A ROSSBY-WAVE DYNAMO FOR THE SUN, I*

PETER A. GILMAN

Dept. of Astro-Geophysics, University of Colorado, Boulder, Colo., U.S.A.

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Abstract. There is increasing interest in the possible existence of large horizontally flowing eddies or 'Rossby waves' in the sun's convection zone and photosphere. We present here and in Part II a mathematical model which shows that flows of this type, driven by an assumed latitudinal temperature gradient, can act as hydromagnetic dynamos to induce magnetic fields that periodically reverse.

In this part, we discuss the assumptions for the model, review earlier linear analyses that demonstrate the ability of Rossby waves to induce solar-like magnetic fields, and finally derive the non-linear equations that govern the model. The analysis is simplified by confining the fluid and magnetic fields to a thin rotating annulus. The flow is taken to be nearly incompressible, heliostrophic and hydrostatic. Induced magnetic fields are allowed to react upon the inducing motions. Transports of momentum and magnetic flux by smaller scale convective motions, and the transport of heat by these motions and radiation, are parameterized by diffusion coefficients. The solar convection is also assumed to be responsible for the latitudinal temperature gradient.

1. Introduction

Recently several workers have suggested that there exist in the solar photosphere and convection zone motions of much larger size than supergranules or sunspots, say 10^5 km or larger. For example, Ward (1964, 1965a, b) has postulated that these motions comprise in essence a 'Rossby type' general circulation in which nearly horizontally flowing waves or eddies carry angular momentum toward the equator from higher latitudes to maintain the equatorial acceleration. He supports this view with extensive statistical analysis of sunspot motions. There are some difficulties in interpretation of the sunspot statistics however (see Ward, 1965b), and it is desirable to have corroborative evidence, of which there is some. For example, Starr and Gilman (1965) have pointed out that the size and shape of large scale bipolar magnetic regions on the sun are consistent with Ward's picture. Also, Plaskett (1966) has deduced the existence of Rossby waves from Doppler shift measurements, but, as Ward (1967) has pointed out, his data sample is too small for conclusive results.

On the other hand, Bumba *et al.* (1964), Howard (1967) and Simon and Weiss (1968), citing the structure of bipolar magnetic regions, have suggested that the motions may instead be giant convective cells (super-supergranules). It is also possible that the motions are a combination of Rossby wave and convective cell.

We present here and in Part II a mathematical model demonstrating the hydromagnetic effects of Rossby type motions, which we assume to be driven by a latitudinal temperature gradient. We find that the model gives us many of the properties of a solar cycle, particularly magnetic field reversals. A short qualitative account of a model very similar to that described below has been given in Gilman (1968).

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In Part I, we discuss the numerous physical assumptions made for the model, qualitative results of earlier calculations, and derive the basic equations governing the model. In Part II, we simplify these equations to a more tractable form, present results of numerical integrations of the resulting system and compare these results to the solar cycle.

2. Definition of a Rossby Wave

For present purposes, we shall consider a Rossby wave to be a nearly, but not entirely, horizontal wavy or eddying flow pattern in a rotating fluid in which Coriolis forces nearly balance horizontal pressure forces (the so-called heliostophic balance). It is important that the motion not be entirely horizontal, for at least two reasons. First, the vertical motions, while small compared to the horizontal flow, will play an important role in generating kinetic energy to sustain the horizontal motion against frictional dissipation. Second, purely horizontal motions cannot give us the stretching and twisting of magnetic field lines we need to simulate a solar cycle.

It is worth noting that our definition of a Rossby wave is more general than in C. G. Rossby's original work, which he applied to planetary-scale flow in the earth's atmosphere. Plaskett (1966) has discussed this early Rossby model in the context of the sun. It describes the propagation of small-amplitude vorticity conserving purely horizontal oscillations in a rotating spherical shell of homogeneous fluid. All of the waves would propagate in longitude toward the east limb on the sun relative to any basic zonal current, e.g., the differential rotation, at a rate dependent on their wavelength and on the variation with latitude of the vertical component of rotation. However, the original Rossby model, while of fundamental importance, was only a beginning, in that it is basically only a kinematic model. That is, no means of excitation for the wave is included, nonlinear and dissipative effects are ignored as are important effects of vertical and horizontal variations in temperature and density, and vertical motions. When these effects are included, as has been done recently in more sophisticated models for describing planetary scale flow in the earth's atmosphere, the kinematic properties found by Rossby tend to be overshadowed by the more important dynamical effects. Our model will contain these effects.

3. Assumptions for the Model

A. LATITUDINAL TEMPERATURE GRADIENT

In the earth's atmosphere, Rossby waves as we have defined them above arise in response to latitudinally non-uniform solar heating. From initially small perturbations on the existing wind pattern, they grow to finite-amplitude disturbances which transport heat from warm to cold latitudes. At the same time, they transport angular momentum into middle latitudes to maintain the mean 'jet stream'. Ward (1964, 1965a) supposed that the Rossby waves which he postulated for the transport of angular momentum into the equatorial acceleration (the sun's 'jet stream') are also thermally driven by an existing latitudinal temperature gradient.

Unfortunately, the observational evidence for a temperature difference in latitude on heliopotential surfaces is conflicting (see, e.g., Beckers, 1960). Perhaps stronger latitudinal gradients exist inside the convection zone where they can not be seen. We know such gradients must exist if local rotation significantly influences the outward heat transport by the solar convection, since the relative orientation of rotation and gravity varies with latitude. In this regard, Chandrasekhar (1953) has shown the inhibiting effect of rotation on the onset of convective instability; Weiss (1964) has estimated its inhibitive effect on the heat transport of fully developed cells, and Roxburgh (1967) has applied his results to the solar convection zone to estimate the latitude gradient. In addition, Durney (1968a, b) has demonstrated the inhibition of convective rolls preferentially near the equator of a convecting spherical shell. For the present paper, we shall *assume* that a latitudinal gradient large enough to excite Rossby waves exists in the convection zone of the sun and examine the hydromagnetic effects of the waves that would arise.

It should be pointed out here that Rossby waves are not the only possible response of the fluid to a latitudinal temperature gradient. We can also get a purely axisymmetric circulation in meridian planes, coupled with a zonal circulation. Generally speaking, for a given temperature difference, the faster the basic rotation, the more likely we are to get a Rossby wave rather than a symmetric meridian circulation, because the symmetric circulation becomes unstable to wave-like disturbances. However, which circulation occurs depends also on the eddy viscosity and other parameters which are not well known for the sun, so that it is not possible to predict in advance the type of motion to expect. From the point of view of dynamo action, however, the steady symmetric circulations are much less interesting, because by Cowling's theorem they cannot by themselves act as a dynamo.

B. GEOMETRY, COMPRESSIBILITY AND BOUNDARY CONDITIONS

Rather than attempt to construct a Rossby wave model that can be applied in detail to the sun, it seems wise at this early stage instead to make a hydromagnetic generalization of a model already well studied and understood in the geophysical context. For example, we shall deal with a perfect gas of uniform composition. The thermodynamic effects of partial ionization will be ignored. In addition, rather than deal with a rotating spherical shell of gas, we shall simplify the problem geometrically and consider instead a thin annulus rotating about its axis (see Figure 1). Gravity, g, (assumed constant) acts vertically downward on the fluid in the annulus. The annulus is thin in two respects. First, the spacing between inner and outer walls is assumed small compared to the mean radius. This allows us to use a local Cartesian coordinate system. Second, the depth of the fluid is assumed to be not large compared to a scale height, so that most effects of compressibility are excluded. This is not quite as strong an assumption as the so-called Boussinesq approximation, for which the depth is much smaller than a scale height.* On the sun, of course, it is quite possible that the

* In geophysical fluid dynamics, our assumption is therefore called the 'quasi'-Boussinesq approximation (see Charney and Stern, 1962).

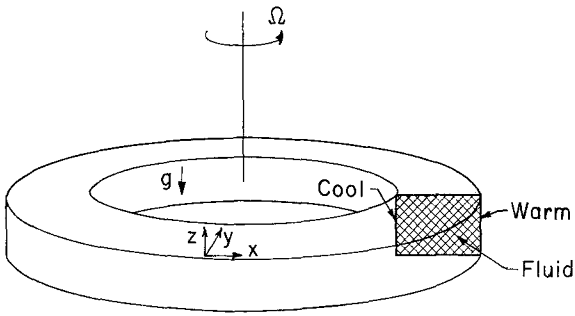

Fig. 1. Schematic representation of thin annulus geometry assumed for the Rossby-wave dynamo model.

Rossby waves extend many scale heights into the convection zone, perhaps even to the bottom.

The inner and outer side walls of the annulus, then, correspond to two latitude circles within a single hemisphere on the sun. Laboratory models of the rotating annulus (filled with water with the temperature fixed at the side walls) have been extensively used to study thermally driven Rossby waves (see, e.g., Hide 1953, 1958; Fultz *et al.*, 1959). Many theoretical models with this geometry have also been devised.

We must specify boundary conditions on the side walls, and the top and bottom of the annulus. For simplicity, we take all four boundaries to be solid and perfectly electrically conducting. This requires that the normal component of motion and magnetic field, and the tangential components of electric current, vanish on the walls. The motion and magnetic fields are then confined entirely to the annular region. We further assume that the top and bottom are 'free', in the sense that the viscous stress vanishes there. Finally, we assume the temperature perturbations vanish at top and bottom.

C. FILTERING APPROXIMATIONS

In addition to assumptions regarding geometry, compressibility and boundary conditions, given above, we also make several assumptions which restrict the kind of motion that can be described by the equations. To integrate the fluid equations in a form general enough to allow all the scales of motion to develop would require much larger computers than now in existence. The final equations will describe explicitly only Rossby wave type motion (and also low frequency Alfvén waves); sound and gravity waves and convection are 'filtered out'. This technique has been an essential ingredient in modeling Rossby waves in the earth's atmosphere.

We can not totally ignore the effects of these smaller-scale motions, however. The convection (granulation and supergranulation) in particular obviously is important in the transport of heat, momentum and magnetic flux. The most we can reasonably do with our model is to represent these processes parametrically with diffusion coefficients.

The detailed assumptions made to screen out convection and gravity and sound waves are:

(a) the horizontal motion is in a state of near heliostrophic balance. That is, the Coriolis force (which for the annulus is horizontal) is nearly balanced by the horizontal gas pressure gradient.

(b) the flow is hydrostatic. That is, the vertical motions, while not zero, are small enough so that vertical accelerations are very small compared to the vertical gas pressure gradient and gravity.

(c) the *vertical* temperature gradient is *less* than the adiabatic gradient. That is, the convection zone appears to Rossby wave disturbances to be gravitationally stable. This assumption helps to ensure that the motion will be hydrostatic.

The implications of (a), (b), and (c) need to be spelled out in greater detail. If the horizontal motion is in near heliostrophic balance, then horizontal accelerations of fluid particles must be small compared to the Coriolis force. If U is a characteristic horizontal velocity, L a relevant horizontal length, and Ω the angular velocity, an equivalent statement is that the Rossby number $Ro \equiv U/2\Omega L$ is small compared to unity. This also says that the time scale L/U for variations in the motion is long compared to the rotational time $1/\Omega$. For the sun, we can take $U \approx 100$ m/sec, a typical shearing velocity in the differential rotation, and $L \approx 1.5 \times 10^5$ km, which is a reasonable scale for latitude variations in the differential rotation and for gross latitude and longitude changes in bipolar magnetic regions. To estimate Ro, we should use the component of rotation in the local vertical direction, so for middle latitudes $2\Omega \approx 3 \times 10^{-6}$ sec^{-1}. For these values, $Ro \approx 10^{-1}$, and $L/U \approx 1.5 \times 10^6$ sec, or two-thirds of a solar rotation. Present knowledge (Bumba and Howard, 1965) of bipolar regions indicates this time is reasonable for their gross evolution.

Near heliostrophic balance also requires that electromagnetic forces, due to induced currents crossing induced magnetic fields, be small compared to the Coriolis force. Given that $Ro \ll 1$, this will be true if $P \equiv M^2/4\pi\varrho U^2 \lessapprox 1$ where M is a characteristic horizontal field strength and ϱ a typical gas density. Equivalently, Maxwell stresses are assumed to be no larger than Reynolds stresses. This assumption proves to be internally consistent in the model: it is not capable of inducing magnetic fields so large that $P \gg 1$. For the sun, if the mean density of the layer we are considering is $\varrho \approx 10^{-4}$ g cm^{-3}, magnetic field strengths M as large as about 300 G can be tolerated. Although there are no direct observations available, large scale subsurface toroidal fields on the sun are probably not much larger than this (Babcock, 1961).

Finally, turbulent viscous forces must also be small compared to Coriolis forces. For eddy kinematic viscosity ν, vertical disturbance scale D, and defining $\delta \equiv D/L$, this puts limits on two Reynolds numbers: $R_D \equiv \delta UD/\nu \gtrapprox 1$; $R_L \equiv UL/\nu \gtrapprox 1$. Here R_D measures the effectiveness of vertical diffusion of horizontal momentum, and R_L that of horizontal diffusion.

For our choice of parameters for the sun, $R_L \gtrapprox 1$ will be satisfied if $\nu \lessapprox 1.5 \times 10^{14}$ cm^2/sec. A much smaller limit for ν is obtained from R_D if D is comparable to a scale height. That is, for $D = 2 \times 10^3$ km (middle of the convection zone), from $R_D \gtrapprox 1$, we get $\nu \lessapprox 1.5 \times 10^{10}$ cm^2/sec. This is smaller by at least an order of magnitude than most estimates of ν from mixing length arguments. However, the actual disturbances on the sun would have $D \approx L$ if they extend through the entire depth of the convection zone, in which case the larger limit of ν, which is more reasonable, would be allowed.

As for hydrostatic balance [assumption (b) above], all other terms in the vertical equation of motion will be small compared to gravity and the vertical pressure gradient so long as $P \lessapprox 1$, $Ro \ll 1$, which we have already assumed, and $\delta^2 Ro^2 \ll 1$, together with an assumption about the magnitude of the subadiabatic temperature gradient, given below.

Assumption (c) above, that the mean vertical temperature gradient is less than the adiabatic gradient, at first glance would seem inappropriate for the solar convection zone. However, there now is both theoretical and laboratory evidence for this occurring as a nonlinear effect in the interior of convecting fluids (Veronis, 1966; Gille, 1967). If we assume that the solar convection is the principal determinant of the vertical temperature structure in the convection zone, then all larger, slower scales of motion would 'feel' this mean gradient.

It is convenient to represent the difference between the actual vertical temperature gradient and the adiabatic gradient by a meteorological variable θ, known as the potential temperature. The log of θ is proportional to the specific entropy s of the gas; that is,

$$s = c_p \ln \theta = c_p \ln T - R \ln p = c_p \ln \alpha + c_v \ln p$$

where T is the ordinary temperature, p the gas pressure, α the specific volume, c_p and c_v are the specific heats at constant pressure and volume, and R is the gas constant. For adiabatic flow, then, the potential temperature of a fluid parcel is conserved. If the vertical temperature gradient were exactly adiabatic, $\ln \theta$ would be independent of height. For a subadiabatic gradient it increases with height. For our model we will assume that $\zeta \equiv \partial \ln \theta / \partial z$ (z is the vertical coordinate made dimensionless with respect to D) is positive and sufficiently large that $\zeta \approx F$, where $F \equiv 4\Omega^2 L^2 / gD$. For the sun, with $D \gtrapprox 2000$ km, $F \lessapprox 4 \times 10^{-5}$, so we need $\zeta \lessapprox 4 \times 10^{-5}$. On the other hand, if the appropriate D is comparable to the depth of the convection zone, F, and therefore $\zeta \approx 6 \times 10^{-7}$. The laboratory and theoretical work cited above indicate ζ as large as 10^{-2} can be achieved. It is clear from the definition of potential temperature that, in general, a latitudinal gradient of temperature is accompanied by a similar gradient of potential temperature.

Finally we put upper limits on the diffusion of heat and magnetic fields by the smaller scales of motion. That is, we require the Peclet numbers $C_L \equiv UL/\kappa \gtrapprox 1$; $C_D \equiv \delta UD/\kappa \gtrapprox 1$, where κ is the thermal diffusion coefficient (which includes effects of both convection and radiation). Similarly, we assume the magnetic Reynolds numbers

$$G_L \equiv UL/\lambda \gtrapprox 1; \qquad G_D \equiv \delta UD/\lambda \gtrapprox 1$$

where λ is the magnetic diffusion coefficient. Clearly the same limits discussed for the viscous diffusion coefficient ν apply also to κ and λ.

4. Qualitative Hydromagnetic Effects of Rossby Waves

Before deriving the detailed equations, it is useful to set the stage by describing qualitatively the hydromagnetic properties of Rossby waves found by the writer in an earlier study (Gilman, 1967a, b, c, d). This work is primarily a linear perturbation analysis of unstable (i.e., exponentially growing) Rossby waves, for which all of the assumptions listed in Sec. 3 are made. The analysis was done for the 'ideal' case of an inviscid, adiabatic, perfectly conducting fluid. Figure 2 schematically summarizes the results. In Figure 2 we are in effect looking down at a section of the annulus, with the lower edge of each sketch corresponding to the outer rim. We have taken the inner wall of the annulus to be colder than the outer, but the hydromagnetic effects follow equally well from the opposite arrangement.

The basic state about which the system is perturbed (Gilman, 1967c) is one in which the flow and magnetic fields are purely zonal, i.e., parallel to the side walls, corresponding to a differential rotation and toroidal magnetic field. The vertical structure of the fluid is represented by two superimposed layers. The zonal flow is parabolic in the upper layer (Figure 2a) roughly like the observed solar differential rotation, while in the lower layer the flow is zero relative to the rotating coordinate system. The zonal (toroidal) magnetic field is uniform throughout.

The unstable Rossby waves that grow on this basic state transport heat horizontally from the outer toward the inner rim (equator toward the pole), as seen in Figure 2a

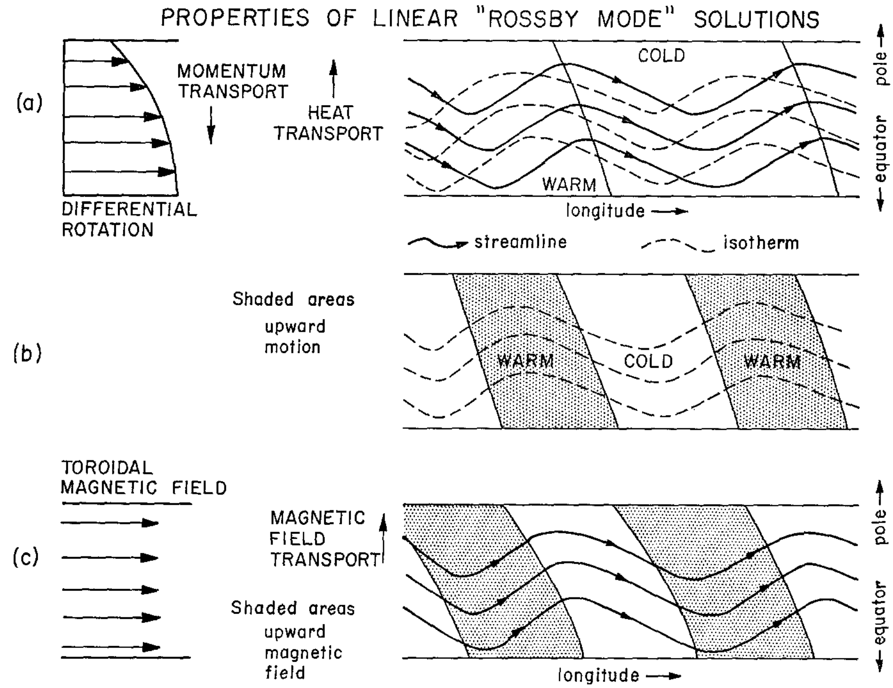

Fig. 2 Schematic summary of linear, unstable Rossby waves and their hydromagnetic effects.

from the relative position of horizontal streamlines and isotherms. At the same time, the waves transport momentum toward the equator; this effect can be demonstrated from the upstream 'tilt' of the waves. This momentum flux is in the same direction as reported by Ward (1964, 1965), and could be responsible for the maintenance of the equatorial acceleration.

The instability is sustained by the relatively small vertical motions, which lift up (Figure 2b; shaded area indicates upward motion) warm gas and bring down cold at different longitudes thereby lowering the center of gravity, releasing potential energy to the kinetic energy of motion. Note that the regions of vertical motion also tilt upstream.

With the toroidal field present (Figure 2c), the vertical motions pick up toroidal flux lines at evenly spaced longitudes, depressing them in between. This gives rise to the vertical magnetic fields shown (shaded and clear areas of Figure 2c). The regions of upward and downward field also tilt upstream, qualitatively like observed bipolar magnetic regions. The horizontal motions of the wave, as indicated by the phase of the wavy stream lines relative to the shaded areas in Figure 2c, carry predominantly upward-directed vertical fields (shaded areas) toward the pole, and downward-directed fields toward the equator. This gives a residual positive vertical field near the poles, and negative near the equator, that is independent of longitude. This symmetric poloidal field should then be stretched out into a new toroidal field by the differential rotation to complete the dynamo cycle. However, the linear study precludes this feedback. We propose in subsequent sections and in Part II to construct a nonlinear model that includes this feedback, and demonstrates that the toroidal and poloidal fields can be maintained and periodically reversed by Rossby-type motions.

With the feedback of poloidal into toroidal field, it is noteworthy that the evolution of magnetic fields in our model will be qualitatively similar in its major elements to that put forth for the sun by Babcock (1961) except, of course, that our model can not produce highly concentrated features like sunspots.

5. Equations for the Model

A. SCALED VARIABLES

To approximate the basic equations of magnetohydrodynamics for our problem, we first introduce dimensionless variables using the parameters given in Sec. 3. Henceforth denoting dimensional variables by asterisks, we define $x = x^*/L$ to be the dimensionless downstream coordinate (i.e., longitude), $y = y^*/L$ the cross-channel coordinate (i.e., latitude), $z = z^*/D$ the vertical coordinate, and $t = (U/L)t^*$ dimensionless time. The dimensionless pressure p, density ϱ, specific volume α, and potential temperature θ are defined as departures from a horizontally averaged state $\bar{p}^*$, $\bar{\varrho}^*$, $\bar{\alpha}^*$, $\bar{\theta}^*$ (all functions of z only), according to the equations

$$\begin{aligned} p^* &= \bar{p}^* (1 + \mu F Ro p) \\ \varrho^* &= \bar{\varrho}^* (1 + F Ro \varrho) \\ \alpha^* &= \bar{\alpha}^* (1 + F Ro \alpha) \\ \ln \theta^* &= \ln \bar{\theta}^* + F Ro \theta \, . \end{aligned} \qquad (1)$$

In the above, $\mu = D/S$, where $S = \bar{p}^*\bar{\alpha}^*/g$, is the scale height. The horizontally averaged variables are in hydrostatic balance; that is,

$$\bar{\alpha}^* \partial \bar{p}^* / \partial z^* + g = 0 .$$

It is convenient to split the velocities and magnetic fields into their horizontal and vertical components, and, further, to represent the horizontal components as the sum of an irrotational and a nondivergent part. That is, if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the x^*, y^*, z^* directions, and $\nabla^* = \mathbf{i}\,\partial/\partial x^* + \mathbf{j}\,\partial/\partial y^*$ is the dimensional *horizontal* gradient operator, we write the total velocity $\mathbf{V}^* = \mathbf{V}_\psi^* + \mathbf{V}_\sigma^* + w^*\mathbf{k}$, where $\mathbf{V}_\psi^* = \mathbf{k} \times \nabla^* \psi^*$, $\mathbf{V}_\sigma = \nabla^* \sigma^*$ make up the horizontal velocity, and w^* is the vertical velocity ($\times$ denotes the vector product). Note that $\nabla^* \cdot \mathbf{V}_\psi^* = 0$, and $\nabla^* \times \mathbf{V}_\sigma^* = 0$, so that $\mathbf{V}_\psi^*$ is the nondivergent horizontal velocity, and $\mathbf{V}_\sigma$ is the irrotational horizontal velocity. In terms of dimensionless variables $\mathbf{V}_\psi = \mathbf{k} \times \nabla \psi$, $\mathbf{V}_\sigma = \nabla \sigma$, and w, we assume a scaling that gives

$$\mathbf{V}^* = U\mathbf{V}_\psi + RoU\mathbf{V}_\sigma + \delta RoUw\mathbf{k} . \tag{2}$$

This scaling is a natural consequence of the assumption that the motion is nearly heliostrophic (see Gilman, 1967a)

The total magnetic field $\mathbf{H}^*$ is written in scaled form as

$$\mathbf{H}^* = M\mathbf{H}_\chi + M\mathbf{H}_\gamma + \delta Mh\mathbf{k} \tag{3}$$

in which the dimensionless horizontal field has two parts, given by $\mathbf{H}_\chi = \mathbf{k} \times \nabla \chi$ and $\mathbf{H}_\gamma = \nabla \gamma$, and in which the dimensionless vertical field is h†.

We note that the flow and fields associated with the 'stream functions' ψ and χ respectively are *purely* horizontal. The horizontal flow and fields specified by σ and γ are, on the other hand, linked to the vertical flow w and field h [see continuity equations (6) and (9) below]. That part of ψ that is independent of the longitudinal coordinate x may be thought of as specifying the axisymmetric 'differential rotation' in our annulus. Similarly, the part of χ independent of x gives an axisymmetric field in the x direction, which corresponds to the toroidal field of Babcock's (1961) and other solar models. The axisymmetric parts of σ and w represent what is usually called meridian circulation, being entirely in meridian planes. Similarly, the part of γ and h independent of x represents an axisymmetric magnetic field in meridian planes. This corresponds to the poloidal field of Babcock's model that is stretched out by the differential rotation into a toroidal field. The *asymmetric* part of h corresponds perhaps to 'bipolar magnetic regions'.

B. SCALED EQUATIONS

The above scaling is chosen so that the largest terms in the equations of motion, which represent the heliostrophic and hydrostatic balances, are of order unity. Given this scaling, and noting that for the sun $F \lesssim 10^{-4} \approx (Ro)^4$ we may write down the

† Given $\mathbf{V}$ and $\mathbf{H}$, ψ, σ, χ and γ are determined to within arbitrary *additive* functions of z, t. Since in the dynamical equations given below ψ, σ, χ and γ always appear differentiated at least once with respect to a horizontal coordinate, these additive functions have no effect and may be set equal to zero.

governing equations correct to second order in Ro. In these equations, previously undefined terms are $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, the *horizontal* Laplacian, Q a dimensionless heating function (to force the latitudinal temperature gradient) assumed of order unity or less, $\bar{P} \equiv M^2/4\pi\bar{\rho}^* U^2$, and $\varepsilon \equiv F/\bar{\zeta} \approx 1$. The equations are, respectively, the horizontal and vertical equations of motion, the equation of continuity, the horizontal and vertical magnetic field induction equations, the magnetic field continuity equation, and, finally, the thermodynamic equation (written in terms of potential temperature).

$$
\begin{aligned}
Ro\left[\frac{\partial}{\partial t} + (\mathbf{V}_\psi + Ro\mathbf{V}_\sigma)\cdot\nabla + Row\frac{\partial}{\partial z}\right](\mathbf{V}_\psi + Ro\mathbf{V}_\sigma)& \\
= -\mathbf{k}\times(\mathbf{V}_\psi + Ro\mathbf{V}_\sigma) - \nabla p + \bar{P}Ro\Big[(\nabla\times\mathbf{H}_\chi)\times(\mathbf{H}_\chi + \mathbf{H}_\gamma)& \\
+ h\frac{\partial \mathbf{H}_\chi}{\partial z} - \delta^2\nabla\frac{h^2}{2}\Big] + \frac{Ro}{R_L}\nabla^2(\mathbf{V}_\psi + Ro\mathbf{V}_\sigma) + \frac{Ro}{R_D}\frac{\partial^2}{\partial z^2}(\mathbf{V}_\psi + Ro\mathbf{V}_\sigma)&
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\delta^2 Ro^2\left[\frac{\partial}{\partial t}\quad \mathbf{V}_\psi\cdot\nabla\right]w = -\bar{\alpha}^*\frac{\partial}{\partial z}\frac{p}{\bar{\alpha}^*} + \alpha& \\
+ \bar{P}Ro\left[-\frac{\partial}{\partial z}\frac{|\mathbf{H}_\chi + \mathbf{H}_\gamma|^2}{\alpha} + \delta(\mathbf{H}_\chi + \mathbf{H}_\gamma)\cdot\nabla h\right]& \\
+ \frac{\delta^2 Ro^2}{R_L}\nabla^2 w + \frac{\delta^2 Ro^2}{R_D}\frac{\partial^2 w}{\partial z^2}&
\end{aligned}
\tag{5}
$$

$$
\bar{\varrho}^* Ro\nabla\cdot\mathbf{V}_\sigma + Ro\frac{\partial}{\partial z}\bar{\varrho}^* w = 0 \tag{6}
$$

$$
\begin{aligned}
Ro\frac{\partial}{\partial t}(\mathbf{H}_\chi + \mathbf{H}_\gamma) = Ro\nabla\times(\mathbf{V}_\psi\times\mathbf{H}_\chi) + Ro\nabla\times(\mathbf{V}_\psi\times\mathbf{H}_\gamma)& \\
+ Ro\frac{\partial}{\partial z}(h\mathbf{V}_\psi) + Ro^2\nabla\times[\mathbf{V}_\sigma\times(\mathbf{H}_\chi + \mathbf{H}_\gamma)]& \\
- Ro^2\frac{\partial}{\partial z}[w(\mathbf{H}_\chi + \mathbf{H}_\gamma)] + \frac{Ro}{G_L}\nabla^2(\mathbf{H}_\chi + \mathbf{H}_\gamma)& \\
+ \frac{Ro}{G_D}\frac{\partial^2}{\partial z^2}(\mathbf{H}_\chi + \mathbf{H}_\gamma)&
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
Ro\left[\frac{\partial}{\partial t} + \mathbf{V}_\psi\cdot\nabla\right]h = Ro^2\mathbf{H}_\chi\cdot\nabla w + \frac{Ro}{G_L}\nabla^2 h + \frac{Ro}{G_D}\frac{\partial^2 h}{\partial z^2} + Ro^2(\mathbf{H}_\gamma\cdot\nabla)w& \\
- Ro^2(\mathbf{V}_\sigma\cdot\nabla)h - Ro^2 w\frac{\partial h}{\partial z} - Ro^2 h\nabla\cdot\mathbf{V}_\sigma&
\end{aligned}
\tag{8}
$$

$$
Ro\left(\nabla\cdot\mathbf{H}_\gamma + \frac{\partial h}{\partial z}\right) = 0 \tag{9}
$$

$$
Ro\left[\frac{\partial}{\partial t} + (\mathbf{V}_\psi + Ro\mathbf{V}_\sigma)\cdot\nabla + Row\frac{\partial}{\partial z}\right]\theta + \frac{Row}{\varepsilon} = \frac{Ro}{C_L}\nabla^2\theta + \frac{Ro}{C_D}\frac{\partial^2\theta}{\partial z^2} + RoQ. \tag{10}
$$

C. BOUNDARY CONDITIONS

In terms of the dimensionless variables we have defined, the boundary conditions stated in Sec. 3 require

$$\frac{\partial\psi}{\partial x}, \frac{\partial\sigma}{\partial y}, \frac{\partial\chi}{\partial x}, \frac{\partial\gamma}{\partial y}, \frac{\partial^2\chi}{\partial y^2}, \frac{\partial h}{\partial y} = 0 \tag{11}$$

at the sides of the annulus, and

$$\frac{\partial\psi}{\partial z}, \frac{\partial\sigma}{\partial z}, \frac{\partial\chi}{\partial z}, \frac{\partial\gamma}{\partial z}, h, w, \theta = 0 \tag{12}$$

at the top and bottom.

D. ROSSBY NUMBER EXPANSION: ZEROTH ORDER SYSTEM

The next step is to expand all dependent variables in a power series in Ro, assumed small. If K represents any of the variables, we write

$$K = K^{(0)} + RoK^{(1)} + Ro^2K^{(2)} + 0(Ro^3) \tag{13}$$

and substitute into Eqs. (4)–(10). Eqs. (4) and (5) alone produce 'zeroth-order' relations, which are, respectively

$$\mathbf{k} \times \mathbf{V}_\psi^{(0)} = -\nabla\psi^{(0)} = -\nabla p^{(0)} \tag{14}$$

$$\frac{\partial}{\partial z}\left(\frac{p^{(0)}}{\bar{\alpha}^*}\right) = \frac{\alpha^{(0)}}{\bar{\alpha}^*}. \tag{15}$$

The first of these, as expected, gives the heliostrophic balance (we may speak of $\mathbf{V}_\psi^{(0)}$ as the heliostrophic wind), and the second, hydrostatic balance. It is clear from (14) that $\psi^{(0)} = p^{(0)}$. (This can be taken as a definition of $\psi^{(0)}$.) From this fact, plus (15), (1) and the definition of potential temperature given in Sec. 3, we can also show that

$$\frac{\partial\psi^{(0)}}{\partial z} = \theta^{(0)}. \tag{16}$$

Taking the horizontal gradient of (16) gives

$$\frac{\partial}{\partial z}\nabla\psi^{(0)} = -\mathbf{k} \times \frac{\partial}{\partial z}\mathbf{V}_\psi^{(0)} = \nabla\theta^{(0)}. \tag{17}$$

This relates the vertical shears in the heliostrophic wind to the horizontal gradients in potential temperature, and is sometimes called the thermal wind relation. Related forms involving the ordinary temperature have been discussed for the sun by Plaskett (1959, 1962, 1966) and others.

E. FIRST ORDER SYSTEM

Collecting first order terms from Eqs. (4)–(10) [excepting (8) which requires special discussion] and using (6), we get

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_\psi^{(0)}\cdot\nabla\right)\mathbf{V}_\psi^{(0)} = -\,\mathbf{k}\times\mathbf{V}_\psi^{(1)} - \mathbf{k}\times\mathbf{V}_\sigma^{(0)} - \nabla p^{(1)}$$
$$+\bar{P}\left[(\nabla\times\mathbf{H}_\chi^{(0)})\times(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}) + h^{(0)}\frac{\partial \mathbf{H}_\chi^{(0)}}{\partial z} - \delta^2\nabla\frac{h^{(0)2}}{2}\right]$$
$$+\frac{1}{R_L}\nabla^2\mathbf{V}_\psi^{(0)} + \frac{1}{R_D}\frac{\partial^2}{\partial z^2}\mathbf{V}_\psi^{(0)} \tag{18}$$

$$\bar{\alpha}^*\frac{\partial}{\partial z}\bar{\varrho}^* p^{(1)} - \alpha^{(1)} + \bar{P}\left[\frac{\partial}{\partial z}\frac{|\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}|^2}{2} - \delta(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)})\cdot\nabla h^{(0)}\right] = 0 \tag{19}$$

$$\nabla\cdot\mathbf{V}_\sigma^{(0)} + \bar{\alpha}^*\frac{\partial}{\partial z}\bar{\varrho}^* w^{(0)} = 0 \tag{20}$$

$$\frac{\partial}{\partial t}(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}) = \nabla\times(\mathbf{V}_\psi^{(0)}\times\mathbf{H}_\chi^{(0)}) + \nabla\times(\mathbf{V}_\psi^{(0)}\times\mathbf{H}_\gamma^{(0)}) + \frac{\partial}{\partial z}h^{(0)}\mathbf{V}_\psi^{(0)}$$
$$+\frac{1}{G_L}\nabla^2(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}) + \frac{1}{G_D}\frac{\partial^2}{\partial z^2}(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}) \tag{21}$$

$$\nabla\cdot\mathbf{H}_\gamma^{(0)} + \frac{\partial h^{(0)}}{\partial z} = 0 \tag{22}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_\psi^{(0)}\cdot\nabla\right)\frac{\partial\psi^{(0)}}{\partial z} + \frac{w^{(0)}}{\varepsilon} = \frac{1}{C_L}\nabla^2\frac{\partial\psi^{(0)}}{\partial z} + \frac{1}{C_D}\frac{\partial^3\psi^{(0)}}{\partial z^3} + Q^{(0)}. \tag{23}$$

We can eliminate $p^{(1)}$, $\mathbf{V}_\psi^{(1)}$ and $\mathbf{V}_\sigma^{(0)}$ from (18) by first taking the vertical component of the curl of it, i.e., applying $\mathbf{k}\cdot\nabla\times$, and then substituting from (20) for $\nabla\cdot\mathbf{V}_\psi^{(0)}$, to get a prediction equation for the vorticity $\nabla^2\psi^{(0)}$ of the heliostrophic wind $\mathbf{V}_\psi^{(0)}$:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_\psi^{(0)}\cdot\nabla\right)\nabla^2\psi^{(0)} = \bar{P}\mathbf{k}\cdot\nabla$$
$$\times\left[(\nabla\times\mathbf{H}_\chi^{(0)})\times(\mathbf{H}_\chi^{(0)}+\mathbf{H}_\gamma^{(0)}) + h^{(0)}\frac{\partial\mathbf{H}_\chi^{(0)}}{\partial z}\right]$$
$$+\bar{\alpha}^*\frac{\partial}{\partial z}\bar{\varrho}^* w^{(0)} + \frac{1}{R_L}\nabla^4\psi^{(0)} + \frac{1}{R_D}\frac{\partial^2}{\partial z^2}\nabla^2\psi^{(0)}. \tag{24}$$

Operating on (21) with $\mathbf{k}\cdot\nabla\times$ will give us a prediction equation for χ:

$$\frac{\partial}{\partial t}\nabla^2\chi^{(0)} = -\,\nabla^2(\mathbf{k}\cdot\mathbf{V}_\psi^{(0)}\times\mathbf{H}_\chi^{(0)}) - \nabla^2\mathbf{k}\cdot(\mathbf{V}_\psi^{(0)}\times\mathbf{H}_\gamma^{(0)}) + \mathbf{k}\cdot\nabla\times\frac{\partial}{\partial z}h^{(0)}\mathbf{V}_\psi^{(0)}$$
$$+\frac{1}{G_L}\nabla^2(\nabla^2\chi^{(0)}) + \frac{1}{G_D}\frac{\partial^2}{\partial z^2}(\nabla^2\chi^{(0)}). \tag{25}$$

In order for the dynamo to work, the equations must contain sufficient provision for induction of toroidal field $\mathbf{H}_\chi^{(0)}$, from the poloidal field $\mathbf{H}_\gamma^{(0)}$, $h^{(0)}$, and vice versa.

We note that (25), which is our prediction equation for the toroidal field, does contain two terms on the right hand side representing the stretching of poloidal into toroidal fields. Our prediction equation for the poloidal field, which we find from (8), must contain the other links.

Turning to (8), the first order terms collect to give

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_{\psi}^{(0)}\cdot\nabla\right) h^{(0)} = \frac{1}{G_L}\nabla^2 h^{(0)} + \frac{1}{G_D}\frac{\partial^2 h^{(0)}}{\partial z^2}. \tag{26}$$

To this order, then, there is no term giving induction of poloidal fields from toroidal. From (26), it is clear that any initial $h^{(0)}$, and consequently $\mathbf{H}_{\gamma}^{(0)}$, can not be sustained and the dynamo would fail. To overcome this, it is necessary to violate the strict Rossby number ordering of the terms and include the first term on the right of (8), which represents the induction of poloidal field from the toroidal field (accomplished by the vertical motion twisting toroidal field lines up into the vertical) giving us

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_{\psi}^{(0)}\cdot\nabla\right) h^{(0)} = Ro\mathbf{H}_{\chi}^{(0)}\cdot\nabla w^{(0)} + \frac{1}{G_L}\nabla^2 h^{(0)} + \frac{1}{G_D}\frac{\partial^2 h^{(0)}}{\partial z^2}. \tag{27}$$

In the present study, we do not include the other Ro^2 terms from (8) because none of them links toroidal with poloidal fields. Also, to include additional second order terms in this and the other equations would considerably increase the computational effort required to solve them. Our results in Part II indicate only the one term we have included is crucial to the success of the dynamo.

Mathematically what we have done is not rigorous, and the effect of the terms left out should be assessed. We show in Part II that in the solution to the dynamo equations we find the 'source term' $Ro\mathbf{H}_{\chi}^{(0)}\cdot\nabla w^{(0)}$ in (27) is no smaller, on the average, than the other terms. Furthermore, for the model in Part II, we can show that all of the Ro^2 terms neglected in (8) to get (27) actually vanish.

The smallness of the source term for poloidal fields is a consequence of the nearly heliostrophic character of the flow, for which vertical motions are smaller than might be expected from continuity of mass alone. These smaller but nonzero vertical motions are therefore slower at producing vertical fields, but they still produce them. Since for the dynamo to work, $Ro\mathbf{H}_{\chi}^{(0)}\cdot\nabla w^{(0)}$ in (27) must overcome the diffusion terms, the dynamo requires somewhat higher magnetic Reynolds numbers G_L and G_D than otherwise.

F. DISCUSSION

The five Eqs. (22)–(25) and (27), then, comprise the equations for our dynamo, from which we compute $\psi^{(0)}$, $w^{(0)}$, $\chi^{(0)}$, $\gamma^{(0)}$, $h^{(0)}$. These equations, linearized about a differential rotation and toroidal magnetic field, yield the growing Rossby waves and their hydromagnetic effects described qualitatively in Sec. 4. The nonlinear system we have obtained will behave in much the same way, except that when the waves reach finite amplitude further growth will be checked by non-linear interactions and dissipation.

The original equations of motion, induction and thermodynamics conserve total energy in the absence of dissipation and thermal forcing. Our dynamo system should in some sense retain this property. Since we raised the order of the system by forming (24) and (25), we must strengthen the boundary conditions at the sides to retain an energy integral. If we require, in addition to the conditions (11) and (12) already imposed, that

$$\psi^{(0)}, \chi^{(0)} = 0 \tag{28}$$

at the sides, then in the absence of dissipation and thermal forcing, the energy integral

$$\int \left[\frac{\varepsilon}{2} (\partial \psi^{(0)}/\partial z)^2 + \tfrac{1}{2} |\nabla \psi^{(0)}|^2 + \tfrac{1}{2} \bar{P} |\nabla \chi^{(0)}|^2 \right] \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

(where the integral is over the entire volume of the annulus) is conserved. The three energies involved are, respectively, the potential energy associated with the horizontal temperature gradient, the kinetic energy of $\mathbf{V}_{\psi}^{(0)}$, the heliostrophic part of the motion, and finally, the magnetic energy of the toroidal field $\mathbf{H}_{\chi}^{(0)}$. We note that the energy of neither the divergent part of the motion $\mathbf{V}_{\sigma}^{(0)}$, $w^{(0)}$, nor of the poloidal field $\mathbf{H}_{\gamma}^{(0)}$, $h^{(0)}$ are included, but from the governing equations it is reasonably clear that they will be bounded. If the toroidal field can not be sustained, then from (27) neither can the poloidal fields.

In the general context of dynamo theory, we note that our system of equations allows for two effects not usually included. First, the motion is not given *a priori*, but rather arises in response to specified thermal forcing, in this case latitudinally nonuniform heating. Second, the induced magnetic fields are allowed to react upon the inducing motions. We should therefore expect some kind of balance to be reached, in which, on the average, induction of new fields is balanced by dissipation.

In principle, the most straightforward way to integrate our dynamo equations would be to set up a three-dimensional grid of points in the annulus, and replace our differential equations by difference equations.* This would give us, however, a formidable computing job, requiring many hours on the largest machines available. Instead, it seems wiser at the outset to conduct a pilot study with a still further simplified form of the equations, using techniques successful for the corresponding problem without magnetic fields which require much less computing effort. This study is the subject of Part II.

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* Strictly speaking, because of the form of the diffusion terms in Eqs. (23) and (24), we would have to specify further boundary conditions or temperature and velocity at the sides of the annulus in order to solve the problem using a three-dimensional grid. This is, however, not necessary with the approximate model developed in Part II.

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