ON THE TIME EVOLUTION OF FORCE-FREE FIELDS

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Abstract. Using the adiabatic approximation, it is shown that the problem of the continuous deformation of a force-free (f.f.) field, in general, has no solution. This means that f.f. fields are nonevolutionary and even small perturbations may produce drastic changes in them. By analogy with a special case of f.f. field, the current-free field, we conclude that perturbations of a f.f. field in general produce pinch (current) sheets.

The term evolutionarity is used in MHD to define the property of a solution to conserve its typical features when subjected to small perturbations. For example, there exist situations with MHD shocks when the perturbation problem does not have any solution (Akhiezer *et al.*, 1975; Syrovatskii, 1961). This means that the original shock solution is not a physically well defined regime because a small perturbation produces a drastic change of the initial state. Note, that nonevolutionarity and instability are quire different properties of a system. In contrast to the first one, the second means a unique type of linear evolution of a system for a given perturbation.

It is of interest to examine the evolutionarity of other MHD-systems, especially that of f.f. fields which are now widely used in theoretical considerations of solar and astrophysical problems. Fields of this type present the zero approximation of magnetohydrostatics of low-β plasmas ($\beta = 8\pi p/B^2$ is a known plasma parameter, p-pressure, B-magnetic field). This approximation leads to equations (Lundquist, 1950)

$$[\mathbf{B}\,\mathrm{rot}\,\mathbf{B}] = 0, \qquad \mathrm{div}\,\mathbf{B} = 0 \tag{1}$$

and is supposed to be appropriate for magnetic fields in the diffuse atmospheres of stars and planets.

The problems of interest here may be formulated in the following way: does the force-free field stay a f.f. one in lowest order of magnitude if we disturb it by small perturbations of a general type? We underline that the question here does not concern small non f.f. deviations because of the general type of perturbation, but the existence of a disturbed solution at all, i.e. the existence of a solution which differs only slightly (in the same order as the perturbation) from the original state. We shall see later that a f.f. field is just such a field for which the latter behavior is generally impossible. This means that there exists a large class of perturbations for which a f.f. field cannot conserve its property and is destroyed in the sense that there must appear regions in which magnetic field becomes essentially non force-free. Our experience of the current free field allows us to suppose that these regions are usual cylindrical pinches or pinch current sheets.

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To prove this let us consider adiabatically slow changes of a field due to slow changes of external conditions. Namely, we will consider the flow with a small value of the parameter

$$\varepsilon = V_0/V_A \ll 1\,, \tag{2}$$

where V_0 – a typical plasma velocity do to slowly changing external (boundary) conditions and $V_A = B/\sqrt{4\pi\rho}$ – the Alfvén velocity.

We consider a plasma of infinitely high conductivity and take as the units of length, velocity, time, density, magnetic and electric field some characteristic values:

$$R_0,\; V_0,\; t_0 = R_0/V_0,\; \rho_0,\; B_0,\; v_0 B_0/c\,. \tag{3}$$

Then for the case $\beta \to 0$ we have equations in nondimensional form (Syrovatskii, 1976):

$$\varepsilon^2 \frac{d\mathbf{v}}{dt} = \frac{1}{\rho}[\operatorname{rot}\mathbf{B}\times\mathbf{B}]\,, \tag{4}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\operatorname{rot}\mathbf{E}, \qquad \operatorname{div}\mathbf{B} = 0\,, \tag{5}$$

$$\mathbf{E} = -[\mathbf{v}\times\mathbf{B}]\,, \tag{6}$$

$$\frac{\partial\rho}{\partial t} = -\operatorname{div}\rho\mathbf{v}\,, \tag{7}$$

where

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\,.$$

Because of the neglected plasma pressure, there is no need for equations of state and energy.

Let us expand all the values in power series of ε^2 for example $\mathbf{v} = \mathbf{v}_0 + \varepsilon^2\mathbf{v}_1 + \cdots$, etc, and restrict ourselves to the zero-order equations. Of these, only the equation

$$\rho_0 \frac{d\mathbf{v}_0}{dt} = [\operatorname{rot}\mathbf{B}_1\times\mathbf{B}_0] + [\operatorname{rot}\mathbf{B}_0\times\mathbf{B}_1] \tag{8}$$

contains the terms of the next order. However these terms can be easily excluded when considering the scalar product of Equation (8) with $\mathbf{B}_0$. As a result we obtain the full system of equations defining the time evolution of a f.f. field in the zero order adiabatic approximation:

$$[\mathbf{B}_0\times\operatorname{rot}\mathbf{B}_0] = 0\,, \qquad \operatorname{div}\mathbf{B}_0 = 0\,, \tag{9}$$

$$\mathbf{B}_0\cdot\frac{d\mathbf{v}_0}{dt} = 0\,, \tag{10}$$

$$\mathbf{E}_0+[\mathbf{V}_0\times\mathbf{B}_0]=0\,, \tag{11}$$

$$\text{rot}\,\mathbf{E}_0=-\frac{\partial\mathbf{B}_0}{\partial t_0}\,, \tag{12}$$

$$\frac{\partial\rho_0}{\partial t}=-\text{div}\,\rho_0\mathbf{v}_0\,. \tag{13}$$

Note that often simply the sequence of independent solutions of Equation (9) for continuously changing boundary conditions is considered as the time evolution of f.f.f. (Barnes and Sturrock, 1972; Jockers, 1977). This approach neglects the frozen-in condition and corresponding motion of a plasma defined by equations (10)–(13). That this approach does not give in general a true time evolution is easily seen by the particular case of current-free field with X-type zero point. For that case the sequence of static solutions of boundary problem will evidently contain zero points while the true time evolution is the current sheet-formation (Syrovatskii, 1971).

For f.f.f. practically every magnetic force line is of X-type or of degenerate X-type and becomes X-type line for small perturbations (see below). So we can expect the singular behavior not only for current-free field but also for f.f.f. of general type.

This can be really seen if we study carefully the full system of Equations (9) to (13) for adiabatically slow evolution of f.f.f. Here Equations (9) define the equilibrium f.f. magnetic field $\mathbf{B}_0$ independently from the other equations. As it is usually supposed and was formally shown by Molodensky (1968) the boundary problem for this equation has a definite f.f.f. solution for given boundary values of $\mathbf{B}_0$ if some supplementary condition is fulfilled (this condition restricts the boundary values of $\mathbf{B}_0$ at points which are connected by the same magnetic force line).*

The solution $\mathbf{B}_0=\mathbf{B}_0(\mathbf{r},t)$ of Equations (9) for a given boundary problem depends on the time only parametrically because of the time dependence of the boundary conditions. With the known $\mathbf{B}_0(\mathbf{r},t)$ Equation (10) determines the velocity component along the magnetic field and Equation (11) across field lines if $\mathbf{E}_0$ is also known. With the given velocity field $\mathbf{v}_0(\mathbf{r},t)$ Equation (13) defines the density of the plasma.

To define the electric field $\mathbf{E}_0$ we have two equations,

$$\text{rot}\,\mathbf{E}_0=-\frac{\partial\mathbf{B}_0}{\partial t}\,, \tag{14}$$

$$\mathbf{B}_0\cdot\mathbf{E}_0=0\,. \tag{15}$$

The latter follows from (11). Equation (14) determines the inductive part of the

* The uniqueness of the solution was proved by Molodensky (1968) for the case of f.f.f. which differs only slightly from current free field. In what follows our arguments are based only on the assumption of existence of any f.f.f. solution, even not unique, but continuously changing with boundary values. If the last condition does not fullfield than the nonevolutionarity follows immediately.

electric field. Let $\mathbf{E}_c$ be a particular solution of Equation (14). Then the general solution for the electric field is

$$\mathbf{E}_0 = \mathbf{E}_c - \nabla\phi\,, \tag{16}$$

where the electric potential ϕ must be defined in such a way that the field $\mathbf{E}_0$ should satisfy Equation (15) and a given boundary condition. In particular, along the magnetic field line we must have $\mathbf{B}_0 \cdot \nabla\phi = \mathbf{B}_0 \cdot \mathbf{E}_c$ or

$$\frac{\partial\phi}{\partial l} = \mathbf{b}\mathbf{E}_c\,. \tag{17}$$

Here $\mathbf{b}$ is a unit vector in the magnetic field direction and l is the coordinate along the magnetic force line.

Now it can easily be seen that Equation (17) is contradictory for perturbations of a general type. Indeed both sides of (17) are determined independently and need not be equal. In general we can arbitrarily change the magnetic field $\mathbf{B}_0$ producing the inductive field E_c, but the potential ϕ is determined independently by the boundary condition for the electric field or by the magnetic field line structure.

There are two types of magnetic field lines of interest: those closing inside the volume considered and those intersecting the boundary.* For a line of the first type, when integrating (17) along a closed path consistent with the force line, we obtain for the left side of (17)

$$\oint \frac{\partial\phi}{\partial l}\,\mathrm{d}l = 0\,, \tag{18}$$

a well-known property of a potential field. However, the integral of the right-hand part of (17) gives

$$\oint \mathbf{E}_c\,\mathrm{d}\mathbf{l} = \int\!\!\int_S \operatorname{rot}\mathbf{E}_c\,\mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t}\int\!\!\int_S \mathbf{B}\,\mathrm{d}\mathbf{S}\,, \tag{19}$$

where S is a fixed surface bounded by a considered closed force line. Note that we consider a fixed path of integration and suppose that the component of magnetic field transverse to this path does not vanish identically in the vicinity of the considered closed force line. It follows then that for a general type of perturbation this integral need not be zero. Hence Equation (17) cannot be fulfilled and in the general case there appears an electric field along the closed magnetic field lines.

The essential point in this proof is the fact that in adiabatic approximation equation for magnetic field (9) gets detached from other equations. Hence the magnetic field can be determined independently from the frozen-in condition (Equations (11) and (12)) and this is the reason for contradiction.

* There may also exist lines (ergodic) which are not closed and do not intersect the boundary. Our conclusions do not concern these lines.

Formally, the result is equivalent to the statement that the frozen-in condition, i.e. equation

$$\mathrm{rot}\,[\mathbf{v}\times\mathbf{B}]=\frac{\partial\mathbf{B}}{\partial t}\,, \tag{20}$$

has in general no solution for velocity $\mathbf{v}$ if an arbitrary changing magnetic field $\mathbf{B}(\mathbf{r}, t)$ is given. Indeed, the integral form of Equation (20) for the surface S bounded by the closed magnetic force line L,

$$\oint_L [\mathbf{v}\times\mathbf{B}]\,\mathrm{d}\mathbf{l}=\frac{\partial}{\partial t}\int\!\!\int_S \mathbf{B}\,\mathrm{d}\mathbf{S}\,, \tag{21}$$

shows that the right-hand side is in general non-zero while there does not exist velocity field which can make non-zero left-hand side because of $\mathrm{d}l$ is along $\mathbf{B}$.

It means that the magnetic force lines with properties discussed are singular in the sense that adiabatic approximation does not work in their vicinity and small boundary perturbations produce drastic changes in the magnetic field and plasma flow.

The simple example of singular line is the circular zero line between two magnetic dipoles of the same orientation placed in line one after other. Using the usual approach based on the Equation (9) only we obtain, when changing the distance between dipoles, the sequence of fields with circular zero lines while the true time evolution results in current sheet development (Syrovatskii, 1969).

For a f.f. field practically every force line is of the X-type or degenerate X-type. This means that in the vicinity of these force lines the transverse component of the magnetic field behaves like a two-dimensional magnetic field near a zero (neutral) line or near a neutral sheet. In the latter case the force line lies on the magnetic surface of sheared field and becomes X-type one for small perturbations. For a two-dimensional force-free (really potential) field the zero-line is a singular line in the sense that the problem of an adiabatically slow deformation in general has no solution. Here perturbations produce discontinuities – pinch (current) sheets (Syrovatskii, 1971). We see that force-free singular lines differ from zero lines of a potential field only by the presence of a longitudinal magnetic field.

For the lines intersecting the boundary, the situation is even simpler. We need to consider only perturbations of the electric field at the boundary which produce a non-zero electric field along a given static magnetic force line. Then we have a zero right-hand side of Equation (17), but a non-zero left-hand one.

Thus, a large class of field lines of a f.f. field can be singular in the sense that there exist perturbations which produce an electric field along these lines in contradiction with (15). In this respect singular lines of a f.f. field are a particular case of the general singular line for an ideal magnetohydrodynamic system (Syrovatskii, 1977a, b). By analogy with a potential field we may conclude that disturbances of a force-free field produce in general pinch (current) sheets in which the external magnetic pressure is balanced by the inner gas pressure and $\beta \geq 1$. Hence the f.f.

field ceases to be f.f. in the whole volume and includes non-force-free features – pinch sheets. In this respect a f.f. field does not differ essentially from a potential field. This does not contradict the assumption that on the average we have $\beta \ll 1$ in the volume considered because large pressures ($\beta \gtrsim 1$) occur only in very thin sheets.

Note here that f.f.field may also contain a number of 0-type magnetic force lines, for which the transverse magnetic field has an 0-type neutral (zero) point. If an 0-type force line is singular, i.e. there exist electric field along this line, then there develops evidently a usual cylindrical pinch.

We see that for perturbations of a general type, an initial f.f. field does not stay f.f. in the whole volume considered and in this sense is non-evolutionary.

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