ON PRACTICAL REPRESENTATION OF MAGNETIC FIELD

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Abstract. Various manners of determination of a magnetic field are reviewed briefly from the standpoint of practicality and uniqueness. Then a practical representation of magnetic fields in terms of a class of force-free magnetic field is described. The proposed scheme is based on the physical consideration that in the chromosphere and lower corona a quasistatic magnetic field must be nearly force-free and that for the class of force-free magnetic field, i.e., $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ with $\alpha =$ constant, the magnetic field can be determined uniquely from the observed distribution of the vertical component of a magnetic field. The applicability of the representation is demonstrated by examples and the limitations are discussed.

1. Introduction

The importance of the magnetic field configuration has been illustrated in a number of astrophysical problems, such as pulsars (Goldreich and Julian, 1969; Ostriker and Gunn, 1969), magnetic stars (e.g. Mestel, 1967), stellar winds (Mestel, 1968a, b; Mestel and Selley, 1970) and many others. In solar physics, many examples of the magnetic field configurations have been proposed in connection to specific models of sunspots (Schlüter and Temesvary, 1958; Chitre, 1963; Deinzer, 1965; Yun, 1971; Simon and Weiss, 1970), the solar wind (Weber and Davis, 1967), prominences (Kippenhahn and Schlüter, 1957; Rust and Roy, 1971), coronal structures (Altschuler and Newkirk, 1969; Pneuman and Kopp, 1971), as well as chromospheric structures (Schatzman, 1961; Nakagawa *et al.*, 1971, hereafter referred to as Paper I; Raadu and Nakagawa, 1971, referred to as Paper II). The manner of determination of the magnetic field in these studies can be classified into the following three categories: (1) the potential (current-free) magnetic field, (2) the force-free magnetic field, and (3) the magnetostatic equilibrium field.

We may review briefly these various manners of determination in terms of the practical applicability and unique representation of a magnetic field on the basis of observation. A magnetic field $\mathbf{B}$ is characterized by the solenoidal condition, $\nabla \cdot \mathbf{B} = 0$, which leads to the general representation by a vector potential $\mathbf{A}$, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. It is known that a unique representation then requires the choice of a 'gauge' for $\mathbf{A}$, for example the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$. The magnetic field at a point $\mathbf{r}$ is given by

$$\mathbf{B}(\mathbf{r}) = \int \nabla' \times \left[\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right] \mathrm{d}\mathbf{r}', \tag{1}$$

which follows by solving $\mathbf{A}$ through the equation for the current density $\mathbf{J}$, i.e.,

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$\nabla^2 \mathbf{A} = 4\pi \mathbf{J}$. In Equation (1), the differentiation and integral refer to the coordinate of the volume current at $\mathbf{r}'$. Equation (1) illustrates the basic difficulty of the determination of a magnetic field in a medium carrying an electric current, namely, that a magnetic field cannot be determined unless the electric current distribution is known.

The potential field representation avoids this difficulty by placing, in essence, the electric current on the boundary surfaces. Then the determination of the magnetic field is replaced by a mathematically straightforward boundary value problem for the potential (Schmidt, 1964; Rust and Roy, 1971; Altschuler and Newkirk, 1969). On the other hand in the determination of a magnetic field through the magnetostatic equilibrium, the distribution of the electric current is deduced from the requirement of equilibrium (Schlüter and Temesvary, 1958; Chitre, 1963; Deinzer, 1965; Yun, 1970, 1971; Simon and Weiss, 1970; Pneuman and Kopp, 1971). This manner of determination is realistic, but requires an elaborate numerical solution for each specific set of the physical parameters, such as the distribution of pressure, density, temperature, opacity, as well as velocity. Consequently, this method has been confined mostly to the construction of theoretical models rather than solving the practical boundary value problem. In this view, the force-free magnetic field approach is an intermediate approximation, as the electric current is given by the magnetic field through the force-free condition ($\mathbf{J} \parallel \mathbf{B}$), i.e.,

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{2}$$

where α is in general a scalar function of coordinates. It is evident in Equation (2) that $\alpha = 0$ corresponds to the potential and that α is a constant along a magnetic line of force; the latter follows from the solenoidal condition by taking the divergence of the equation. For a general spatial function of α, however, Equation (2) leads to a mathematical problem more complex than the magnetostatic case.

The boundary value problem is discussed by Grad and Rubin (1958). There are intrinsic difficulties in specifying the boundary condition from observations. Natural boundary conditions would fix the end points of field lines and would depend on the past history of photospheric motions (Schmidt, 1966, 1968). If the current on the boundary is used this can only be prescribed over the region of positive flux or over the region of negative flux (Schmidt, 1968), as the value of α must be the same at both ends of a field line. Hence the possible application to the practical boundary value problem is confined to the case of $\alpha =$ constant.

In the solar atmosphere an examination of the force balance readily revelas the dominance of the magnetic field in the chromosphere and lower corona (Sturrock and Woodbury, 1967). A negligible departure from a force-free structure is sufficient to balance all non-magnetic forces. Thus for a quasistatic magnetic field in these layers it is plausible to assume the prevalence of a force-free magnetic field. Further it has been shown in Papers I and II, that reasonable topological similiarities can be obtained between the Hα observed features and the force-free magnetic field for an appropriate constant value of α.

Therefore, in this paper, we describe a generalized and practical representation of a constant α force-free magnetic field suitable to solve the boundary value problem. Although limited by the assumption of constant α, the present formulation can provide practical means of comparison of the features observed in Hα filtergrams and the configuration of a magnetic field, including the possible estimate of the energy content of the magnetic field. The applicability of the present formulation is demonstrated through the examples of the topological comparison of the Hα features and the local configuration of the magnetic field similar to those considered in Papers I and II. The limitations and possible improvements of the representation are also discussed.

2. Theoretical Formulations

It is known that a magnetic field can be specified by two scalar functions, as the requirement of the choice of a gauge for a vector potential and the solenoidal condition imply (Lüst and Schlüter, 1954; Chandrasekhar, 1961). We shall, therefore, represent a magnetic field in terms of two arbitrary scalar functions and write for a system of Cartesian coordinates,

$$\mathbf{B} = \nabla \times \nabla \times (P\mathbf{l}_z) + \nabla \times (T\mathbf{l}_z) = \\ = \left(\frac{\partial^2 P}{\partial x \partial z} + \frac{\partial T}{\partial y}\right)\mathbf{l}_x + \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial T}{\partial x}\right)\mathbf{l}_y - \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)\mathbf{l}_z, \tag{3}$$

where P and T are the two arbitrary scalar functions, and $\mathbf{l}_x$, $\mathbf{l}_y$ and $\mathbf{l}_z$ are the unit vectors in the three principal directions.

For a given magnetic field $\mathbf{B}$, the functions P and T are not unique. Suppose that $(P+\phi)$ and $(T+\theta)$ also give the same magnetic field $\mathbf{B}$. Then from Equation (3)

$$\nabla \times \nabla \times (\phi\mathbf{l}_z) + \nabla \times (\theta\mathbf{l}_z) = 0. \tag{4}$$

The vertical component of this equation is,

$$\nabla_H^2 \phi = 0, \tag{5}$$

where the subscript H indicates that only derivatives with respect to x and y are taken. The solution to this equation is in general

$$\phi = \mathcal{R}e\left(f(x+iy, z)\right) \tag{6}$$

where f is an analytic function of the variable $(x+iy)$. The horizontal component of Equation (4) is

$$\nabla_H \theta \times \mathbf{l}_z + \nabla_H \frac{\partial \phi}{\partial z} = 0. \tag{7}$$

Now $\partial\phi/\partial z$ is also an analytic function of $(x+iy)$ and by Equation (7) θ is an orthogonal function in the xy plane. Hence,

$$\theta = -\mathcal{I}m\left(\frac{\partial f}{\partial z}(x+iy, z)\right). \tag{8}$$

Thus the choice of P and T for a given magnetic field $\mathbf{B}$ is arbitrary to within the choice of a function $f(x+iy, z)$ analytic in $(x+iy)$. In particular we notice that the choice of z dependence for f is quite arbitrary. Also from condition (7) it follows that

$$\nabla_H^2 \theta = 0. \tag{9}$$

Both θ and ϕ are harmonic functions of x and y.

From Equation (3) the curl of the magnetic field $\mathbf{B}$ is given by

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times T\mathbf{l}_z) + \nabla \times (-\nabla^2 P\mathbf{l}_z) \tag{10}$$

We now consider the special case of the force-free condition (2) for which α is a constant. In this case Equation (2) may be written in terms of poloidal and toroidal components,

$$\nabla \times \nabla \times \{(T - \alpha P)\,\mathbf{l}_z\} + \nabla \times \{(\alpha T - \nabla^2 P)\,\mathbf{l}_z\} = 0. \tag{11}$$

By comparison with Equation (4) we see that the general solution to Equation (11) is,

$$T - \alpha P = \phi \tag{12}$$

and

$$\alpha T - \nabla^2 P = \theta, \tag{13}$$

where ϕ, θ are as defined by Equations (6) and (8). For the particular choice of the arbitrary function, $f=0$, the equations reduce to

$$T = \alpha P. \tag{14}$$

$$\nabla^2 P = -\alpha^2 P. \tag{15}$$

The choice of a constant α force-free field leads to particularly simple equations for P and T. In general for variable α, the force-free Equation (2) cannot be immediately written in poloidal and toroidal components and the analysis does not proceed in the simple way outlined above. The force-free Equation (11) breaks up into a poloidal and toroidal part which are each separately equal to zero, only for the particular choice of a solution given by Equations (14) and (15).

In seeking the solution P, we note that the observation can provide the boundary condition in terms of the horizontal distribution of the line of sight component, say, the vertical component $B_z(x, y)$, of the magnetic field at the level of the atmosphere, say $z=0$. This suggests readily the separation of the horizontal and the vertical variables in the solution P. Further by demanding the solution to be bounded for $z\to\infty$, we find the general solution of Equation (15) can be written in the form

$$P = \sum_{\mathbf{k}\neq 0} \frac{1}{k^2} B_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x} - (k^2-\alpha^2)^{1/2} z}, \tag{16}$$

where $\mathbf{k}=k_x+\mathbf{l}_x+k_y\mathbf{l}_y$, $k^2=k_x^2+k_y^2$, $\mathbf{x}=x\mathbf{l}_x+y\mathbf{l}_y$, and $\mathbf{k}\neq 0$ excludes $k_x=k_y=0$. In Equation (16) B_k's are the Fourier coefficients of the observed $B_z(x, y, z=0)$, i.e.,

$$B_z(x, y, z=0) = B_{00} + \sum_{k\neq 0} B_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{17}$$

where B_{00} denotes the value of $B_z(z=0)$ averaged over the domain of the Fourier expansion. It should be noted that in the present formulation the solenoidal condition $\nabla \cdot \mathbf{B}=0$ is strictly satisfied, thus the domain of analysis must be chosen so that B_{00} should be zero or negligible in comparison with other coefficients. With the solution P given in Equation (16), the components of the magnetic field become from Equation (3),

$$B_x = \sum_{k \neq 0} \frac{i}{k^2} \left[\alpha k_y - k_x (k^2 - \alpha^2)^{1/2}\right] B_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - (k^2-\alpha^2)^{1/2} z}, \tag{18}$$

$$B_y = \sum_{k \neq 0} \frac{-i}{k^2} \left[\alpha k_x + k_y (k^2 - \alpha^2)^{1/2}\right] B_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - (k^2-\alpha^2)^{1/2} z}, \tag{19}$$

$$B_z = \sum_{k \neq 0} B_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - (k^2-\alpha^2)^{1/2} z}. \tag{20}$$

For the topological comparison, we must evaluate the configuration of the magnetic lines of force. This is achieved by evaluating the successive spatial coordinates $x_i (i=x, y, z)$ of a magnetic line of force in terms of the arc length s. Then, for a specific line of force, the successive coordinates x_i can be given by

$$x_i (s + \Delta s) = x_i (s) + \frac{\mathrm{d}x_i}{\mathrm{d}s} \Delta s + \frac{1}{2!} \frac{\mathrm{d}^2 x_i}{\mathrm{d}s^2} (\Delta s)^2 + \cdots. \tag{21}$$

The direction cosines $\mathrm{d}x_i/\mathrm{d}s$ are

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = \frac{B_i (s)}{B (s)}, \tag{22}$$

where $B(s)=\left[\sum_i B_i^2 (s)\right]^{1/2}$. Equation (22) follows by eliminating a function $f(s)$ between the equation of the magnetic line of force,

$$\frac{\mathrm{d}x}{B_x (s)} = \frac{\mathrm{d}y}{B_y (s)} = \frac{\mathrm{d}z}{B_z (s)} = f (s),$$

i.e.,

$$\mathrm{d}x_i = f (s) B_i (s), \tag{23}$$

and the defining equation

$$\mathrm{d}s^2 = \sum_i \mathrm{d}x_i^2. \tag{24}$$

3. Physical and Topological Characteristics of Solution

It follows from Equation (2) that the parameter α is related to the magnitude as well as the direction of the electric current density. Topologically, a large value of α induces a strong twist of a magnetic field line, with the direction of twist depending on the sign α. Some additional physical as well as topological characteristics of the solution follow from Equation (18)–(20). First the vertical dependence of

$\exp\left[-(k^2-\alpha^2)^{1/2}z\right]$ imposes a condition on the maximum value of $\alpha(\alpha_{max})$ in terms of the minimum value of $k(k_{\min})$. Namely, for a physically realizable solution we must have $\alpha_{max}<k_{min}$. The topological consequence of this requirement is that for a given domain $(k_{\min})$, the small scale (i.e. large k) magnetic features would appear like those of the potential field ($\alpha=0$), as $\alpha_{\max}$ can be neglected in comparison with large $k \gg k_{\min}$. In other words, in the present representation only features of large scale lengths could appear to be affected strongly by the variation of the value of parameter α, and those affected features are associated with the magnetic lines of force reaching the height comparable to the horizontal scale. These topological characteristics seem to be in agreement with observations. Foukal (1971), showed that newly emerged small scale features always resemble the potential field configuration, while Saito and Billings (1964) showed the presence of a complex twist for a long-lived large scale coronal magnetic field.

To be more specific, let us consider the topology of the solution given by a single term in the Fourier series, say a representative horizontal wave number k_0. Then the vertical dependence $\beta=(k_0^2-\alpha^2)^{1/2}$ becomes a constant, and we can rewrite Equation (3) in the form

$$\mathbf{B} = \alpha(\nabla_H P) \times \mathbf{1}_z - \beta(\nabla_H P) - k_0^2 P \mathbf{1}_z, \tag{25}$$

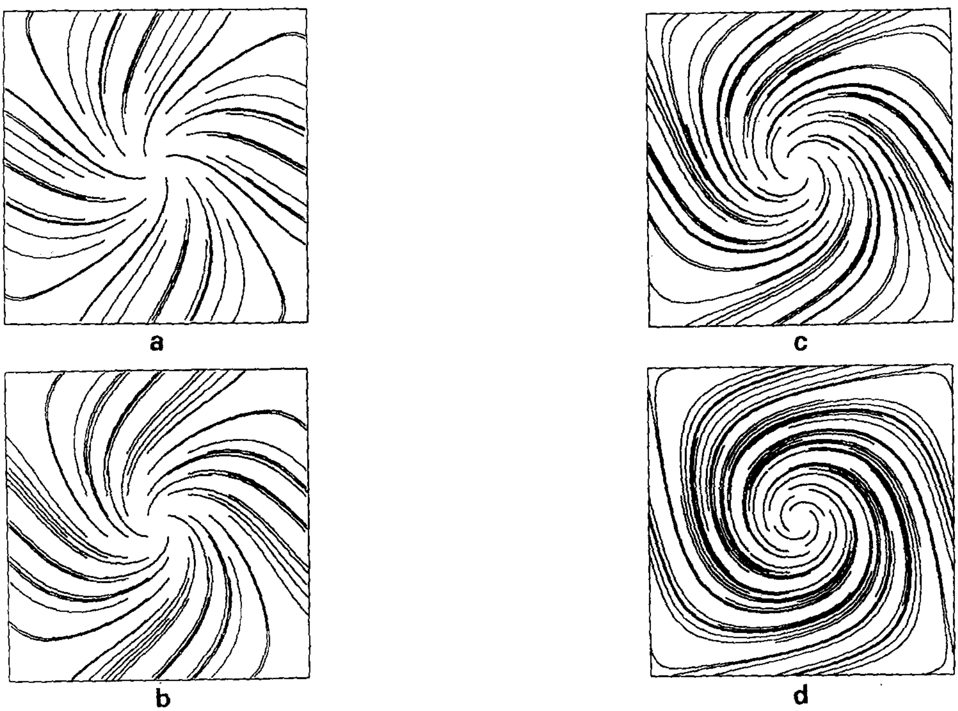

Fig. 1. The topological comparison of Hα observation (Big Bear Solar Observatory, 9 September 1970) and the lines of force of force-free magnetic fields for various twist angles γ; a, $\gamma=75°$, b, $\gamma=60°$, c, $\gamma=45°$ and d, $\gamma=30°$.

where

$$\nabla_H P = \frac{\partial P}{\partial x}\mathbf{l}_x + \frac{\partial P}{\partial y}\mathbf{l}_y, \tag{26}$$

$\nabla_H P$ is a vector perpendicular to the contour of B_z (i.e. P=const) and $(\nabla_H P)\times \mathbf{l}_z$ is a vector tangential to the contour. Thus the twist angle γ by which a magnetic line of force intersects the contour of B_z is given by

$$\tan\gamma = \frac{\beta}{\alpha}, \quad \text{i.e.,} \quad \alpha = k_0 \cos\gamma. \tag{27}$$

Note $\alpha=0$ corresponds $\gamma=\pi/2$, and the twist angle is independent of z. The latter topological character is similar to those given by the simplest solutions in Papers I and II.

In Figures 1 and 2, the magnetic field configurations given by such a single term solution are shown with Hα observations (courtesy of the Big Bear Solar Observatory) for topological comparison. The example considered in Figure 1 for a unipolar sunspot is given by $P=\cos x \cos y$ $(-\pi/2 \leqslant x \leqslant \pi/2;\ -\pi/2 \leqslant y \leqslant \pi/2)$ with $\pi/2-\gamma=15°, 30°, 45°, 60°$, and this illustration is comparable to Figure 1 of Paper 1. The example considered in Figure 2 is given by $P=\sin 2x\cos y(-\pi/2\leqslant x<\pi/2;\ -\pi/2\leqslant y(\pi/2))$ and

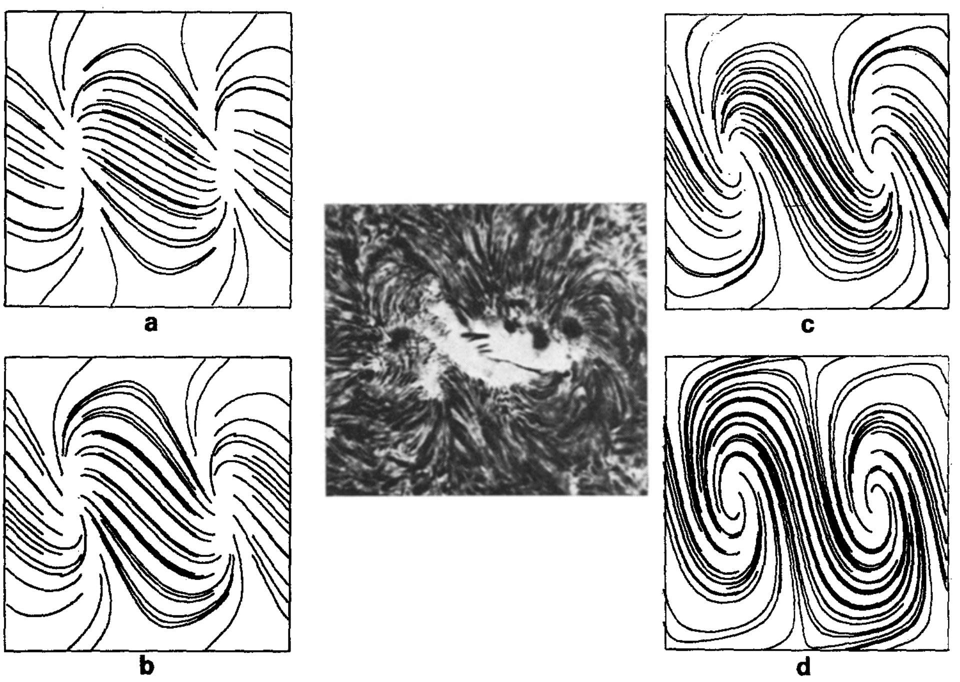

Fig. 2. The topological comparison of Hα observation (Big Bear Solar Observatory, 27 May 1970) and the lines of force of force-free magnetic field for various twist angle γ; a, $\gamma=75°$, b, $\gamma=60°$, c, $\gamma=45°$ and d, $\gamma=30°$.

this example is comparable to Figure 6 of Paper II for a pair of bipolar sunspots of equal strength. Again $\pi/2-\gamma=15°$, 30°, 45°, and 60° are considered. In both figures, we find that a number of magnetic lines of force can be superposed exactly on the observed Hα features. These examples thus demonstrate clearly the applicability of the present representation.

4. Discussions and Remarks

In the practical application of the present formulation, perhaps the physically most significant quantity is the magnetic energy content M within the volume of analysis which is given by

$$M = \int_v \frac{|\mathbf{B}|^2}{8\pi} \mathrm{d}\mathbf{r} = \frac{A}{64\pi} \sum_{\mathbf{k}\neq 0} \frac{B_{\mathbf{k}} B_{\mathbf{k}}^*}{(k^2 - \alpha^2)^{1/2}}, \tag{28}$$

where A denotes the surface area of analysis and $B_{\mathbf{k}}^*$ the complex conjugate of $B_{\mathbf{k}}$. Since $\alpha=0$ for a potential field, it is clear from Equation (28) that the magnetic energy content increases with the value of α. Hence examining the variation of α, it is possible to learn the growth or decay of the magnetic field. In Paper I, it was shown that a slight change of α can provide the magnetic energy sufficient for a solar flare. Also in Paper II it was shown that the loop prominences formed after flares with small value of α. It should be noted that in Equation (28), the quantity $1/(k^2-\alpha^2)^{1/2}$ denotes the characteristic scale height for a specific wave number k. In other words, the increase of the magnetic energy content with increasing values of α is, in essence, due to the increase of the effective volume of integral for each specific wave number k. This last point may be significant as a larger volume is usually associated with a larger flare.

The limitation of the present formulation can be examined in terms of the constancy of α over a domain as well as the presence of a non force-free field. Both of these can be checked through the following equality,

$$\frac{\dfrac{\partial B_z}{\partial y} - \dfrac{\partial B_y}{\partial z}}{B_x} = \frac{\dfrac{\partial B_x}{\partial z} - \dfrac{\partial B_z}{\partial x}}{B_y} = \frac{\dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}}{B_z} = \alpha, \tag{29}$$

which follows from Equation (2). Equation (29) indicates that the possible test of the presence of a non force-free magnetic field requires the measurements of all components of the magnetic field at two different levels in the solar atmosphere. The test of constant α can be achieved, from the last equality of Equation (29) if a vector magnetograph observation is available at a certain level of the atmosphere. However, for a non-constant α, there is no general formulation, thus in such a case, the value of α for the domain should be chosen by averaging.

Finally it must be stated that in the topological comparison of the magnetic field and the Hα observation, the observed Hα features must be interpreted with careful considerations of the radiative transfer which produces the observed contrast. Nevertheless, as discussed in the Introduction, the dominance of the magnetic field in the solar atmosphere strongly suggests the presence of a force-free magnetic field in the chromosphere and lower corona. Thus the present formulation could be used for a quantitative analysis of the magnetograph and Hα observations, including the potential magnetic field.

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