is close to the convex domain Ω_0^L (resp. Ω_0^R) for sufficiently small ε. Our main aim is to study the dependency of $\alpha \in (0, 1)$ on equilibrium solutions of (1.5), (1.6) when ε is sufficiently small.

The approach we will use here is the theory of invariant manifolds (for instance, Carr [2] and Morita [20]). When ε is sufficiently small, we are able to reduce the PDE problem (1.5), (1.6) to the associated ODE problem, which is described by the essentially same type as (1.4) including additional parameters $\bar{d}_1, d_2, \theta$ and α. Thus, its analysis enables us to study the problem (1.5), (1.6). In Sect. 4, we consider the simple case when $\bar{d}_1 = \bar{d}_2 = d$, $R_1 = R_2 = R$, $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $a < b$ (this inequality indicates that two competing species never coexist if Ω is convex, as already mentioned), and draw the picture of global bifurcation of equilibrium solutions with respect to d and α for suitably fixed a, b, R and θ (Figs. 2–5). Here let us briefly explain the case when $\theta = 1$. When d is large, there are only four solution branches (I, I) (II, II), (III, III) and (IV, IV) for $\alpha \in (0, 1)$, which correspond to the spatially homogeneous equilibrium solutions $(0, 0)$, $(R/a, 0)$, $(0, R/a)$ and $(R/(a + b), R/(a + b))$, respectively (Fig. 5a). This situation corresponds to the one mentioned by Conway et al. [4]. When d is intermediate, there appear two new branches bifurcating from the (IV, IV)-branch which exist for $\alpha \in (\alpha_*, 1 - \alpha_*)$, where α_* is some critical value depending on d (Fig. 5b). These are unstable spatially inhomogeneous nonnegative equilibrium solutions, that is, coexistence of two competing species cannot be maintained. Finally, when d is small, each of these bifurcating branches folds over at $\alpha = \alpha^*$ and $1 - \alpha^*$, where α^* is some critical value depending on d (Fig. 5c). As the result, stable spatially inhomogeneous nonnegative equilibrium solutions exist for $\alpha \in (\alpha^*, 1 - \alpha^*)$. These solutions correspond to the ones shown by Matano and Mimura [16] and also by Jimbo [13]. In summary, one could clearly understand the dependency of shape of domains as well as diffusion rates on coexistence of two competing species. The precise discussions will be stated in Sects. 3 and 4.

2. Setting of the problem

We consider the following ε-family of reaction-diffusion systems in $(L^2(\Omega_\varepsilon))^2$ which is more general than (1.5):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\bar{d}_1}{\varepsilon^\theta} \Delta u_1 + f_1(u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = \dfrac{\bar{d}_2}{\varepsilon^\theta} \Delta u_2 + f_2(u_1, u_2), \end{cases} \quad (t, x) \in (0, \infty) \times \Omega_\varepsilon \qquad (2.1)_\varepsilon$$

with the zero-flux boundary and initial conditions:

$$\frac{\partial u_1}{\partial n} = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon \qquad (2.2)$$

and

$$u_1(0, x) = \phi(x), \quad u_2(0, x) = \psi(x), \quad x \in \bar{\Omega}_\varepsilon, \qquad (2.3)$$

where $u_1, u_2 \in \mathbb{R}_+ = [0, \infty)$ and $\phi, \psi \in \mathbb{R}_+$ and $\phi \not\equiv 0, \psi \not\equiv 0$ and $f_1, f_2 : \mathbb{R}^2_+ \to \mathbb{R}$ are smooth, $\bar{d}_1, \bar{d}_2$ are positive constants, $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega_\varepsilon$. As was stated in (A-1), the domain Ω_ε ($\subset \mathbb{R}^2$) is symmetric with respect to the x_1-axis with smooth boundary $\partial\Omega_\varepsilon$ (see Fig. 1).

From an ecological viewpoint, we impose two assumptions on f_1 and f_2.

(A-3) $$\frac{\partial f_1}{\partial u_2} \leqslant 0, \quad \frac{\partial f_2}{\partial u_1} \leqslant 0 \quad \text{for any } u_1 \geqslant 0, u_2 \geqslant 0.$$

(A-4) $$\begin{cases} f_1(0,0) = f_2(0,0) = 0, \quad f_1(0, u_2) \geqslant 0, \quad f_2(u_1, 0) \geqslant 0, \\ f_1(u_1, u_2) \leqslant 0 \quad \text{for large } u_1 > 0 \text{ and any } u_2 \geqslant 0, \\ f_2(u_1, u_2) \leqslant 0 \quad \text{for large } u_2 > 0 \text{ and any } u_1 \geqslant 0, \end{cases}$$

We call $(2.1)_\varepsilon$ a competition-diffusion system when it satisfies (A-3) and (A-4). Obviously, the Gause–Lotka–Volterra population dynamics in (1.5) satisfies these assumptions.

For the competition-diffusion system $(2.1)_\varepsilon$, it is shown in Hirsch [10] and Matano and Mimura [16] that stable attractors consist only of equilibrium solutions. For this reason, we focus our attention on equilibrium solutions of $(2.1)_\varepsilon$, (2.2) only.

We simply write $(2.1)_\varepsilon \sim (2.3)$ as

$$\frac{\partial U}{\partial t} = D/\varepsilon^\theta \mathscr{L} U + F(U), \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon, \tag{2.1$_\varepsilon$}$$

$$\frac{\partial U}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon, \tag{2.2}$$

$$U(0, x) = \Phi(x), \qquad x \in \bar{\Omega}_\varepsilon, \tag{2.3}$$

where $U = (u_1, u_2)$, $\mathscr{L}U = (\Delta u_1, \Delta u_2)$, $\Phi = (\phi, \psi)$, $F(U) = (f_1(U), f_2(U))$, and D is the diagonal matrix with elements $\bar{d}_i$ $(i = 1, 2)$. We note that the local existence of solutions of $(2.1)_\varepsilon \sim (2.3)$ can be shown in a standard manner because $\mathscr{L}$ with the zero-flux boundary conditions is a generator of analytic semigroup in $(L^2(\Omega_\varepsilon))^2$ (see Henry [9], for instance).

Let Ξ and Σ be $\Xi = [0, K] \times [0, K]$ and $\Sigma = \{U \in (H^1(\Omega_\varepsilon))^2 \cap (L^\infty(\Omega_\varepsilon))^2 \mid U \in \Xi, \|U\|_{(H^1(\Omega_\varepsilon))^2} \leq K'\}$ for some $K > 0$ and $K' > 0$. Then we note that for large K, Ξ is an invariant region of the system $(2.1)_\varepsilon$, (2.2) and that if $\Phi(x)$ is in Ξ, any solution $U(t; \Phi)$ of $(2.1)_\varepsilon \sim (2.3)$ eventually enters Σ for large K' (Chueh et al. [3], Fang [7]). Therefore, we may restrict our discussion to equilibrium solutions in the interior of Σ.

Let $\lambda_\varepsilon^{(1)}$, $\lambda_\varepsilon^{(2)}$ be the first two eigenvalues of $-\Delta$ in Ω_ε with the zero-flux boundary conditions and $\omega_\varepsilon^{(1)}$, $\omega_\varepsilon^{(2)}$ be the corresponding normalized eigenfunctions. Note that $0 = \lambda_\varepsilon^{(1)} < \lambda_\varepsilon^{(2)}$ holds. It is known in Hale and Vegas [8] and Vegas [21] that $\omega_\varepsilon^{(1)} = |\Omega_\varepsilon|^{-1/2}$ and

$$\lim_{\varepsilon \downarrow 0} \omega_\varepsilon^{(2)} = \omega_0^{(2)} = \begin{cases} -(|\Omega_0^R|/(|\Omega_0|\,|\Omega_0^L|))^{1/2} & \text{in } \Omega_0^L \\ (|\Omega_0^L|/(|\Omega_0|\,|\Omega_0^R|))^{1/2} & \text{in } \Omega_0^R \end{cases}$$

in $H^2(\Omega_0)$-topology. For $\alpha = |\Omega_0^R|/(|\Omega_0^R| + |\Omega_0^L|)$, define $\phi_\varepsilon^{(1)}$ and $\phi_\varepsilon^{(2)}$ by

$$\phi_\varepsilon^{(1)} = |\Omega_\varepsilon|^{1/2}((1-\alpha)\omega_\varepsilon^{(1)} - (\alpha(1-\alpha))^{1/2}\omega_\varepsilon^{(2)})$$

and

$$\phi_\varepsilon^{(2)} = |\Omega_\varepsilon|^{1/2}(\alpha\omega_\varepsilon^{(1)} + (\alpha(1-\alpha))^{1/2}\omega_\varepsilon^{(2)}),$$

respectively. It is easy to see that

$$\lim_{\varepsilon \downarrow 0} \phi_\varepsilon^{(1)} = \begin{cases} 1 & \text{in } \Omega_0^L \\ 0 & \text{in } \Omega_0^R \end{cases} \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \phi_\varepsilon^{(2)} = \begin{cases} 0 & \text{in } \Omega_0^L \\ 1 & \text{in } \Omega_0^R \end{cases}. \tag{2.4}$$

Let Q^ε and P^ε be projections from $(L^2(\Omega_\varepsilon))^2$ into $(\text{span}\{\omega_\varepsilon^{(1)}, \omega_\varepsilon^{(2)}\})^2$ and $P^\varepsilon = Id - Q^\varepsilon$, respectively, where Id is the identity operator. Here we show three theorems without proofs, which is shown in Fang [7].

Theorem 2.1. *There exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, $\mathscr{K}_\varepsilon = \{Y_1\phi_\varepsilon^{(1)} + Y_2\phi_\varepsilon^{(2)} + h_\varepsilon(Y_1, Y_2) \mid Y_1, Y_2 \in \mathbb{R}^2\} \cap \Sigma$ is a four-dimensional Lipschitz continuous manifold for some $h_\varepsilon(Y_1, Y_2) \in \mathbb{C}(\mathbb{R}^4, P^\varepsilon((H^1(\Omega_\varepsilon))^2))$ and it is invariant under the semiflow $S(t)(\Phi) = U(t; \varepsilon, \Phi)$ of* $(2.1)_\varepsilon \sim (2.3)$ for $\Phi \in \Sigma$. *h_ε satisfies $|||h_\varepsilon|||_{\varepsilon,\infty} = O(\varepsilon^{(2\theta+1)/2})$ and $h_\varepsilon(Y_1, Y_1) = 0$, where $|||h|||_{\varepsilon,\infty} = \sup\{\|h(Y_1, Y_2)\|_{(H^1(\Omega_\varepsilon))^2} | Y_1, Y_2 \in \mathbb{R}^2\}$ for $h \in \mathbb{C}(\mathbb{R}^4, P^\varepsilon((H^1(\Omega_\varepsilon))^2))$.*

Let us consider the following four-dimensional ODEs:

$$\begin{cases} \dfrac{dY_1}{dt} = F(Y_1) + \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}\alpha D(Y_2 - Y_1) + R_1^\varepsilon(Y_1, Y_2), \\ \dfrac{dY_2}{dt} = F(Y_2) + \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}(1-\alpha)D(Y_1 - Y_2) + R_2^\varepsilon(Y_1, Y_2), \end{cases} \tag{2.5$_\varepsilon$}$$

where R_1^ε and R_2^ε are defined by

$$\begin{cases} R_1^\varepsilon(Y_1, Y_2) = \left\langle F(Y_1\phi_\varepsilon^{(1)} + Y_2\phi_\varepsilon^{(2)} + h_\varepsilon(Y_1, Y_2)), \dfrac{\phi_\varepsilon^{(1)}}{(1-\alpha)|\Omega_\varepsilon|}\right\rangle - F(Y_1) \\ R_2^\varepsilon(Y_1, Y_2) = \left\langle F(Y_1\phi_\varepsilon^{(1)} + Y_2\phi_\varepsilon^{(2)} + h_\varepsilon(Y_1, Y_2)), \dfrac{\phi_\varepsilon^{(2)}}{\alpha|\Omega_\varepsilon|}\right\rangle - F(Y_2) \end{cases} \tag{2.6}$$

with

$$\langle F, \phi_\varepsilon^{(i)}\rangle = (\langle f_1, \phi_\varepsilon^{(i)}\rangle_{L^2(\Omega_\varepsilon)}, \langle f_2, \phi_\varepsilon^{(i)}\rangle_{L^2(\Omega_\varepsilon)}) \quad (i = 1, 2).$$

It is known in [7] that there is some $K_1 > 0$ such that $|R_i^\varepsilon|_{K_1,\infty} = O(\varepsilon^{1/2})$ as $\varepsilon \downarrow 0$ and $R_i^\varepsilon(Y, Y) = 0$ for $Y \in \mathbb{R}^2$ $(i = 1, 2)$, where $|R|_{K_1,\infty} = \sup(|R(Y_1, Y_2)| \,|\, |(Y_1, Y_2)| \leqslant K_1)$ for $R \in \mathbb{C}(\mathbb{R}^4, \mathbb{R}^2)$.

Theorem 2.2. *There exist $\varepsilon_2 > 0$ ($\varepsilon_2 < \varepsilon_1$), $v > 0$ and $N > 0$ such that for any $\varepsilon \in (0, \varepsilon_2)$, if $U(t) \in \Sigma$ is a solution of (2.1)$_\varepsilon$, (2.2), then there exist $Y_1^0, Y_2^0 \in \mathbb{R}^2$ such that*

$$\|U(t) - (Y_1(t;\varepsilon)\phi_\varepsilon^{(1)} + Y_2(t;\varepsilon)\phi_\varepsilon^{(2)} + h_\varepsilon(Y_1(t;\varepsilon), Y_2(t;\varepsilon)))\|_{(H^1(\Omega_\varepsilon))^2} \leqslant Ne^{-vt} \quad \text{for } t \geqslant 0,$$

where $(Y_1(t;\varepsilon), Y_2(t;\varepsilon))$ is the solution of (2.5)$_\varepsilon$ with the initial value (Y_1^0, Y_2^0).

Theorem 2.3. *There exists a positive constant τ independently of α and ε such that*

$$\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} = \frac{\tau}{\alpha(1-\alpha)}.$$

3. Equilibrium solutions of (2.1)$_\varepsilon$, (2.2)

In the previous section, we have shown that the dynamics of solutions to (2.1)$_\varepsilon$, (2.2) is approximated by that of (2.5)$_\varepsilon$ when ε is sufficiently small. In this section, by using this result, we study the existence and stability of equilibrium solutions of (2.1)$_\varepsilon$, (2.2). We rewrite again the problem (2.1)$_\varepsilon$, (2.2) and (2.3) as

$$\frac{\partial U}{\partial t} = D/\varepsilon^\theta \mathscr{L}U + F(U), \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon \tag{3.1$_\varepsilon$}$$

$$\frac{\partial U}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon \tag{3.2}$$

$$U(0, x) = \Phi(x), \qquad x \in \bar{\Omega}_\varepsilon, \tag{3.3}$$

respectively.

First, we put $\gamma(\theta) = \lim_{\varepsilon \downarrow 0} (\lambda_\varepsilon^{(2)}/\varepsilon^\theta)$. Then it is obvious from Theorem 2.3 that $\gamma(\theta) = \infty$ if $\theta > 1$, $\gamma(\theta) = \tau/[\alpha(1-\alpha)]$ if $\theta = 1$ and $\gamma(\theta) = 0$ if $0 < \theta < 1$. Suppose $\theta > 1$. Since the coefficients of the second terms in the right-hand sides of (2.5)$_\varepsilon$

diverge to ∞ as $\varepsilon \downarrow 0$, we easily find that the solution of $(2.5)_\varepsilon$, (2.6) satisfies

$$\lim_{t \to \infty} (Y_1(t; \varepsilon) - Y_2(t; \varepsilon)) = 0.$$

By using this fact and Theorem 2.2, we have the following theorem which corresponds to that of Conway et al. [4]:

Theorem 3.1. *Let $U(t, x; \varepsilon, \Phi)$ be a solution of $(3.1)_\varepsilon \sim (3.3)$ with $\theta > 1$. There exist $\varepsilon_3 > 0$, $M > 0$ and $\kappa > 0$ such that for any $\varepsilon \in (0, \varepsilon_3)$.*

$$\| U(t, \cdot\, ; \varepsilon, \Phi) - W(t) \|_{(L^2(\Omega_0))^2} \leqslant M e^{-\kappa t}, \quad t \geqslant 0$$

holds. Here $W(t)$ is the spatial average of $U(t, \cdot\, ; \varepsilon, \Phi)$ which satisfies

$$\frac{dW}{dt} = F(W) + L(t), \quad t > 0,$$

where $L(t)$ is some function satisfying

$$|L(t)| \leqslant M_1 e^{-\kappa_1 t}$$

for some constants M_1 and κ_1.

As an immediate consequence of this theorem, we see that any equilibrium solutions of $(3.1)_\varepsilon$, (3.2) with $\theta > 1$ must be spatially homogeneous, that is, these are equilibria of

$$\frac{dW}{dt} = F(W), \quad t > 0.$$

For the other case $0 < \theta \leqslant 1$, $(2.5)_\varepsilon$ can be written as

$$\begin{cases} \dfrac{dY_1}{dt} = F(Y_1) + \gamma(\theta)\alpha D(Y_2 - Y_1) + o(1) \\ \dfrac{dY_2}{dt} = F(Y_2)\gamma(\theta)(1 - \alpha)D(Y_1 - Y_2) + o(1) \end{cases} \tag{3.4}$$

as $\varepsilon \downarrow 0$. Now define the limit equation of $(2.5)_\varepsilon$ with $0 < \theta \leqslant 1$ by

$$\begin{cases} \dfrac{dY_1}{dt} = F(Y_1) + \gamma(\theta)\alpha D(Y_2 - Y_1) \\ \dfrac{dY_2}{dt} = F(Y_2) + \gamma(\theta)(1 - \alpha)D(Y_1 - Y_2). \end{cases} \tag{3.5}$$

Theorem 3.2. *Assume $0 < \theta \leqslant 1$. Let $(\bar{Y}_1, \bar{Y}_2)$ be a nondegenerate equilibrium of (3.5). Then there exists $\varepsilon_4 > 0$ such that the equilibrium solution $\bar{U}(\varepsilon)$ of $(3.1)_\varepsilon$, (3.2) exists for $0 < \varepsilon \leqslant \varepsilon_4$, which satisfies*

$$\lim_{\varepsilon \downarrow 0} \bar{U}(\varepsilon) = \begin{cases} \bar{Y}_1 & \text{in } \Omega_0^L \\ \bar{Y}_2 & \text{in } \Omega_0^R \end{cases} \tag{3.6}$$

with respect to the norm $\| \cdot \|_{(L^2(\Omega_0))^2}$.

Proof. By applying the Implicit Function Theorem to $(2.5)_\varepsilon$, it is easy to show the existence of the equilibrium $(\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon))$ of $(2.5)_\varepsilon$ with $0 < \theta \leqslant 1$ satisfying $\lim_{\varepsilon \downarrow 0} (\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon)) = (\bar{Y}_1, \bar{Y}_2)$. Therefore, Theorem 2.2 says that $\bar{U}(\varepsilon) =$

$\bar{Y}_1(\varepsilon)\phi_\varepsilon^{(1)} + \bar{Y}_2(\varepsilon)\phi_\varepsilon^{(2)} + h_\varepsilon(\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon))$ is an equilibrium solution of $(3.1)_\varepsilon$, (3.2) for sufficiently small $\varepsilon > 0$. (3.6) can be easily shown by using (2.4) and $\|\|h_\varepsilon\|\|_{\varepsilon,\infty} = O(\varepsilon^{3/2})$ as $\varepsilon \downarrow 0$. □

Theorem 3.3. *Assume* $0 < \theta \leqslant 1$. *Let* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ $(\operatorname{Re}\lambda_1 \geqslant \cdots \geqslant \operatorname{Re}\lambda_4)$ *be the eigenvalues of the linearized matrix of* (3.5) *at* $(\bar{Y}_1, \bar{Y}_2)$ *and let* $\mu_\varepsilon^{(1)}, \mu_\varepsilon^{(2)}, \ldots$ $(\operatorname{Re}\mu_\varepsilon^{(1)} \geqslant \operatorname{Re}\mu_\varepsilon^{(2)} \geqslant \cdots)$ *be the spectra of the linearized operator of* $(3.1)_\varepsilon$, (3.2) *at* $\bar{U}(\varepsilon)$. *Then there exist positive constants* ε_5 *and* c *such that* $\lim_{\varepsilon\downarrow 0}\mu_\varepsilon^{(i)} = \lambda_i$ *for* $1 \leqslant i \leqslant 4$ *and* $\operatorname{Re}\mu_\varepsilon^{(i)} < -c/\varepsilon^\theta$ $(i \geqslant 5)$ *for* $0 < \varepsilon \leqslant \varepsilon_5$.

Proof. We write $(3.1)_\varepsilon$, (3.2) as follows:

$$\frac{\partial U}{\partial t} = D/\varepsilon^\theta A_\varepsilon U + F(U), \tag{3.7}$$

where $A_\varepsilon U = \mathscr{L}U$ with the boundary condition (3.2). The eigenvalue problem of the linearized equation of (3.7) at $\bar{U}(\varepsilon)$ is

$$L^\varepsilon V = ((D/\varepsilon^\theta)A_\varepsilon + F'(\bar{U}(\varepsilon)))V = \mu V, \tag{3.8}$$

where F' denotes the Fréchet derivative. By putting $J^\varepsilon = \varepsilon^\theta L^\varepsilon = DA_\varepsilon + \varepsilon^\theta F'(\bar{U}(\varepsilon))$ and $\lambda = \varepsilon^\theta\mu$, (3.8) is written as

$$(\lambda I - J^\varepsilon)U = V \quad \text{for any } V \in (L^2(\Omega_\varepsilon))^2. \tag{3.9}$$

By an argument similar to Theorem 3.5 in Ei and Mimura [6], we find that for a sufficiently small $\varepsilon > 0$, the spectral set of J^ε is contained in $\{\lambda \in \mathbb{C} \mid |\lambda| \leqslant c_1\varepsilon^\theta\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -c_2\}$ for some constants $c_1 > 0$ and $c_2 > 0$. That is, the spectral set of L^ε consists of $\sigma_1(\varepsilon)$ and $\sigma_2(\varepsilon)$, where $\sigma_1(\varepsilon) \subset \{\lambda \in \mathbb{C} \mid |\mu| \leqslant c_1\}$ and $\sigma_2(\varepsilon) \subset \{\mu \in \mathbb{C} \mid \operatorname{Re}\mu < -c_2\varepsilon^{-\theta}\}$. Let μ_ε be the spectrum in $\sigma_1(\varepsilon)$. Substituting $V = V_\varepsilon^{(1)} + V_\varepsilon^{(2)}$ with $V_\varepsilon^{(1)} = Q^\varepsilon V = V_1\omega_\varepsilon^{(1)} + V_2\omega_\varepsilon^{(2)}$ and $V_\varepsilon^{(2)} = P^\varepsilon V$ into (3.8), we obtain

$$-\lambda_\varepsilon^{(2)}\omega_\varepsilon^{(2)}DV_2 + \varepsilon^\theta Q^\varepsilon F'(\bar{U}(\varepsilon))(V_\varepsilon^{(1)} + V_\varepsilon^{(2)}) = \varepsilon^\theta\mu_\varepsilon V_\varepsilon^{(1)} \tag{3.10a}$$

and

$$DB_\varepsilon V_\varepsilon^{(2)} + \varepsilon^\theta P^\varepsilon F'(\bar{U}(\varepsilon))(V_\varepsilon^{(1)} + V_\varepsilon^{(2)}) = \varepsilon^\theta\mu_\varepsilon V_\varepsilon^{(2)}, \tag{3.10b}$$

where B_ε means $A_\varepsilon|_{P^\varepsilon(L^2(\Omega_\varepsilon))^2}$ and $\lambda_\varepsilon^{(2)}$ is the second eigenvalue of $-\Delta$ in Ω_ε with the zero-flux boundary condition. Since (3.10b) is solvable for $V_\varepsilon^{(2)}$ for sufficiently small $\varepsilon > 0$, we rewrite it as

$$V_\varepsilon^{(2)} = K_\varepsilon V_\varepsilon^{(1)}, \tag{3.11}$$

where $K_\varepsilon = \varepsilon^\theta(\varepsilon^\theta\mu_\varepsilon - DB_\varepsilon - \varepsilon^\theta P^\varepsilon F'(\bar{U}(\varepsilon)))^{-1}P^\varepsilon F'(\bar{U}(\varepsilon))$. Here we note that $K_\varepsilon \to 0$ in $(L^2(\Omega_\varepsilon))^2$ as $\varepsilon \downarrow 0$. Substituting (3.11) into (3.10a), we have

$$-(\lambda_\varepsilon^{(2)}/\varepsilon^\theta)\omega_\varepsilon^{(2)}DV_2 + Q^\varepsilon F'(\bar{U}(\varepsilon))(I + K_\varepsilon)(V_1\omega_\varepsilon^{(1)} + V_2\omega_\varepsilon^{(2)}) = \mu_\varepsilon(V_1\omega_\varepsilon^{(1)} + V_2\omega_\varepsilon^{(2)}). \tag{3.12}$$

Taking the inner production of (3.12) with $\omega_\varepsilon^{(1)}$ and $\omega_\varepsilon^{(2)}$, respectively, we know that μ_ε is an eigenvalue of the matrix

$$S_\varepsilon = \begin{pmatrix} S_{11}^\varepsilon & S_{12}^\varepsilon \\ S_{21}^\varepsilon & S_{22}^\varepsilon \end{pmatrix},$$

where

$$S_{11}^{\varepsilon} = \int_{\Omega_\varepsilon} Q^{\varepsilon}(F'(\bar{U}(\varepsilon))(I + K_\varepsilon)\omega_\varepsilon^{(1)})\omega_\varepsilon^{(1)}\, dx;$$

$$S_{12}^{\varepsilon} = -\int_{\Omega_\varepsilon} Q^{\varepsilon}(F'(\bar{U}(\varepsilon))(I + K_\varepsilon)\omega_\varepsilon^{(2)})\omega_\varepsilon^{(1)}\, dx;$$

$$S_{21}^{\varepsilon} = -\int_{\Omega_\varepsilon} Q^{\varepsilon}(F'(\bar{U}(\varepsilon))(I + K_\varepsilon)\omega_\varepsilon^{(1)})\omega_\varepsilon^{(2)}\, dx;$$

$$S_{22}^{\varepsilon} = -(\lambda_\varepsilon^{(2)}/\varepsilon^{\theta})D + \int_{\Omega_\varepsilon} Q^{\varepsilon}(F'(\bar{U}(\varepsilon))(I + K_\varepsilon)\omega_\varepsilon^{(2)})\omega_\varepsilon^{(2)}\, dx.$$

Lemma 3.3. *There is $\varepsilon_0 > 0$ such that S_ε is continuous for $\varepsilon \in [0, \varepsilon_0)$ and $J \cdot {}^tS_\varepsilon \cdot J^{-1}$ converges to the linearized matrix of (3.5) at $(\bar{Y}_1, \bar{Y}_2)$ as $\varepsilon \downarrow 0$, where*

$$J = \begin{pmatrix} 1-\alpha & -(\alpha(1-\alpha))^{1/2} \\ \alpha & (\alpha(1-\alpha))^{1/2} \end{pmatrix}. \tag{3.13}$$

Proof. The continuity of S_ε in $\varepsilon \in (0, \varepsilon_0)$ has been already shown by Hale and Vegas [8]. Therefore, it suffices to show the existence of $\lim_{\varepsilon \downarrow 0} S_\varepsilon$. Here we only prove the continuity of S_{22}^{ε} on $[0, \varepsilon_0)$, since the continuity of other terms of S_ε can be proved quite similarly. Since $\lim_{\varepsilon \downarrow 0} \omega_\varepsilon^{(2)} = \omega_0^{(2)}$ and $\lim_{\varepsilon \downarrow 0} \lambda_\varepsilon^{(2)}/\varepsilon^{\theta} = \gamma(\theta)$, it turns out from (3.6) that

$$\left| S_{22}^{\varepsilon} - \begin{pmatrix} -\lambda(\theta)d_1 + C_{11} & C_{12} \\ C_{21} & -\gamma(\theta)d_2 + C_{22} \end{pmatrix} \right| \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

where

$$C_{ij} = \alpha \frac{\partial f_j}{\partial u_i}(\bar{Y}_1) + (1-\alpha)\frac{\partial f_j}{\partial u_i}(\bar{Y}_2) \quad (i, j = 1, 2),$$

which shows the proof of this lemma. □

Thus, we know that the eigenspace corresponding to $\sigma_1(\varepsilon)$ is four dimensional and that the eigenvalues satisfy $\lim_{\varepsilon \downarrow 0} \mu_\varepsilon^{(i)} = \lambda_i$ for $1 \leqslant i \leqslant 4$ and $\operatorname{Re} \mu_\varepsilon^{(i)} < -c_2/\varepsilon^{\theta}$ $(i \geqslant 5)$ for sufficiently small ε, because $\mu_\varepsilon^{(i)}$ is in $\sigma_2(\varepsilon)$ $(i \geqslant 5)$. This gives the proof of Theorem 3.2. □

When $0 < \theta \leqslant 1$, we could find the existence and stability of equilibrium solutions of $(3.1)_\varepsilon$, (3.2) which correspond to the nondegenerate equilibria of (3.5) for sufficiently small $\varepsilon > 0$. Therefore, we may consider the limit equation (3.5) which is classified into two cases: $0 < \theta < 1$ and $\theta = 1$.

When $0 < \theta < 1$, Theorem 2.3 implies that $\gamma(\theta) = \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon^{(2)}/\varepsilon^{\theta} = 0$ so that (3.5) becomes

$$\begin{cases} \dfrac{dY_1}{dt} = F(Y_1), \\ \dfrac{dY_2}{dt} = F(Y_2), \end{cases} \quad t > 0. \tag{3.14}$$

On the other hand, when $\theta = 1$, (3.5) becomes

$$\begin{cases} \dfrac{dY_1}{dt} = F(Y_1) + \dfrac{\tau}{1-\alpha} D(Y_2 - Y_1), \\ \dfrac{dY_2}{dt} = F(Y_2) + \dfrac{\tau}{\alpha} D(Y_1 - Y_2), \end{cases} \quad (t > 0. \tag{3.15}$$

Integrating all the theorems stated in this section, we have two important facts. One is that if equilibria of (3.14) or (3.15) are nondegenerate, their existence and stability completely correspond to equilibrium solutions of $(3.1)_\varepsilon$, (3.2). The other is that when $0 < \theta < 1$ and $\theta > 1$, there is no bifurcation phenomenon when α and d are varied, so that the number of equilibrium solutions does not change with respect to α and d, while when $\theta = 1$, it may depend on α as well as d.

In the next section, we precisely discuss the dependency of α and d on equilibrium solutions of $(3.1)_\varepsilon$, (3.2) with the Gause–Lotka–Volterra dynamics for sufficiently small ε, that is, we draw the global branch of equilibrium solutions of $(3.1)_\varepsilon$, (3.2) with respect to α and d.

4. Coexistence problem of a two competing species model

In this section we apply the results obtained in the previous sections to the competition-diffusion system $(3.1)_\varepsilon$, (3.2) with the Gause–Lotka–Volterra dynamics in (1.5) and study the dependency of domain-shape on coexistence of two competing species. For simplicity only, we consider a special case that the biological environment is completely the same for two competing species, that is,

(A-5) $\quad \bar{d}_i = d, \quad R_i = R, \quad a_i = a, \quad b_i = b \ (i = 1, 2) \quad$ and $a < b$.

The last condition assures that two species can never coexist if the domain is convex.

First we discuss the nonnegativity of equilibrium solutions of $(3.1)_\varepsilon$, (3.2). We consider only the case $0 < \theta \leqslant 1$. Let $(\bar{Y}_1, \bar{Y}_2)$ with $\bar{Y}_1 = (\bar{y}_{11}, \bar{y}_{12})$, $\bar{Y}_2 = (\bar{y}_{21}, \bar{y}_{22})$ be a nondegenerate equilibrium of (3.5) and let $\bar{U}(\varepsilon) = (\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ be the corresponding equilibrium solution of $(3.1)_\varepsilon$, (3.2) constructed in Theorem 3.2. From Theorem 3.2 it immediately follows that if $\bar{Y}_1$ and $\bar{Y}_2$ are positive, then $\bar{U}(\varepsilon)$ is in the nonnegative quadrant for sufficiently small $\varepsilon > 0$, and that if at least one of the components of $(\bar{Y}_1, \bar{Y}_2)$ is negative, then $\bar{U}(\varepsilon)$ is not in the nonnegative quadrant. Therefore, we may consider that at least one of the components of $(\bar{Y}_1, \bar{Y}_2)$ is zero and the others are positive. To do it, we classify the forms of $(\bar{Y}_1, \bar{Y}_2)$ into the following five cases:

Case 1. Only one component is positive and the others are zero.
Case 2. Two components are positive and the others are zero, under which $(\bar{Y}_1, \bar{Y}_2)$ can be further classified into the following:
Case 2.1. $((0, R/a), (0, R/a))$ or $((R/a, 0), (R/a, 0))$.
Case 2.2. $((0, R/a), (R/a, 0))$ or $((R/a, 0), (0, R/a))$.
Case 2.3. $((0, 0), (R/(a+b), R/(a+b)))$ or $((R/(a+b), R/(a+b)), (0, 0))$.
Case 3. Three components are positive and the other is zero.

If $(\bar{Y}_1, \bar{Y}_2)$ is of the form of Case 2.1, it easily follows from Theorems 2.1 and 2.2 that the corresponding equilibrium solution is spatially constant in Ω_ε, so that its nonnegativity is obvious. We will consider the other four cases.

Proposition 4.1. *Assume* $0 < \theta \leqslant 1$ *and* (A3).
(i) *If* $(\bar{Y}_1, \bar{Y}_2)$ *is of Case* 1 *or Case* 2.3, *then* $\bar{U}(\varepsilon)$ *is not in the nonnegative quadrant for sufficiently small* $\varepsilon > 0$.
(ii) *If* $(\bar{Y}_1, \bar{Y}_2)$ *is of Case* 2.2 *or Case* 3, *then* $\bar{U}(\varepsilon)$ *is in the nonnegative quadrant for sufficiently small* $\varepsilon > 0$.

Proof. (i) We consider Case 1. Without loss of generality, we may take the case that $\bar{y}_{11} = \bar{y}_{12} = \bar{y}_{22} = 0$ and $\bar{y}_{21} = R/a$. Let $(\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon))$ be the equilibrium of $(2.5)_\varepsilon$ which tends to $(\bar{Y}_1, \bar{Y}_2)$ as $\varepsilon \downarrow 0$. The u_2-component of $(\bar{Y}_1, \bar{Y}_2)$ is zero, so that the second component of $(\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon))$ is zero. Therefore, Proposition A in the Appendix shows that the second component of $h_\varepsilon(\bar{Y}_1(\varepsilon), \bar{Y}_2(\varepsilon))$ is zero, which implies $\bar{u}_2(\varepsilon) \equiv 0$. We now consider the stationary problem for the first equation of $(3.1)_\varepsilon$, (3.2)

$$\begin{cases} \dfrac{d}{\varepsilon} \Delta u_1 + (R - a u_1) u_1 = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} . \tag{4.1}$$

Applying the discussion in Sect. 3 again, we know that (4.1) has at least four equilibrium solutions: $u_1(\varepsilon) \equiv 0$, $u_1(\varepsilon) \equiv R/a$, $\bar{u}_1^*(\varepsilon)$ and $\bar{u}_1^{**}(\varepsilon)$, which satisfy

$$\lim_{\varepsilon \downarrow 0} \bar{u}_1^*(\varepsilon) = \begin{cases} 0 & \text{in } \Omega_0^L \\ R/a & \text{in } \Omega_0^R \end{cases}, \qquad \lim_{\varepsilon \downarrow 0} \bar{u}_1^{**}(\varepsilon) = \begin{cases} R/a & \text{in } \Omega_0^L \\ 0 & \text{in } \Omega_0^R \end{cases}.$$

Here we note that $(\bar{u}_1^*(\varepsilon), 0)$ is the equilibrium solution of $(3.1)_\varepsilon$, (3.2) which tends to $(\bar{Y}_1, \bar{Y}_2)$ as $\varepsilon \downarrow 0$. If $\bar{u}_1^*(\varepsilon)$ is nonnegative for sufficiently small $\varepsilon > 0$, the maximum principle shows that $\max_{x \in \Omega_\varepsilon} \bar{u}_1^*(\varepsilon) \leqslant R/a$ and so we have $\Delta \bar{u}_1^*(\varepsilon) \leqslant 0$. Then, the strongly maximum principle says that either $\bar{u}_1^*(\varepsilon)$ is constant in Ω_ε or it attains its minimum at some point of $\partial\Omega_\varepsilon$ at which

$$\frac{\partial \bar{u}_1^*(\varepsilon)}{\partial n} < 0.$$

Since $\bar{u}_1^*(\varepsilon)$ is spatially inhomogeneous, we have the latter case. However it is impossible because the boundary condition is Neumann type. This shows that $\bar{u}_1^*(\varepsilon)$ should be negative at some points of Ω_ε for sufficiently small $\varepsilon > 0$.

Next, we consider Case 2.3. Without loss of generality, we may assume that $\bar{y}_{11} = \bar{y}_{12} = 0$ and $\bar{y}_{21} = \bar{y}_{22} = R/(a+b)$. Since $\bar{u}_1(\varepsilon) \equiv \bar{u}_2(\varepsilon)$ holds (see the Appendix), it suffices to study the stationary problem for the first equation of $(3.1)_\varepsilon$, (3.2)

$$\begin{aligned} &\frac{d}{\varepsilon} \Delta u + (R - (a+b)u)u = 0 && \text{in } \Omega_\varepsilon, \\ &\frac{\partial u}{\partial n} = 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

A similar argument to the above shows that any spatially inhomogeneous equilibrium solution must be negative at some points of Ω_ε for sufficiently small $\varepsilon > 0$. Other cases can be discussed in a quite similar manner. So we omit the proof.

(ii) We consider Case 2.2. Without loss of generality, we may assume that $\bar{y}_{11} = \bar{y}_{22} = R/a$ and $\bar{y}_{12} = \bar{y}_{21} = 0$. Let $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ be the corresponding equilibrium solution of $(3.1)_\varepsilon$, (3.2) which tends to $(R/a, 0)$ in Ω_0^L and $(0, R/a)$ in Ω_0^R as $\varepsilon \downarrow 0$. Consider the stationary problem for $\bar{u}_2(\varepsilon)$

$$\frac{d}{\varepsilon} \Delta\bar{u}_2(\varepsilon) + (R - b\bar{u}_1(\varepsilon) - a\bar{u}_2(\varepsilon))\bar{u}_2(\varepsilon) = 0 \quad \text{in } \Omega_0^L. \tag{4.2}$$

If $\bar{u}_2(\varepsilon)$ attains its negative minimum at some point of Ω_0^L, the second term in the left-hand side of (4.2) is positive and $\Delta\bar{u}_2(\varepsilon)$ is also nonnegative at the minimum point. This is a contradiction to that $\bar{u}_2(\varepsilon)$ is an equilibrium solution of (4.2), which implies that $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is positive in Ω_0^L. Similarly, we find that $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is also positive in Ω_0^R. We will show that $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is positive in R_ε. To do it, we assume, in addition to the assumption (A-1), that $r(x_1, \varepsilon)$ is represented by $r(x_1, \varepsilon) = \varepsilon r(x_1)$, where $r(x_1)$ is a smooth, positive and even function from $[-1, 1]$ into $\mathbb{R}$ with $r(-1) = r(1) > r(0) > 0$. Then it is known in [8] and [21] that $\|\omega_\varepsilon^{(2)}\|_{H^2(R_\varepsilon)} = O(\varepsilon^{1/2})$, and, by the Sobolev imbedding theorem, that $\|\omega_\varepsilon^{(2)}\|_{L^\infty(R_\varepsilon)} \leqslant M_1$ for some constant $M_1 > 0$. By putting $W_\varepsilon(x_1, \zeta) = \omega_\varepsilon^{(2)}(x_1, \varepsilon\zeta)$ for $(x_1, \zeta) \in R_1$, it is shown that W_ε is relatively compact in $C^2(R_1)$, and by using an argument similar to Jimbo [12, Theorem 3], that W_ε tends to some W_0 as $\varepsilon \downarrow 0$ in $C^2(R_1)$ which is independent of ζ, where $W_0(x_1)$ satisfies

$$\begin{cases} d^2W_0(x_1)/dx_1^2 = 0 \\ W_0(-1) = -(|\Omega_0^R|/(|\Omega_0|\,|\Omega_0^L|))^{1/2} \\ W_0(1) = (|\Omega_0^L|/(|\Omega_0|\,|\Omega_0^R|))^{1/2}. \end{cases} \tag{4.3}$$

As mentioned in the proof of Theorem 3.2, $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is given by

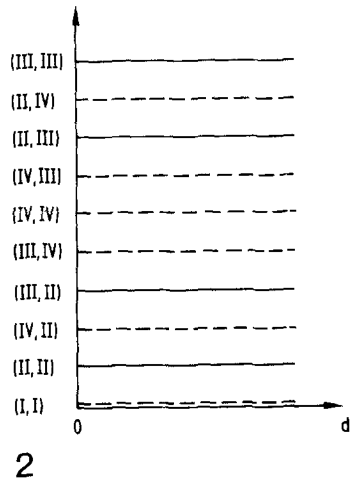
$$\begin{aligned}(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon)) &= (\bar{y}_{11}(\varepsilon)\phi_\varepsilon^{(1)} + \bar{y}_{12}(\varepsilon)\phi_\varepsilon^{(2)}, \bar{y}_{21}(\varepsilon)\phi_\varepsilon^{(1)} + \bar{y}_{22}(\varepsilon)\phi_\varepsilon^{(2)}) \\ &\quad + h_\varepsilon(\bar{y}_{11}(\varepsilon), \bar{y}_{12}(\varepsilon), \bar{y}_{21}(\varepsilon), \bar{y}_{22}(\varepsilon))\end{aligned}$$

so that (4.3) and $(\bar{y}_{11}, \bar{y}_{12}) = (R/a, 0)$, $(\bar{y}_{21}, \bar{y}_{22}) = (0, R/a)$ imply that $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is near $(R(1 - x_1)/2a,\ R(1 + x_1)/2a)$ in R_ε for sufficiently small $\varepsilon > 0$. Therefore, we know that $(\bar{u}_1(\varepsilon), \bar{u}_2(\varepsilon))$ is positive in R_ε. Since Case 3 can be also discussed in a similar way, the proof is complete. □

We next draw the global pictures of nonnegative equilibrium solutions of the limit equations of $(2.5)_\varepsilon$, taking d as a bifurcation parameter and fixing other parameters.

(i) $0 < \theta < 1$

Each of (3.14) has four equilibria (I): $(0, 0)$, (II): $(R/a, 0)$, (III): $(0, R/a)$ and (IV): $(R/(a + b), R/(a + b))$, which are all nondegenerate. Therefore, (3.14) has 16 equilibria, which are given by the combinations of (I), (II), (III) and (IV), that is, (I, I), (I, II), . . . , where (I, I) means $\bar{Y}_1 = (0, 0)$ in Ω_0^L and $\bar{Y}_2 = (0, 0)$ in Ω_0^R. Other equilibria are similarly defined. By Proposition 4.1 we know that 6 equilibrium solutions of $(3.1)_\varepsilon$, (3.2) corresponding to (I, II), (I, III), (I, IV), (II, I), (III, I) and (IV, I), are not in nonnegative quadrants. The global picture of the other 10 equilibria of (3.14) is shown in Fig. 2 when d is varied. For the stability of these equilibria, we already know that (II, II), (III, III), (II, III) and (III, II) are stable, while (I, I), (IV, IV), (II, IV), (IV, II), (III, IV) and (IV, III) are unstable.



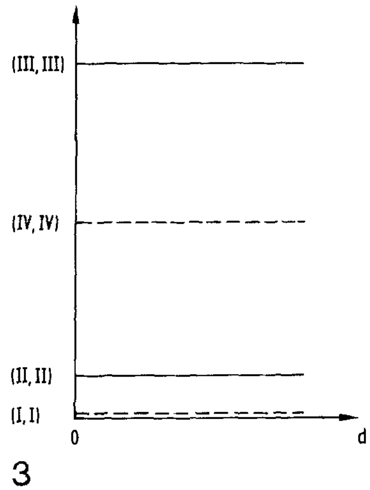


Fig. 2. The global diagram of equilibria of (3.14) with respect to d

Fig. 3. The global diagram of equilibria of $(2.5)_\varepsilon$ with $\theta > 1$ with respect to d for sufficiently small $\varepsilon > 0$

(ii) $\theta > 1$
By Theorem 3.1, we immediately find that equilibria of $(2.5)_\varepsilon$ are (I, I), (II, II), (III, III) and (IV, IV). The global picture of these equilibria is shown as in Fig. 3. Obviously, (II, II) and (III, III) are stable, while (I, I) and (IV, IV) are unstable.
(iii) $\theta = 1$
For this case, existence and stability of equilibria of (3.15) depend on α and d. Here we note that the following 6 equilibria of (3.15) are not in nonnegative quadrants:

(I, IV): $(m_1^-, m_1^-; m_2^+, m_2^+)$, (IV, I): $(m_1^+, m_1^+; m_2^-, m_2^-)$,

(III, I): $(0, n_1^+; 0, n_2^-)$, (II, I): $(n_1^+, 0; n_2^-, 0)$,

(I, III): $(0, n_1^-; 0, n_2^+)$, (I, II): $(n_1^-, 0; n_2^+, 0)$,

where

$$m_1^\pm = \frac{1}{2(a+b)}\left(R - \frac{2\tau}{1-\alpha}d \pm \left(R^2 - \frac{4\tau^2 d^2}{\alpha(1-\alpha)}\right)^{1/2}\right), \qquad n_1^\pm = \frac{a+b}{a}m_1^\pm,$$

$$m_2^\pm = \frac{1}{2(a+b)}\left(R - \frac{2\tau}{\alpha}d \pm \left(R^2 - \frac{4\tau^2 d^2}{\alpha(1-\alpha)}\right)^{1/2}\right), \qquad n_2^\pm = \frac{a+b}{a}m_2^\pm,$$

respectively, and m_i^-, n_i^- are negative for $i = 1, 2$. So, we are interested in the other 10 equilibria.

When $\alpha = \frac{1}{2}$, Mimura and Kawasaki [18] and Levin [15] already showed the global bifurcation diagram of equilibria of (3.15) with respect to d as in Fig. 4a, in which there appear the primary and secondary bifurcations of pitchfork type at $d = d^*$ and d^{**}, respectively. On the other hand, when $\alpha \neq \frac{1}{2}$, there exists

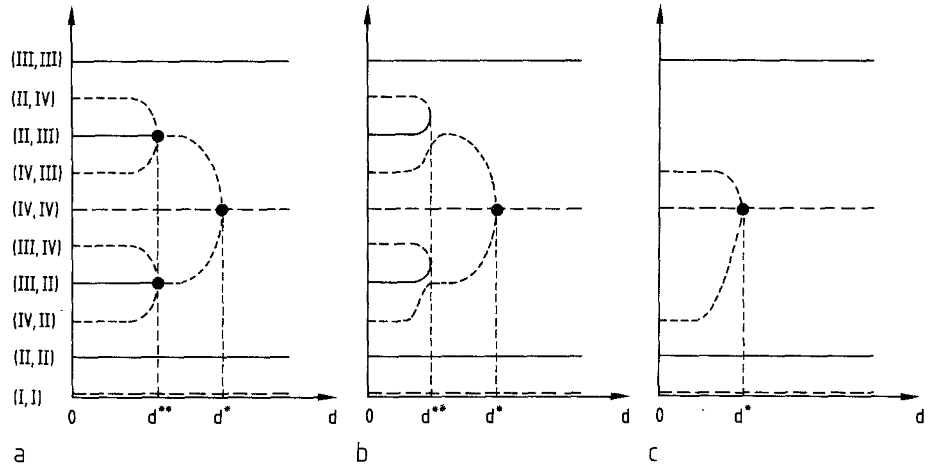


Fig. 4. The global bifurcation diagrams of nonnegative equilibria of (3.15) with respect to d when **a** $\alpha = \frac{1}{2}$; **b** $\alpha^* < \alpha < \frac{1}{2}$; **c** $0 < \alpha < \alpha^*$, where the *solid lines* denote stable equilibria and the *broken lines* denote unstable ones. ●, Bifurcation point

$\alpha^* \in (0, \frac{1}{2})$ such that when $\alpha^* < \alpha < \frac{1}{2}$, as drawn in Fig. 4b, the primary bifurcation still exists, while the secondary bifurcation is deformed as the imperfection of the symmetry case $\alpha = \frac{1}{2}$ so that two knees appear. When α is decreasing from α^*, these knees disappear as in Fig. 4c. Since the case when $\frac{1}{2} < \alpha < 1$ is similar to the above, we omit the discussion.

By Fig. 4a,b we can see that when α is not near 0 and 1, there are stable branches which tend to (II, III) or (III, II) as $d \downarrow 0$. This indicates the coexistence of two competing species.

On the other hand, when d is fixed, the global bifurcation diagrams of equilibria of (3.15) with respect to $\alpha \in (0, 1)$ can be also shown as in Fig. 5.

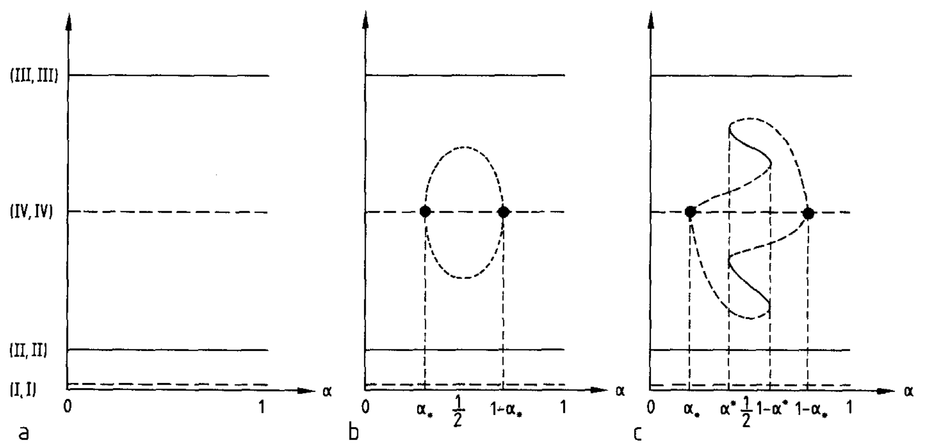


Fig. 5. The global diagrams of equilibria of (3.15) with respect to α when $\theta = 1$, and **a** d is large, **b** d is intermediate, **c** d is small, respectively, where the *solid lines* denote stable equilibria and the *broken lines* denote unstable ones. ●, Bifurcation point

Let us come back to the original problem $(3.1)_\varepsilon$, (3.2), for which the following question naturally arises: Is the global picture of equilibrium branches of (3.14) or (3.15) qualitatively inherited to those of $(3.1)_\varepsilon$ for sufficiently small ε? One easily finds that this is true for the cases when $0 < \theta < 1$ and $\theta > 1$, because all of the branches are nondegenerate. However, for the case when $\theta = 1$, there are pitchfork bifurcation points and limiting points in the solution branches as in Fig. 4. Therefore, the behavior of solution branches of $(3.1)_\varepsilon$, (3.2) in a vicinity of these degenerate points is not necessarily similar to those in Figs. 2 and 3.

We now show that there are bifurcation points in the solution branch of $(3.1)_\varepsilon$, (3.2), which correspond to those of (3.15), when d is varied. Use the transformation from (Y_1, Y_2) to (Z_1, Z_2) through $(Y_1, Y_2) = (Z_1, Z_2)J^{-1}$ for $Z_1, Z_2 \in \mathbb{R}^2$, where J is defined in (3.13), and rewrite $(2.5)_\varepsilon$ with $\theta = 1$ as

$$\begin{cases} \dfrac{dZ_1}{dt} = G_1(Z_1, Z_2; d, \varepsilon), \\ \dfrac{dZ_2}{dt} = G_2(Z_1, Z_2; d, \varepsilon), \end{cases} \tag{4.4}$$

where

$$G_1(Z_1, Z_2; d, \varepsilon) = (1-\alpha)F(Y_1) + \alpha F(Y_2) + (1-\alpha)R_1^\varepsilon(Y_1, Y_2) + \alpha R_2^\varepsilon(Y_1, Y_2),$$

$$G_1(Z_1, Z_2; d, \varepsilon) = \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} DZ_2 + (\alpha(1-\alpha))^{1/2}(-F(Y_1) + F(Y_2) - R_1^\varepsilon(Y_1, Y_2) + R_2^\varepsilon(Y_1, Y_2)).$$

Let us consider the equilibria of (4.4) instead of $(2.5)_\varepsilon$. We note that $(\mathrm{IV}, \mathrm{IV}) = (Z_1^*, Z_2^*)K^{-1}$, where $Z_1^* = [R/(a+b),\ R/(a+b)]$, $Z_2^* = (0, 0)$, and that (Z_1^*, Z_2^*) is an equilibrium of (4.4) independent of d and ε. Then $G_i(Z_1^*, Z_2^*; d, \varepsilon) = 0$ $(i = 1, 2)$ hold and $\partial G_1(Z_1^*, Z_2^*; d^*, 0)/\partial Z_1$ is represented by

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u_1} & \dfrac{\partial f_2}{\partial u_1} \\ \dfrac{\partial f_1}{\partial u_2} & \dfrac{\partial f_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -\dfrac{2a+b}{a+b}R & -\dfrac{b}{a+b}R \\ -\dfrac{b}{a+b}R & -\dfrac{2a+b}{a+b}R \end{pmatrix},$$

which has no zero eigenvalue. Thus, it turns out by the Implicit Function Theorem that there exists a unique function $Z_1 = Z_1(Z_2; d, \varepsilon)$ defined for sufficiently small $\varepsilon > 0$ and for (Z_2, d) in a neighborhood of (Z_2^*, d^*), which satisfies $G_1(Z_1(Z_2; d, \varepsilon), Z_2, d, \varepsilon) = 0$ and $Z_1(Z_2^*; d^*, 0) = Z_1^*$.

Putting $Z_1(Z_2; d, \varepsilon)$ into the second equation of (4.4), we find that $Z_2 \equiv 0$ is the equilibrium of (4.4). By cancelling Z_2, the Implicit Function Theorem shows that there exists a unique function $d = d(Z_2; \varepsilon)$ with $d(Z_2^*, 0) = d^*$ for sufficiently small $\varepsilon > 0$ such that equilibrium solutions are exactly given by two pairs (Z_1^*, Z_2^*) and $(Z_1(Z_2; d(Z_2, \varepsilon), \varepsilon), Z_2)$. A calculation by implicit differentiation shows that the Hessian of $d(Z_2; \varepsilon)$ is positive definite at $Z_2 \equiv 0$ and $\varepsilon = 0$. This means that the pitchfork bifurcation of (3.15) is inherited to $(3.1)_\varepsilon$, (3.2) for sufficiently small $\varepsilon > 0$. A similar argument can be given to the bifurcation diagram around the bifurcating points with respect to α as in Fig. 5.

The problem is whether or not limiting points appear in the solution branches of $(3.1)_\varepsilon$, (3.2). This situation is generically structurally stable so that it is inherited to the problem $(3.1)_\varepsilon$, (3.2) when ε is sufficiently small. Although we need more rigorous discussions for this conclusion, we do not touch on it here.

In summary, we can conclude as follows: for sufficiently small $\varepsilon > 0$, Fig. 2 shows that the case $0 < \theta < 1$ corresponds to the result of Matano and Mimura [16], while Fig. 3 shows that the case $\theta > 1$ corresponds to the result of Conway et al. [4]. The case $\theta = 1$ is the critical value for which there is a connection of the above two cases. Furthermore, the global pictures (including stability property) of equilibrium solutions of $(3.1)_\varepsilon$, (3.2) are almost similar to Figs. 4 and 5.

The method which we used here can be also applied to other ecological models. For prey-predator equations, for instance, the solution structures, which include not only equilibrium solution but also periodic (and aperiodic, in some situations) ones, are more complex. This is also an interesting problem from the domain-shape viewpoint.

Appendix

To understand the construction of the invariant manifold $\mathscr{K}_\varepsilon$, we show here two simple properties of h_ε which have been used in the proof of Proposition 4.1.

Let V^ε be a set defined by

$$V^\varepsilon = \{h \in \mathbb{C}(\mathbb{R}^4; P^\varepsilon((H^1(\Omega_\varepsilon))^2)) \mid |||h|||_{\varepsilon,\infty} \leqslant \beta_0 \varepsilon^{(2\theta+1)/2}, \quad |||h|||_{\varepsilon,L} \leqslant \beta_1 \varepsilon^{(2\theta+1)/2}, h(Y_1, Y_2) = 0 \text{ if } Y_1 = Y_2, Y_1, Y_2 \in \mathbb{R}^2\}, \tag{A1}$$

where

$$|||h|||_{\varepsilon,\infty} = \sup\{\|h(Y_1, Y_2)\|_{(H^1(\Omega_\varepsilon))^2} \mid Y_1, Y_2 \in \mathbb{R}^2\},$$

$$|||h|||_{\varepsilon,L} = \sup\left\{\frac{\|h(Y_1, Y_2) - h(\bar{Y}_1, \bar{Y}_2)\|_{(H^1(\Omega_\varepsilon))^2}}{|(Y_1, Y_2) - (\bar{Y}_1, \bar{Y}_2)|} \,\middle|\, (Y_1, Y_2) \neq (\bar{Y}_1, \bar{Y}_2)\right\}.$$

Let $\mathscr{R}^\varepsilon$ be an operator on V^ε defined by

$$(\mathscr{R}^\varepsilon h)(Y_1^0, Y_2^0) = \int_{-\infty}^0 \exp\{-\varepsilon^\theta D A_\varepsilon s\} P^\varepsilon F(Y_1 \phi_\varepsilon^{(1)} + Y_2 \phi_\varepsilon^{(2)} + h(Y_1, Y_2))\, ds \tag{A2}$$

for $h \in V^\varepsilon$ and $(Y_1^0, Y_2^0) \in \mathbb{R}^4$, where $A_\varepsilon = \mathscr{L}$ with the Neumann boundary condition and $(Y_1, Y_2) = (Y_1(t; Y_1^0, Y_2^0, h), Y_2(t; Y_1^0, Y_2^0, h))$ is the solution of the following equation

$$\begin{cases} dY_1/dt = (\lambda_\varepsilon^{(2)}/\varepsilon^\theta)\alpha D(Y_2 - Y_1) + F^{(1)}(Y_1, Y_2), \\ dY_2/dt = (\lambda_\varepsilon^{(2)}/\varepsilon^\theta)(1-\alpha) D(Y_1 - Y_2) + F^{(2)}(Y_1, Y_2), \\ (Y_1, Y_2)(0) = (Y_1^0, Y_2^0), \end{cases} \tag{A3}$$

where

$$F^{(1)}(Y_1, Y_2) = \left\langle F(Y_1\phi_\varepsilon^{(1)} + Y_2\phi_\varepsilon^{(2)} + h(Y_1, Y_2)), \frac{\phi_\varepsilon^{(1)}}{(1-\alpha)|\Omega_\varepsilon|}\right\rangle_{(L^2(\Omega_\varepsilon))^2},$$

$$F^{(2)}(Y_1, Y_2) = \left\langle F(Y_1\phi_\varepsilon^{(1)} + Y_2\phi_\varepsilon^{(2)} + h(Y_1, Y_2)), \frac{\phi_\varepsilon^{(2)}}{\alpha|\Omega_\varepsilon|}\right\rangle_{(L^2(\Omega_\varepsilon))^2}.$$

It is known in Fang [7] that for suitable positive constants β_0 and β_1, $\mathscr{R}^\varepsilon$ is a contraction in V^ε for sufficiently small $\varepsilon > 0$ and h_ε is the unique fixed point of $\mathscr{R}^\varepsilon$ in V^ε. Note that in our case of Sect. 4 $f_1(u_1, u_1) = f_2(u_1, u_1)$, $f_1(0, u_2) = 0$ and $f_2(u_1, 0) = 0$ hold for any $u_1, u_2 \in \mathbb{R}$. Writing $Y_1 = (y_{11}, y_{12})$, $Y_2 = (y_{21}, y_{22})$, we have the following:

Proposition A. *For the competition-diffusion system* $(3.1)_\varepsilon$, (3.2) *with the Gause–Lotka–Volterra dynamics as in* (1.5) *in which* $R_i = R$, $a_i = a$, $b_i = b$ *and* $\bar{d}_i = d$, *the following hold*:
(i) *The first* (resp. second) *component of* $h_\varepsilon(Y_1, Y_2)$ *is zero if* $y_{11} = y_{21} = 0$ (resp. $y_{12} = y_{22} = 0$).
(ii) *The first and second components of* h_ε *are equal if* $y_{11} = y_{12}$ *and* $y_{21} = y_{22}$.

Proof. For the case (i) we confine $\mathscr{R}^\varepsilon$ in the following function set

$$\bar{V}^\varepsilon = \{h = (h_1, h_2) \in V^\varepsilon \,|\, h_1 = 0 \text{ if } y_{11} = y_{21} = 0 \text{ and } h_2 = 0 \text{ if } y_{12} = y_{22} = 0\}. \quad \text{(A4)}$$

It suffices to show that $\mathscr{R}^\varepsilon$ maps $\bar{V}^\varepsilon$ into $\bar{V}^\varepsilon$. In fact, by the uniqueness of solution of (A3) and $f_1(u_1, 0) = 0$, we find that if $y_{11}^0 = y_{21}^0 = 0$ then $y_{11}(t) = y_{21}(t) = 0$. So the first component of $(\mathscr{R}^\varepsilon h)(Y_1^0, Y_2^0)$ is zero by $f_1(u_1, 0) = 0$. Similarly, if $y_{21}^0 = y_{22}^0 = 0$ we see that the second component of $(\mathscr{R}^\varepsilon h)(Y_1^0, Y_2^0)$ is zero, which completes the proof. Case (ii) can be shown in a similar way. □

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