A NOTE ON THE SOLUTION OF THE SATURATION FLUX LIMITED SOLAR WIND EQUATIONS

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Abstract. The solution curves of the differential equations determining the behavior of the solar wind are calculated for the case where the heat flux has its maximum value $\frac{3}{2}\, nkTv_{th}$. All the supersonic solutions are asymptotically adiabatic, $T \sim r^{-4/3}$.

1. Introduction

The simplest model of the solar wind treats the plasma in a simple collision dominated approximation yielding (Chapman, 1954) for the conductive flux

$$F = -\,\kappa_0 T^{5/2} \frac{\mathrm{d}T}{\mathrm{d}r}. \tag{1}$$

Such a model is only valid if the plasma remains collision dominated, that is, if the mean-free path is small compared to the scale of variation of the temperature, T. The mean-free path is readily calculated to be

$$l = \frac{(3kT)^2}{5.7\pi n e^4 \ln \Lambda}, \qquad \Lambda = \frac{3}{2e^3}\left(\frac{K^3 T^3}{\pi n e}\right)^{1/2} \tag{2}$$

with $\ln \Lambda$ lying between 20 and 25 for densities and temperatures of the solar wind. Now the collision dominated solutions have two types of asymptotic behavior $T \sim r^{-2/7}$ (Parker, 1958) and $T \sim r^{-4/3}$ (Durney, 1971) and an asymptotically constant supersonic flow, the number density $n \sim r^{-2}$. If for the temperature and density in the solar corona the $\frac{2}{7}$ solution is the relevant one then the ratio of mean-free path to scale height is

$$\varepsilon = \frac{(3kT)^2\, \mathrm{d}T/\mathrm{d}r}{400\, n e^4 T} \sim r^{3/7} \tag{3}$$

which increases as r increases; clearly ε ultimately exceeds unity and the plasma is no longer collision dominated. (The $r^{-4/3}$ solution is collision dominated.) Indeed, in many models of solar and stellar winds this happens before the asymptotic regime is reached.

Ideally we need a non-local theory of heat transport in the weak collision case but progress can be made by noting that the maximum heat flux is that when all the particles carry their thermal energy with the thermal speed (this corresponds to a

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delta function distribution function), and gives an energy flux

$$F = \beta \tfrac{1}{2} n_e v_e \, \tfrac{1}{2} n_e v_e^2 = \beta \frac{1}{4} \frac{n_e}{n_e^{1/2}} (3kT)^{3/2}, \tag{4}$$

where β is a factor of order unity which is due to the non-delta function type of distribution function. The problem analyzed in this paper is the solution of the solar wind when the heat flux is given by this expression.

2. Equations Governing the Flow

The motion of the solar wind is governed by Eulers equation, mass and energy conservation which are

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{-1}{mn} \frac{\mathrm{d}}{\mathrm{d}r} (nkT) - \frac{GM}{r^2} \tag{5}$$

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} (r^2 F) = \frac{-3\varrho}{2} v \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{nkT}{(mn)^{5/3}} \right) \tag{6}$$

$$nvr^2 = \text{constant}. \tag{7}$$

We make the standard transformation to the dimensionless variables τ, ψ, λ, first introduced by Chamberlain (1961) and modified by Roberts (1971),

$$\tau = \frac{T}{T_1 \varepsilon_\infty}, \qquad \psi = \frac{mv^2}{kT_1 \varepsilon_\infty}, \qquad \lambda = \frac{GMm}{kT_1 \varepsilon_\infty r}, \tag{8}$$

where $\varepsilon_\infty kT_1$ is the residual energy per particle at infinity, T_1 is a reference temperature, M the mass of the Sun and m the average mass of the particles, $= m_{\mathrm{H}}/2$ for hydrogen. In terms of these variables Equations (5) and (6) transform to

$$\frac{1}{2} \frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = \frac{1 - 2\tau/\lambda - \mathrm{d}\tau/\mathrm{d}\lambda}{(1 - \tau/\psi)} \tag{9}$$

$$\alpha \left(\frac{\tau}{\psi} \right)^{1/2} \tau = 1 + \lambda - \frac{5\tau}{2} - \frac{\psi}{2} \tag{10}$$

where $\alpha = \beta 3^{3/2} (m/m_e)^{1/2}/4$ taken for convenience to be 23 in the worked example.

3. Solutions of the Equations

To simplify the equations it is convenient to introduce $\omega = (\tau/\psi)^{1/2}$, and to eliminate $\mathrm{d}\psi/\mathrm{d}\lambda$, yielding

$$\tfrac{1}{2} (\alpha\omega^3 + 5\omega^2 - 3\alpha\omega - 3) \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = (1 - \alpha\omega) \left(1 - \frac{2\tau}{\lambda} \right) - (1 - \omega^2) \tag{11}$$

$$\alpha\omega = \frac{1 + \lambda}{\tau} - \frac{5}{2} - \frac{1}{2\omega^2}. \tag{12}$$

We consider now the general properties of the solutions.

3.1. Locus of Infinite Derivatives

$d\tau/d\lambda$ is infinite when its coefficient in Equation (11) vanishes, that is

$$f(\omega) \equiv \alpha\omega^3 + 5\omega^2 - 3\alpha\omega - 3 = 0. \tag{13}$$

This cubic has two turning points at $5/3\alpha(\sqrt{1+(9\alpha^2/25)} \pm 1)$, and $f(\omega) \to \infty$, $\omega \to \infty$, $f(0) = -3$, there is therefore only one positive root ω_0. For $\alpha = 23$, $\omega_0 = 1.65$. The locus $d\tau/d\lambda$ infinite is then given by Equation (12) with $\omega = \omega_0$,

$$\tau = \frac{2\omega_0^2}{(2\alpha\omega_0^3 + 5\omega_0^2 + 1)}(1 + \lambda) = 0.024(1 + \lambda). \tag{14}$$

3.2. Locus of Zero Derivatives

$d\tau/d\lambda$ is zero when the right hand side of Equation (11) vanishes and is therefore given parametrically by

$$(1 - \alpha\omega^3)\left(1 - \frac{2\tau}{\lambda}\right) = (1 - \omega^2), \quad \alpha\omega = \frac{1 + \lambda}{\tau} - \frac{5}{2} - \frac{1}{2\omega^2}. \tag{15}$$

This curve has two limiting solutions, one has $\omega \to \infty$, $\tau \to 0$; $\lambda \to 0$, $\tau \to \lambda/2$, the other has $\omega \to \alpha^{-1/3}$ $\tau \to 2/(5 + 3\alpha^{2/3})$ as $\lambda \to 0$. These two limit curves join together giving just one curve in the (τ, λ) plane where $d\tau/d\lambda = 0$.

3.3. Critical Point

The previous two curves intersect at one point, the critical point, which is given by the simultaneous solution of Equations (13) and (15). For our example this is where $\tau = 0.025$, $\lambda = 0.052$.

3.4. Limiting Solution

There is also an exact solution of the equations which have $\omega = \alpha^{-1/3}$ and

$$\tau_e = \frac{2}{(5 + 3\alpha^{2/3})}(1 + \lambda) = 0.068(1 + \lambda). \tag{16}$$

This curve is an upper limit on all possible solutions since if using Equation (12) we maximise τ for a given λ this gives $\omega = \alpha^{-1/3}$.

In the neighbourhood of this solution we write

$$\tau = \tau_e(1 + \tau_1)\ \omega = \alpha^{-1/3}(1 + \omega_1) \tag{17}$$

and determine the equations satisfied τ_1 and ω_1. A little care is needed since τ_1 is of order ω_1^2 and is given by

$$\frac{d\tau_1}{d\lambda} = \frac{8\sqrt{3}\alpha^{1/3}}{(5 + 3\alpha^{2/3})^{3/2}(1 - \alpha^{2/3})} \frac{(-\tau_1)^{1/2}}{\lambda} \tag{18}$$

which has the solution

$$\tau_1 = -\frac{48\alpha^{2/3}}{(5+3\alpha^{2/3})^3 (1-\alpha^{2/3})^2}\, m\,(\lambda/\lambda_0)^2 . \tag{19}$$

As we would expect τ_1 is always negative; is zero for some $\lambda = \lambda_0$ and depart from τ_e for both large and small λ.

3.5. Asymptotic behaviour for $\lambda \to 0$

If we look for a power series expansion with

$$\tau = \tau_0 \lambda^n (1 + \tau_1 \lambda + \cdots), \qquad \omega = \omega_0 \lambda^m (1 + \omega_1 \lambda + \cdots) \tag{20}$$

there are three possibilities $\omega \to 0$, ∞ or ω_1 a constant.

3.5.1. $\omega \to 0$, $\lambda \to 0$

In this case $m > 0$ and from Equation (12)

$$\frac{1+\lambda}{\tau} - \frac{5}{2} - \alpha\omega = \frac{1}{2\omega^2} \to \infty, \qquad \tau \to 2\omega^2 \to 0 . \tag{21}$$

Substituting into Equation (13) the leading terms are

$$-\tfrac{3}{2}\tau_0 n \lambda^{n-1} = -2\tau_0 \lambda^{n-1} \tag{22}$$

so $n = \frac{4}{3}$ and τ_0 is arbitrary. There is hence a one parameter family of solutions which behave asymptotically like

$$\tau = \tau_0 \lambda^{4/3} (1 + \cdots), \qquad \omega = \left(\frac{\tau_0}{2}\right)^{1/2} \lambda^{2/3} (1 + \cdots) \tag{23}$$

with τ_0 the free parameter. These solutions are supersonic $\omega \to 0$, and are asymptotic to the regular adiabatic $\frac{4}{3}$ solution that exists for the normal solar wind (Durney, 1971).

3.5.2. $\omega \to \omega_2$ *a constant*

In this case Equation (14) gives

$$\alpha\omega_2 + \frac{1}{2\omega_2^2} + \frac{5}{2} = \frac{1+\lambda}{\tau} \to \frac{1}{\tau_0} \tag{24}$$

since the left hand side is positive definite, $\tau_0 > 0$. Turning to Equation (13) the term $2\tau/\lambda \to \infty$ as $\lambda \to 0$ and cannot be balanced unless $1 - \alpha\omega^3 \to 0$, which gives

$$\omega \to \alpha^{-1/3}, \qquad \tau \to \frac{2}{(5 + 3\alpha^{2/3})} \tag{25}$$

which is where τ_e meets the τ axis.

However we have already seen that all other solutions diverge from τ_e and so the only possibility is $\tau = \tau_e$.

3.5.3. $\omega \to \infty$

In this case it follows from Equation (14) that

$$\tau \to \frac{1}{\alpha\omega} \tag{26}$$

and then Equation (13) requires, to leading order,

$$\frac{\alpha\omega^3}{2}\frac{d\tau}{d\lambda} = -\alpha\omega^3(1 - 2\tau/\lambda) \tag{27}$$

and since $\tau \to \frac{1}{2}\omega \to 0$ the only solution of this is

$$\tau = \tfrac{2}{3}\lambda + \cdots. \tag{28}$$

This is in fact another limiting solution. To demonstrate this consider first the general power series expansion

$$\tau = \tfrac{2}{3}\lambda(1 + \Sigma\tau_n\lambda^n), \qquad \omega = \frac{3}{2\lambda\alpha}(1 + \Sigma\omega_n\lambda^n). \tag{29}$$

Substitution into Equations (13) and (14) gives the equations

$$\begin{aligned}
&\omega_1 + \tau_1 = -\tfrac{2}{3}\\
&\omega_2 + \tau_2 = \tau_1^2 - \tau_1\\
&\omega_3 + \tau_3 = 2\tau_2\tau_1 - \tau_1^3 - \tau_2 + \tau_1^2 - 4\alpha^2/27\\
&\omega_4 + \tau_4 =\\
&4\tau_1 + 2 = 2\tau_1 + \tfrac{10}{3}\\
&4\tau_2 - 2\omega_1 = 3\tau_2 + 20\tau_1/3 - 10\omega_1/3 - 4\alpha^2/3\\
&4\tau_3 - 2(\omega_2 - \omega_1^2) - 32\alpha^2/27 = 4\tau_3 + 10\tau_2 - 20\tau_1\omega_1/3 - 8\alpha^2\tau_0/3 +\\
&\qquad - 10\omega_2/3 + 10\omega_1^2 - d\omega_1\alpha^2 - 8^2/9\\
&4\tau_4 + \cdots = 5\tau_4 + \cdots
\end{aligned}$$

It is clear that a degeneracy occurs in the third equation of the second set, the τ_3 terms cancel and this equation becomes a consistency condition on τ_2, ω_2, τ_1, ω_1 which are already known. In principle there are two possibilities; this equation is identically satisfied, in which case τ_3 is arbitrary and represents the degree of freedom corresponding to the one parameter family asymptoting to $\tau = \frac{2}{3}\lambda$, or it is not satisfied in which case the degree of freedom enters through a logarithmic branch.

$$\tau = \tfrac{2}{3}\lambda(1 + \tau_1 + \lambda_2)^2 + \tau_3\lambda^3 + \cdots) + s_0\lambda^4 \ln\lambda(1 + s_1\lambda + s_0)^2 + \cdots) \tag{31}$$

$$\omega = \frac{3}{2\alpha\lambda}(1 + \omega_1\lambda + \cdots) + \mu_0{}^4\ln\lambda(1 + \mu_1\lambda + \cdots).$$

In the present problem the constraint is not satisfied and we have the logarithmic branch. All solutions with $\omega \to \infty$ $\lambda \to 0$ converge on $\tau = 2\lambda/3$.

3.6. Asymptotic behaviour $\lambda \to \infty$

Again there are in principle the possibilities $\omega \to \infty$, 0, ω_3. Since we have already identified the $\omega \to 0$ $\omega \to \infty$ behaviour we need only consider $\omega \to \omega_3$. In this case we note that from Equations (14) if $\lambda \to \infty$, $\omega \to \omega_3$ then $\tau \to \tau_0 \lambda$, and it then follows by substitution that

$$\tfrac{1}{2}\tau_0 (\alpha\omega_3^3 + 5\omega_3^2 - 3\alpha\omega_3 - 3) = (1 - \alpha\omega_3^3)(1 - 2\tau_0) - (1 - \omega_3^2) \tag{32}$$

$$\frac{1}{\tau_0} = \alpha\omega_3 + \tfrac{5}{2} + \frac{1}{2\omega_3^2} \tag{33}$$

and this has the unique solution $\omega_3^3 = 1/\alpha$. This is just the solution τ_e we identified previously namely

$$\tau_e = \frac{2(1 + \lambda)}{(5 + 3\alpha^{2/3})}. \tag{34}$$

Since this is the only asymptotic solution it follows that along any solution curve that goes to infinity $\omega \to \alpha^{-1/3}$.

3.7. Other remarks

It is a help in calculating the solutions to notice that τ can vanish for finite λ and that in this case $\omega \to \infty$, and $d\tau/d\lambda \to -2$. The other possibility namely $\tau \to 0$ is just the $\lambda^{4/3}$ family of solutions.

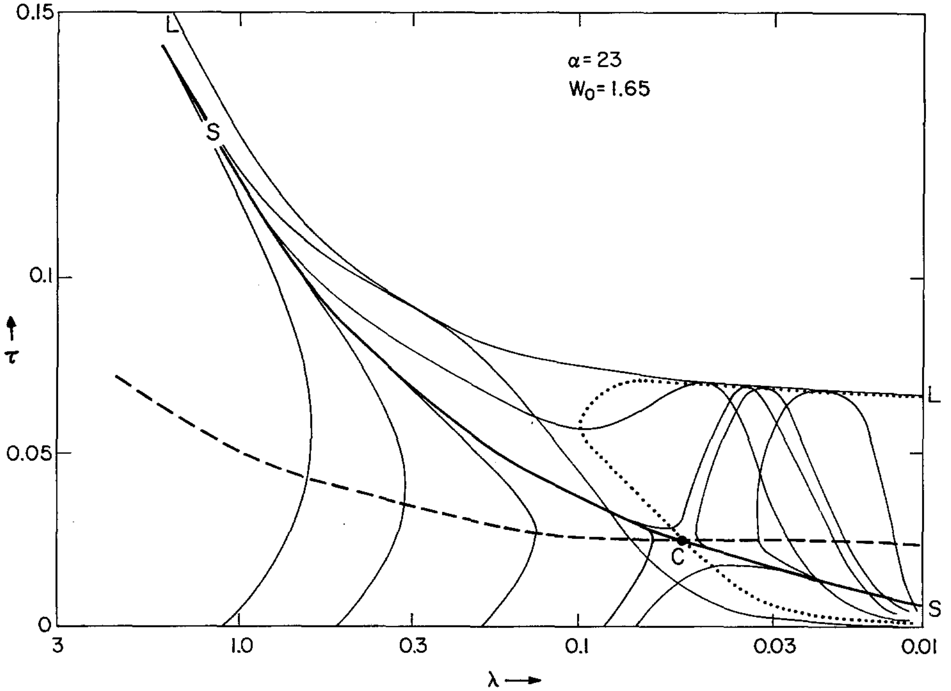

Fig. 1. Solution of the saturation flux equations.

4. The Solution Curves

In Figure 1 we have drawn the family of solutions to the saturation flux equations. The salient features are

(a) the locus $\tau'=0$, the dotted line;

(b) the locus $\tau'=\infty$, the dashed line;

(c) the critical point, C;

(d) the limiting solution, L;

(e) the critical solution, the bold line that extends from $\tau=0$, $\lambda=0$, to $\tau=\infty$, $\lambda=\infty$ and from which all solutions diverge.

Not all solutions take all values of ω. There are a set of solutions beginning and ending at $\tau=\lambda=0$, on which ω varies monotonically from 0 to ∞. Then there is a set whichstarts at $\omega=0$, has ω increasing to $\alpha^{-1/3}$, increases further to a maximum somewhere between $\alpha^{-1/3}$ and ω_1, the value of ω on the curve $\tau'=\infty$, and then ω decreases to $\alpha^{-1/3}$ again. A third family has ω infinite at $\tau=\lambda=0$, ω decreases reaches a minimum and then increases again until the solution curve intersects the λ axis. A fourth set has $\omega=\alpha^{-/13}$ as $\tau=\infty$ $\lambda=\infty$, τ increases monotonically until $\omega=\infty$ when again the solution crosses the λ axis.

5. Conclusions

By far the most important conclusion and the one of relevance to the solar wind is that for any value of τ and λ there is always a supersonic solution that is well behaved, $\omega\to 0$, $\tau\to 0$ $\lambda\to 0$, $\tau\to\lambda^{4/3}$. This is similar to the adiabatic wind. However, the fact that all supersonic solutions have a $\lambda^{4/3}$ dependence is of greater importance; along such a curve the ratio of the mean free path to the scale of variation of the temperature is

$$\varepsilon=\frac{l}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\sim\frac{T^2}{nr}\sim\frac{1}{r^{5/3}}\to 0\,. \tag{35}$$

The plasma becomes collision dominated along the collisionless solution. We therefore have the situation that a solution of the full conducting wind that is Parker like $(T\sim 1/r^{2/7})$ becomes collisionless, the saturation flux then forces the solution over at large distances into a Durney like solution $(T\sim 1/r^{4/3})$ and again becomes collision dominated.

The actual nature of the solutions and the extent of the transition region will be dealt with in a subsequent communication.

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