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Inferring extinction in a declining population

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Abstract. This note describes a test for extinction in a declining population based on a record of sightings. The test assumes that, prior to extinction, the sightings follow a Poisson process with decreasing rate function. An application to a sighting record of the black-footed ferret is presented.

Key words: Sighting data – Non-stationary Poisson process – Black-footed ferret

1 Introduction

The existence of certain rare animal species is known only through occasional chance sightings. In such cases, it is possible to infer extinction from the time of the most recent sighting. Solow (1992) described a test for extinction under the assumption that, prior to extinction, the sighting times follow a stationary Poisson process. This assumption is appropriate for chronically small, but stable, populations subject to rapid extinction. However, it is inappropriate for declining populations (which are subject to elevated extinction risk). In this paper, a test for extinction in a declining population is described.

The problem considered in this paper is similar to problems arising in software reliability (e.g., Raftery 1988). The test described in this paper, however, appears to be new.

2 The test

Suppose that during the period of observation $(0, T)$ sightings occur at times $t_1, t_2, \ldots, t_n$. These sightings are assumed to arise from a non-stationary Poisson process with rate function:

$$\begin{array}{ll} \exp(a_0 + a_1 t) & 0 \leqq t \leqq T_E \\ 0 & t > T_E \end{array} \tag{1}$$

($a_1 < 0$) where the parameters of the pre-extinction sighting rate function a_0 and a_1 and the extinction time (T_{E}) are unknown. Interest centers on testing the null hypothesis H_0: $T_{\mathrm{E}} = T$ (or, equivalently, $T_{\mathrm{E}} > T$) against the alternative H_1: $T_{\mathrm{E}} < T$.

Conditional on n, the sightings represent an ordered sample from an exponential distribution with mean $1/a_1$ truncated on the right at T_{E}:

$$f(u) = a_1 \exp(-a_1 u)/(1 - \exp(-a_1 T_{\mathrm{E}})) \quad 0 \leqq u \leqq T_{\mathrm{E}} \tag{2}$$

(Cox and Lewis 1966). Let T_i be the random variable of which t_i is a realization and let $S = \sum_{i=1}^{n} T_i$ with realized value s. It follows from the results of Beg (1982) that the uniformly most powerful unbiased test of size α of H_0 against H_1 is to reject H_0 if $T_n < c_n(s)$, where the critical value is chosen to satisfy $\mathrm{prob}(T_n < c_n(s) \mid S = s) = \alpha$.

Because S is sufficient for a_1 (Beg 1982), under H_0 the conditional distribution of T_n given $S = s$ is the same as the distribution of the largest gap in $n - 1$ points independently uniformly distributed on $(0, s)$ such that the largest gap does not exceed T. It follows from the results of Fisher (1929) on the distribution of the largest gap that under H_0:

$$\mathrm{prob}(T_n \leqq t_n \mid S = s) = F_s(t_n)/F_s(T) \tag{3}$$

where:

$$F_s(u) = 1 - \sum_{i=1}^{[1/y]} (-1)^{i-1} \binom{n}{i} (1 - iy)^{n-1} \tag{4}$$

where [ ] denotes the integer part and $y = u/s$. The significance level of the observed value of T_n is given by (3).

3 Power considerations

By the same argument that led to (3), the conditional power of the test given $S = s$ is:

$$F_s(\min(c_n(s), T_{\mathrm{E}}))/F_s(T_{\mathrm{E}}) = \begin{array}{ll} 1 & 0 \leqq T_{\mathrm{E}} \leqq c_n(s) \\ \alpha F_s(T)/F_s(T_{\mathrm{E}}) & c_n(s) < T_{\mathrm{E}} \leqq s \,. \end{array} \tag{5}$$

The test has low power if T_{E} is close to T and also if T_{E} is large compared to S. The latter condition holds if, prior to extinction, the population has declined to a level at which the sighting rate is close to 0. Paradoxically, it is easier to detect extinction in a common species than in a rare species.

The unconditional power of the test can be found by integrating (5) over the probability density function of S. The distribution of S (which naturally depends on a_1) is known (e.g., Bartholomew 1963), but it is complicated. Bain et al. (1977) gave an approximation based on the beta distribution. As an alternative, the unconditional power was estimated by simulation. Specifically, 10,000 samples of size n were generated from (2) for selected values of n, $a_1 T$, and $a_1 T_{\mathrm{E}}$ and the test was applied. The results are summarized in Table 1. Provided extinction occurs before the sighting rate has fallen too low, the test has reasonable power even when n is as small as 15.

Table 1. Estimated power based on 10,000 simulated realizations for selected values of n, $a_1 T_E$, and $a_1 T$

$a_1 T_E$	$a_1 T$	$n = 10$	$n = 15$	$n = 20$
0.5	0.5	0.052	0.049	0.051
	1.0	0.439	0.680	0.851
	1.5	0.455	0.702	0.868
	2.0	0.459	0.706	0.863
	2.5	0.469	0.715	0.863
1.0	1.0	0.048	0.051	0.050
	1.5	0.248	0.428	0.595
	2.0	0.309	0.504	0.694
	2.5	0.322	0.519	0.698
	3.0	0.324	0.523	0.707
1.5	1.5	0.050	0.047	0.053
	2.0	0.140	0.212	0.312
	2.5	0.189	0.304	0.446
	3.0	0.211	0.337	0.487
	3.5	0.216	0.351	0.498
2.0	2.0	0.053	0.051	0.050
	2.5	0.084	0.118	0.162
	3.0	0.121	0.186	0.261
	3.5	0.143	0.207	0.300
	4.0	0.144	0.222	0.318

Table 2. Dates of sightings of the black-footed ferret in Wyoming, January 1972–December 1990

Year	Month					
1972	6	7	8	10		
1973	5	6	7	8	9	10
1974	6	7				
1975	5	8	10			
1976	5	9	10			
1977	6					
1978						
1979	6					
1980						
1981	9	10				
1982	2	3	7			
1983	7					
1984	7	9				

4 Example

The US Department of the Interior maintains a record of sightings of the black-footed ferret. In this section, the test described in Sect. 2 is applied to the record of confirmed sightings in the State of Wyoming over the period January 1972 to December 1990. Unit time was taken to be 1 month. For this record, which is given in Table 2, $n = 28$, $T = 229$, $T_n = 153$, and $S = 1714$. The construction of

Table 2 from original documents required some judgement, so these data are approximate.

The rate of sightings appears to have declined from the beginning of the observation period, so the model outlined in Sect. 2 was adopted over the model in which the pre-extinction sighting rate is constant. The test described in Sect. 2 was applied and the significance level of the observed value of T_n was 0.050. Under conventional notions of statistical significance, the record provides moderately strong evidence against the null hypothesis that extinction has not occurred.

5 Discussion

The test described in this paper makes minimal use of biological information. If information about the population dynamics of the species of interest is available, then it should be used in inferring extinction. However, for occult species, the sighting record may be the only information available on which to base such inference.

In addition to a classical test for extinction, Solow (1993) considered a Bayesian approach when the pre-extinction sighting rate is constant. A Bayesian approach is also possible in the case of a declining sighting rate (e.g., Raftery and Akman 1986). Such an approach would, of course, be particularly useful when there is specific prior information about the time of extinction.

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