SOME CALCULATIONS OF THE ENERGY-RELEASE RATE G FOR CRACKS IN MICROPOLAR AND COUPLE-STRESS ELASTIC MEDIA

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A few years ago Sternberg and Muki [1] considered the effect of couple stresses on the stress concentration at the tip of a crack. They treated the problem of a finite crack in an infinite medium under conditions of plane strain with a uniform tension acting at infinity. The main conclusions were that at the crack tip the stress and couple stress fields had singularities of the same order, the order of the stress singularities being the same as those of the classical elastic problem. It was found that the limit of the stress intensity factor as ℓ (the couple stress parameter) tended to zero was different to the usual elastic result (ℓ identically equal to zero). However, their approach which involved the numerical solution of integral equations did not give a precise evaluation of the coefficients involved in the stress and couple stress intensity factors. The couple stress theory has been criticised by Eringen [2] who replaces it by the micropolar theory of elasticity (see [2] for a review). In this note we consider the problem of a semi-infinite crack in a strip using both theories. These solutions which are accomplished by the use of a path independent integral demonstrate that G, the energy release rate, does tend to the elastic result as $\ell \to 0$ even though the stress intensity factor may not.

For both the couple stress and micropolar theories the equations of equilibrium can be written [2]

$$t_{\ell k,\ell} = 0$$
$$m_{\ell k,\ell} + \varepsilon_{kmn} t_{mn} = 0 \tag{1}$$

where $t_{\ell k}$ (the first suffix denoting the direction of the outward normal) are the stress components, $m_{\ell k}$ are the couple stress components, and ε_{kmn} is the permutation tensor. We are to consider plane strain situations so that in a cartesian co-ordinate system (x_1, x_2, x_3) derivatives with respect to x_3 are zero and in the second of (1) only the equation with k = 3 will be relevant since $t_{3\alpha} = t_{\alpha 3} = 0$, $\alpha = 1,2$; $m_{33} = 0$; and $m_{\alpha\beta} = 0$, $\alpha = \beta$ (α, $\beta = 1,2$) for plane strain.

For the couple stress theory in addition to (1) we have the constitutive relations (see [1])

$$m_{\alpha 3} = 4\mu\ell^2 w_{,\alpha}$$
$$2\mu e_{\alpha\beta} = t_{(\alpha\beta)} - \nu\delta_{\alpha\beta}\, t_{\gamma\gamma} \tag{2}$$

where

$$2t_{(\alpha\beta)} = (t_{\alpha\beta} + t_{\beta\alpha});\ 2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha};\ 2w = u_{2,1} - u_{1,2}$$

(α,β,γ taking the values 1 and 2), u_1 and u_2 are the components of the displacement vector in the 1 and 2 directions, μ is the shear modulus, ν is Poisson's ratio, and ℓ is the couple stress parameter. The strain energy function W_1 for the plane problem is given by

$$4\mu W_1 = t_{(\alpha\beta)}t_{(\alpha\beta)} - \nu t_{\alpha\alpha}t_{\beta\beta} + \frac{1}{2\ell^2} m_{\alpha 3}m_{\alpha 3} \quad (\alpha,\beta=1,2) \tag{3}$$

In the micropolar theory the main difference to the above is the introduction of an independent microrotation vector; for the plane case this is θ_3 the component in the 3 direction replacing w above (note that by its definition w was constrained to be the same as the local rotation of the medium). The relevant constitutive relations become

$$\begin{aligned} t_{k\ell} &= \lambda e_{rr}\delta_{k\ell} + (\mu+\kappa)e_{k\ell} + \mu e_{\ell k} \\ e_{11} &= u_{1,1};\; e_{22} = u_{2,2} \\ e_{12} &= u_{2,1} - \theta_3;\; e_{21} = u_{1,2} + \theta_3 \\ m_{13} &= \gamma\theta_{3,1};\; m_{23} = \gamma\theta_{3,2} \end{aligned} \tag{4}$$

with strain energy function W_2 where

$$2W_2 = \lambda e_{kk}e_{\ell\ell} + (\mu+\kappa)e_{k\ell}e_{k\ell} + \mu e_{k\ell}e_{\ell k} + \gamma(\theta_{3,1}^2 + \theta_{3,2}^2) \tag{5}$$

where k, ℓ take the values 1 and 2; λ,μ, and κ are material constants; and γ plays a similar role to $4\mu\ell^2$ in the couple stress theory. Note the difference between the definitions of the strains e_{12} and e_{21} in the two theories.

A great deal of attention has been given recently to finding path independent integrals for elastic media, see Eshelby [3] for a collection of results. Here we generalise these results to the couple-stress and micropolar materials considered above. We merely state the results, as use of the above field equations and an application of the divergence theorem is sufficient to prove path independence. The integrals are defined as

$$G = \int_S (W\delta_{\ell 1} - t_{\ell k}u_{k,1} - m_{\ell 3}\theta_{3,1})\, dS_\ell \tag{6}$$

where the integral over S is to be taken over some surface enclosing the crack tip in the usual manner, dS_ℓ denoting the surface element with normal x_ℓ. The above integrals are applicable to the plane strain problem for either the micropolar or couple-stress theory, θ_3 being replaced by w in the latter theory, and ℓ and k taking the values 1 and 2 in each case. G can be shown to be the energy release by deducing it via the balance of energy in a cylinder enclosing the crack tip. It should be noted however that in the couple stress case when the material can be considered to be an elastic material of grade 2 the above result for G does not appear to be the same as that deduced from a formal application of the results of [3] for grade 2 materials. This may be due to the constrained rotation in the couple stress theory and that the strain energy depends only on $t_{(\alpha\beta)}$ eqn (3). The result (6) for G in the couple-stress case agrees with the formula given by Sih and Liebowitz [4] when the contour S reduces to a small rectangle enclosing the crack tip.

It was shown by Rice [5] that for the case of a semi-infinite crack in an elastic strip with suitable displacement boundary conditions an application of the J integral (equivalent to G above with $m_{\ell 3} \equiv 0$) would give the energy release directly without the need to solve a complicated boundary value problem. Here we consider a strip of thickness 2h, with a crack which is stress free lying on $x_2 = 0$; $x < 0$. The boundary conditions on the sides of the strip $x_2 = \pm h$ are taken to be

$$u_2 = 0, \; u_1 = \pm u_{10}, \; \theta_3 = 0 \text{ for all } x_1 \text{ when } x_2 = \pm h \tag{7}$$

u_{10} is a constant and for the couple-stress case $\theta_3 = 0$ becomes $w = 0$.

Taking the integral G of (6) around a large contour which consists of vertical strips at $x_1 = \pm\infty$, the sides of the strip and the crack faces, it is possible to relate the value of G at the crack tip to its value around the rest of the contour. By virtue of the boundary conditions (7) the integral is zero when taken along the sides of the strip, and by the stress-free crack conditions ($t_{22} = 0$, $m_{23} = 0$ $x_1 < 0$, $x_2 = 0$) the integral is zero when taken along the crack faces. It remains to evaluate the integral along the vertical strips and this is done on the assumption that at $x_1 = \pm\infty$ there will be no variations in the x_1 direction so that derivatives with respect to x_1 can be disregarded in the field equations when the energy stored in the vertical strips at $x_1 = \pm\infty$ is calculated. After some calculation the following results are obtained

(1) For micropolar media

$$G = \frac{(2\mu + \kappa)\, u_{10}^2}{2h - \dfrac{\gamma T}{(2\mu + \kappa)} \tanh(Th)} \tag{8}$$

where $T^2 = \dfrac{\kappa(2\mu + \kappa)}{\gamma(\mu + \kappa)}$, and this relation can be used to rewrite the denominator of (8) as $2h - \dfrac{\kappa}{(\mu + \kappa)T} \tanh(Th)$.

F2) For Couple-stress elasticity

$$G = \frac{\mu u_{10}^2}{h - \ell \tanh(h/\ell)} \tag{9}$$

Clearly when $\ell \to 0$ in (9) this reduces to the result for an elastic medium $G = \mu u_{10}^2/h$, and similarly in (8) when $\gamma \to 0$, G tends to the corresponding elastic result. Note that in (4), $(2\mu + \kappa)$ plays the role of 2μ in the usual elastic stress-strain relations. Note further that the denominator of (9) can never be zero since h is always greater than $\ell \tanh(h/\ell)$ and $\ell > 0$. To show that the denominator of (8) can never be zero we need the restriction on the micropolar elastic moduli which are necessary to ensure a positive internal energy. These are given in [2] as

$$3\lambda + 2\mu + \kappa \geq 0 \qquad 2\mu + \kappa \geq 0 \qquad \kappa \geq 0 \tag{10}$$

The second of these inequalities shows that $2(\mu + \kappa) \geq \kappa$ which with (10) ensures that T is real and that the denominator of (8) is always positive. This can be seen by considering the alternative form of the denominator given beneath (8).

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26 August 1974