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Flocks and Ovals*

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Abstract. An infinite family of q-clans, called the *Subiaco* q-clans, is constructed for $q = 2^e$. Associated with these q-clans are flocks of quadratic cones, elation generalized quadrangles of order (q^2, q), ovals of $\mathrm{PG}(2, q)$ and translation planes of order q^2 with kernel $\mathrm{GF}(q)$. It is also shown that a q-clan, for $q = 2^e$, is equivalent to a certain configuration of $q + 1$ ovals of $\mathrm{PG}(2, q)$, called a *herd.*

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1. Introduction

In $\mathrm{PG}(2, q)$ an *oval* is a set of $q + 1$ points, no three collinear. A *hyperoval* is a set of $q + 2$ points, no three collinear. Hyperovals exist only when q is even. Since $\mathrm{PGL}(3, q)$ is transitive on the ordered quadrangles of $\mathrm{PG}(2, q)$ we can map any hyperoval to an equivalent hyperoval containing the *fundamental quadrangle* $\{(0,0,1), (0,1,0), (1,0,0), (1,1,1)\}$. From this we can represent every hyperoval, $\mathcal{H}$, on the fundamental quadrangle in $\mathrm{PG}(2, q)$ by a permutation, f, of $\mathrm{GF}(q)$, with $f(0) = 0$ and $f(1) = 1$:

$$\mathcal{H} = \{(1, t, f(t)) \mid t \in \mathrm{GF}(q)\} \cup \{(0,0,1), (0,1,0)\}.$$

Permutations that describe hyperovals in this way are called *o-polynomials.* (See [7] for a reference to the above work, noting that the word oval is used for hyperoval.)

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We define trace: GF(q)→GF(2), where $q = 2^e$, by

$$\mathrm{trace}(x) = x + x^2 + x^4 + \cdots + x^{2^{e-1}}.$$

A fact we shall frequently use is that the quadratic equation $ax^2 + bx + c, a, b, c \in \mathrm{GF}(q), a \neq 0$, is irreducible over GF(q) if and only if $b \neq 0$ and $\mathrm{trace}((ac)/b^2) = 1$.

2. Herds

2.1. NORMALIZATION

Let $\mathbf{C} = \{A_t \mid t \in \mathrm{GF}(q)\}$ be a family of 2×2 matrices with entries in GF(q). We define the quadratic form Q_{st} as

$$Q_{st}(x,y) = (x \;\; y)(A_s - A_t)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Following Payne [16], [15], [1], we have $\mathbf{C}$ being a *q-clan* if Q_{st} is anisotropic for all $s \neq t$.

If $\mathbf{C} = \{A_t \mid t \in \mathrm{GF}(q)\}$ is a q-clan, so is $\mathbf{C}' = \{A_t - A_0 \mid t \in \mathrm{GF}(q)\}$; so without loss of generality we let A_0 equal the zero matrix $\mathbf{0}$. Also if

$$A_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad t \in \mathrm{GF}(q),$$

are the matrices of a q-clan then so are the matrices

$$A_t' = \begin{pmatrix} a & b+c \\ 0 & d \end{pmatrix}, \quad t \in \mathrm{GF}(q);$$

hence without loss of generality each A_t is upper triangular. If $A_t = \begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix}$ then

$$Q_{st}(x,y) = (a_s + a_t)x^2 + (b_s + b_t)xy + (c_s + c_t)y^2.$$

As we shall only be concerned with fields of characteristic 2, the above can be rewritten as: Q_{st} is anisotropic for all $s \neq t$ if and only if

$$\mathrm{trace}\left(\frac{(a_s + a_t)(c_s + c_t)}{(b_s + b_t)^2}\right) = 1 \quad \text{for all } s \neq t.$$

Since Q_{st} is anisotropic for $s \neq t$, we have $b_s \neq b_t$ for all $s \neq t$. So $t \mapsto b_t$ is a permutation; we may relabel the subscript so that $b_t = t^{1/2}$. We have

$$\mathbf{C}' = \left\{ A'_t = \begin{pmatrix} a_1^{-1/2} & 0 \\ 0 & a_1^{1/2} \end{pmatrix} A_t \begin{pmatrix} a_1^{-1/2} & 0 \\ 0 & a_1^{1/2} \end{pmatrix} \middle| \, t \in \mathrm{GF}(q) \right\},$$

is also a q-clan with

$$A'_0 = \mathbf{0}, \quad A'_t = \begin{pmatrix} a'_t & t^{1/2} \\ 0 & c'_t \end{pmatrix} \text{ and also } A'_1 = \begin{pmatrix} 1 & 1 \\ 0 & c'_1 \end{pmatrix}.$$

So without loss of generality, $a_1 = 1$.

Let $a = c'_1$ (since Q_{01} is anisotropic, trace$(a) = 1$). Define $f\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$ by $f(t) = a_t$ and $g\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$ by $g(t) = c_t/a$. Then $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Since Q_{st} is anisotropic for all $s \neq t$, we have, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$,

$$\mathcal{T}_a(f,g)\colon \mathrm{trace}\left(\frac{a(f(s)+f(t))(g(s)+g(t))}{s+t}\right) = 1 \quad \text{for all } s \neq t.$$

Conversely, assuming $\mathcal{T}_a(f,g)$, then if $A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & ag(t) \end{pmatrix}$, then $\mathbf{C} = \{A_t \mid t \in \mathrm{GF}(q)\}$ is a q-clan with

$$A_0 = \mathbf{0} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$$

where trace$(a) = 1$. We use this normalization in the next section.

The main theorem of the next section shows one motivation for studying q-clans. Others follows: elation generalized quadrangles from q-clans [14], [9]; flocks of quadratic cones from q-clans [23]; translation planes from flocks [5], [25].

2.2. EQUIVALENCE OF HERDS AND q-CLANS, q EVEN

THEOREM 1. *Let q be even. Let $f, g\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then $\mathcal{T}_a(f,g)$ is true if and only if g is an o-polynomial, f_s is an o-polynomial for all $s \in \mathrm{GF}(q)$ where*

$$f_s(x) = \frac{f(x) + asg(x) + s^{1/2}x^{1/2}}{1 + as + s^{1/2}}$$

and trace$(a) = 1$.

Proof. ($\Rightarrow$) Suppose $\mathcal{T}_a(f,g)$ is true. We also suppose that $f(0) = 0$ and $f(1) = 1$. The function f is one-to-one since if $x \neq y$ and $f(x) = f(y)$ then

$$\frac{a(f(x)+f(y))(g(x)+g(y))}{x+y} = 0,$$

contradicting $\mathcal{T}_a(f,g)$. Let $\mathcal{H}$ be the set of points $\{(1,t,f(t)) \mid t \in \mathrm{GF}(q)\} \cup \{(0,1,0),(0,0,1)\}$. Since f is one-to-one no line on $(0,1,0)$ meets $\mathcal{H}$ in more than two points. Clearly no line on $(0,0,1)$ meets $\mathcal{H}$ in more than two points. We show that no three points of $\{(1,t,f(t)) \mid t \in \mathrm{GF}(q)\}$ are collinear.

Suppose $x,y,z \in \mathrm{GF}(q)$ are distinct and that the points $(1,x,f(x)),(1,y,f(y))$ and $(1,z,f(z))$ are collinear. Then

$$\frac{f(x)+f(y)}{x+y} = \frac{f(x)+f(z)}{x+z} = \frac{f(y)+f(z)}{y+z} = b, \text{ say.}$$

Since trace is additive we have

$$\mathrm{trace}(ab(g(x)+g(y))) + \mathrm{trace}(ab(g(x)+g(z))) = \mathrm{trace}(ab(g(y)+g(z))).$$

But this is contrary to $\mathcal{T}_a(f,g)$. So $\mathcal{H}$ is a hyperoval. As $f(0) = 0$ and $f(1) = 1$, f is an o-polynomial.

Since $\mathcal{T}_a(f,g)$ is true if and only if $\mathcal{T}_a(g,f)$ is true, g is also an o-polynomial.

We now look at $\mathrm{trace}(b(f(x)+f(y))(f_s(x)+f_s(y))/(x+y))$ where $b = a + s^{-1} + s^{-1/2}$:

$$\mathrm{trace}\left(\frac{b(f(x)+f(y))\left(\dfrac{f(x)+asg(x)+s^{1/2}x^{1/2}}{1+as+s^{1/2}} + \dfrac{f(y)+asg(y)+s^{1/2}y^{1/2}}{1+as+s^{1/2}}\right)}{x+y}\right),$$

$$x \neq y$$

$$= \mathrm{trace}\left(\frac{a(f(x)+f(y))(g(x)+g(y))}{x+y}\right)$$

$$+ \mathrm{trace}\left(\frac{1}{s}\frac{(f(x)+f(y))^2}{x+y} + \frac{1}{s^{1/2}}\frac{f(x)+f(y)}{(x+y)^{1/2}}\right), x \neq y$$

$$= \mathrm{trace}\left(\frac{a(f(x)+f(y))(g(x)+g(y))}{x+y}\right), x \neq y, \quad \text{since } \mathrm{trace}(X^2+X) = 0.$$

So $\mathcal{T}_b(f,f_s)$ is true if and only if $\mathcal{T}_a(f,g)$ is true. Since $f_s(0) = 0$ and $f_s(1) = 1$ for all $s \in \mathrm{GF}(q)$, f_s is an o-polynomial. Putting $x = 0$ and $y = 1$ in $\mathcal{T}_a(f,g)$, we see that $\mathrm{trace}(a) = 1$.

($\Leftarrow$) Let

$$f_s(x) = \frac{f(x) + asg(x) + s^{1/2}x^{1/2}}{1 + as + s^{1/2}}$$

for some a with trace$(a) = 1$. Suppose that f_s is an o-polynomial for all $s \in$ GF(q).

Fix $x \neq y$. Then $(1, x, f_s(x)), (1, y, f_s(y))$ and $(0, 1, 0)$ are not collinear for all $s \in$ GF(q). So $f_s(x) \neq f_s(y)$, giving $f_s(x) + f_s(y) \neq 0$, that is,

$$f(x) + f(y) + s(ag(x) + ag(y)) + s^{1/2}(x^{1/2} + y^{1/2}) \neq 0$$

for all $s \in$ GF(q).

The above equation is a quadratic in $s^{1/2}$. Hence this implies that

$$\text{trace}\left(\frac{(f(x) + f(y))(ag(x) + ag(y))}{x + y}\right) = 1.$$

Thus $\mathcal{T}_a(f, g)$ holds. □

A *herd* of ovals in PG$(2, q)$, q is even, is a family of $q + 1$ ovals $\{\mathcal{O}_s \mid s \in$ GF$(q) \cup \{\infty\}\}$, each containing (1, 0, 0), (0, 1, 0) and (1, 1, 1) and with nucleus (0, 0, 1), with

$$\mathcal{O}_\infty = \{(1, t, g(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\},$$

$$\mathcal{O}_s = \{(1, t, f_s(t)) \mid t \in \text{GF}(q)\} \cup \{(0, 1, 0)\}, \quad s \in \text{GF}(q),$$

where

$$f_s(t) = \frac{f_0(x) + asg(x) + s^{1/2}x^{1/2}}{1 + as + s^{1/2}},$$

for some a where trace$(a) = 1$.

Thus the last theorem says that, for q even, a q-clan, **C**, gives rise to a herd of ovals of PG$(2, q)$, which we shall denote by H(**C**), and conversely.

Remarks. 1. The 'only if' part of the theorem is due to Payne [14], although not explicitly stated there. The proof given here is new, and, in particular, does not involve generalized quadrangles.

2. We have provided a proof of the existence of Payne's [14] hyperovals that does not involve the use of generalized quadrangles, as desired by Cherowitzo [3].

3. This theorem is used in [22] to classify 32-clans by computer. Results are also obtained there for q-clans, q even, q small.

4. The sufficiency half of the proof needs only the hypothesis that each f_s is a permutation.

3. Elation Generalized Quadrangles

Let

$$G = \{(\mathbf{a}, c, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathrm{GF}(q)^2, c \in \mathrm{GF}(q)\}$$

with multiplication defined as

$$(\mathbf{a}, c, \mathbf{b})(\mathbf{a}', c', \mathbf{b}') = (\mathbf{a} + \mathbf{a}', c + c' + \mathbf{b} \circ \mathbf{a}', \mathbf{b} + \mathbf{b}'),$$

where

$$\mathbf{b} \circ \mathbf{a} = \sqrt{\mathbf{b} P \mathbf{a}^T} \text{ with } P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathbf{C} = \{A_t \,|t \in \mathrm{GF}(q)\}$ be a q-clan where $A_t = \begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix}$, $a_t, b_t, c_t \in \mathrm{GF}(q)$.

We have the associated 4-gonal family ([19]) $\{A(t) \mid t \in \mathrm{GF}(q) \cup \{\infty\}\}$ given by

$$A(\infty) = \{(0, 0, \mathbf{b}) \in G \mid \mathbf{b} \in \mathrm{GF}(q)^2\},$$

$$A(t) = \{(\mathbf{a}, \sqrt{\mathbf{a} A_t \mathbf{a}^T}, b_t \mathbf{a}) \mid \mathbf{a} \in \mathrm{GF}(q)^2\}, t \in \mathrm{GF}(q).$$

The *centre* of G is

$$Z = \{(0, c, 0) \mid c \in \mathrm{GF}(q)\}.$$

For $t \in \mathrm{GF}(q) \cup \{\infty\}$ the *tangent space* at $A(t)$ is

$$A^*(t) = A(t)Z.$$

The construction of the generalized quadrangle from $\mathbf{C}$ is as follows: Points: (i) elements $g \in G$; (ii) cosets $A^*(t)g, t \in \mathrm{GF}(q) \cup \{\infty\}, g \in G$; (iii) a new symbol (∞). Lines: (a) cosets $A(t)g, t \in \mathrm{GF}(q) \cup \{\infty\}, g \in G$; (b) symbols $[A(t)], t \in \mathrm{GF}(q) \cup \{\infty\}$. Incidence: point (∞) is on the $q + 1$ lines $[A(t)]$; point $A^*(t)g$ is on the line $[A(t)]$ and on the q lines, $A(t)g$, contained in $A^*(t)g$; point g is on the $q + 1$ lines $A^*(t)g$ which contain g; there are no other incidences.

This gives an elation generalized quadrangle, GQ($\mathbf{C}$), of order (q^2, q), q even, whenever $\mathbf{C}$ is a q-clan.

4. Flocks of Quadratic Cones

Let $\mathcal{O}$ be an oval in PG$(2,q)$. Embed PG$(2,q)$ in PG$(3,q)$, and take a point v of PG$(3,q)$ not in the embedded plane PG$(2,q)$. The union of points of the lines incident with the point v and the oval $\mathcal{O}$ is a *cone* with *vertex* v and *base* $\mathcal{O}$. The lines of the cone are sometimes referred to as the *generators* of the cone. A *quadratic cone* is a cone where the base $\mathcal{O}$ is a (nondegenerate) conic. A *flock* of a cone is a set of q planes partitioning the cone minus the vertex v into disjoint ovals. If all the planes of the flock meet in an (external) line we say that the flock is *linear*. There exists linear flocks of a cone in PG$(3,q)$ for all q. The only flocks of cones in PG$(3,q)$, where $q = 2, 3$, and 4, are the linear flocks [23].

Let $\mathcal{K}$ be a quadratic cone in PG$(3,q)$ defined by

$$X_0X_1 = X_2^2.$$

The q planes, π_t, with $t \in \mathrm{GF}(q)$, of a flock $\mathcal{F}$ which do not contain the vertex $(0, 0, 0, 1)$ of $\mathcal{K}$, can be described by the set of equations

$$a_tX_0 + c_tX_1 + b_tX_2 + X_3 = 0 \quad \text{for } t \in \mathrm{GF}(q).$$

THEOREM 2 ([14], [23]). *Let* $q = 2^e$. *We have*

$$\mathcal{F} = \{a_tX_0 + c_tX_1 + b_tX_2 + X_3 = 0 \mid t \in \mathrm{GF}(q)\},$$

being a flock of a quadratic cone $\mathcal{K}$ *if and only if, given* $b_t \neq b_s$ *whenever* $s \neq t$,

$$\operatorname{trace}\left(\frac{(a_s + a_t)(c_s + c_t)}{(b_s + b_t)^2}\right) = 1 \quad \textit{for all } s \neq t.$$

Remark. Thus

$$\mathbf{C} = \left\{\begin{pmatrix} a_t & b_t \\ 0 & c_t \end{pmatrix} \middle| \, t \in \mathrm{GF}(q)\right\}$$

is a q-clan, for $q = 2^e$, if and only if

$$\mathcal{F}(\mathbf{C}) = \{a_tX_0 + c_tX_1 + b_tX_2 + X_3 = 0 \mid t \in \mathrm{GF}(q)\}$$

is a flock of $\mathcal{K}$.

5. Translation Planes

We now briefly sketch the construction of a translation plane from a flock of a quadratic cone, which was independently done by Thas [5] and Walker [25].

Let $\mathcal{F}(\mathbf{C})$ be the flock of a quadratic cone, $\mathcal{K}$, of the q-clan $\mathbf{C}$. Embed $\mathcal{K}$ into the Klein quadric, $\mathcal{Q}$, in $\mathrm{PG}(5, q)$, and let Δ be the polarity of $\mathrm{PG}(5, q)$ arising from $\mathcal{Q}$. Then $\Omega = \bigcup_{\pi_i \in \mathcal{F}(\mathbf{C})} (\Delta(\pi_i) \cap \mathcal{Q})$ is an ovoid of $\mathcal{Q}$.

Let $\mathcal{S}$ be the spread of $\mathrm{PG}(3, q)$ corresponding to Ω by the Klein correspondence. Let $\pi(\mathbf{C})$ be the translation plane of order q^2 with kernel $\mathrm{GF}(q)$ obtained from $\mathcal{S}$ by the Bruck–Brose construction.

6. Known q-Clans for q Even

We will list the known q-clans for $q = 2^e$. The q-clan associated with the linear flocks [23] for even q:

$$\mathbf{C}_1\colon\quad A_t = \begin{pmatrix} t & t \\ 0 & at \end{pmatrix},$$

where $a \in \mathrm{GF}(q)$ and $\mathrm{trace}(a) = 1$. The herd $\mathrm{H}(\mathbf{C}_1)$ consists of $q + 1$ (nondegenerate) conics. The elation generalized quadrangle associated with this q-clan is isomorphic to $\mathrm{H}(3, q^2)$ [19].

The q-clan of Fisher–Thas–Walker–Kantor–Payne [5], [25], [8], [14] for $q = 2^e$, e odd:

$$\mathbf{C}_2\colon\quad A_t = \begin{pmatrix} t & t^2 \\ 0 & t^3 \end{pmatrix}.$$

The flock associated with this q-clan is linear when $q = 2$. The herd $\mathrm{H}(\mathbf{C}_2)$ consists of $q + 1$ non-conical translation ovals if $q > 2$.

The q-clan of Payne [14] for $q = 2^e$, e odd:

$$\mathbf{C}_3\colon\quad A_t = \begin{pmatrix} t & t^3 \\ 0 & t^5 \end{pmatrix}.$$

The flock associated with this q-clan is linear when $q = 2$. The herd $\mathrm{H}(\mathbf{C}_3)$ consists of two Segre–Bartocci ovals (see [20]) and $q - 1$ Payne ovals [14], for $q > 8$. When $q = 8$, $\mathbf{C}_3$ is equivalent to $\mathbf{C}_2$.

The q-clan, $\mathbf{C}_4$, associated with the flock of De Clerck and Herssens [4] for $q = 16$. The herd $\mathrm{H}(\mathbf{C}_4)$ consists of 17 Lunelli–Sce [10] ovals (see Section 8.1).

Payne [16] has shown that given an elation generalized quadrangle $\mathrm{GQ}(\mathbf{C})$ associated with a q-clan, one can construct 'new' flocks via the $\mathrm{GQ}(\mathbf{C})$. These new flocks are constructed by recoordinatizing one of the lines incident with the point labelled (∞) of $\mathrm{GQ}(\mathbf{C})$. These flocks may be isomorphic to the original flock though. In fact, the number of nonisomorphic flocks that are constructed by recoordinatizing $\mathrm{GQ}(\mathbf{C})$ is the number of orbits of the automorphism group of $\mathrm{GQ}(\mathbf{C})$ on the lines incident with (∞). This shows that nonisomorphic flocks can have isomorphic GQ(**C**)'s. For q even each of the above q-clans give rise to a unique flock, except for the q-clan $\mathbf{C}_3$. This gives the nonlinear flock, $\mathcal{F}(\mathbf{C}_5)$, of Payne [16] for $q = 2^e$ with $e > 3$ constructed by recoordinatizing GQ($\mathbf{C}_3$) to obtain $\mathbf{C}_5$. (Of course, $\mathbf{C}_3$ and $\mathbf{C}_5$ are equivalent.)

There are also some q-clans for $q = 64$ and $q = 256$ that appear in [22].

In [24] Thas gave as an open problem the construction of q-clans associated with nonlinear flocks for q even, q square. The first example, $\mathbf{C}_4$, of such a q-clan was found for $q = 16$ by De Clerck and Herssens [4]. The main result of this paper is the construction of an infinite family of q-clans for all q even, which includes $\mathbf{C}_4$ for $q = 16$.

7. The Subiaco q-Clans

7.1. THE CASE $q = 2^e$ WHERE e IS ODD

THEOREM 3. *Let* $q = 2^e$, e *odd. Let*

$$f(x) = \frac{x^2 + x}{(x^2 + x + 1)^2} + x^{1/2} \quad \textit{and} \quad g(x) = \frac{x^4 + x^3}{(x^2 + x + 1)^2} + x^{1/2}.$$

Then

$$\mathbf{S}'' = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} \middle| t \in \mathrm{GF}(q) \right\}$$

is a q-clan.

Proof. We show that the matrices

$$\begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix}, t \in \mathrm{GF}(q),$$

form a q-clan by showing that $(f(x) + f(y))(g(x) + g(y))/(x + y)$ has trace 1 for all $x \neq y$; noting that trace(1) = 1 whenever $q = 2^e$ for e odd. From now on we assume $x \neq y$:

$$\frac{(f(x) + f(y))(g(x) + g(y))}{x + y}$$

$$= \frac{1}{x+y} \left(\frac{x^2 + x}{(x^2 + x + 1)^2} + x^{1/2} + \frac{y^2 + y}{(y^2 + y + 1)^2} + y^{1/2} \right)$$

$$\times \left(\frac{x^4 + x^3}{(x^2 + x + 1)^2} + x^{1/2} + \frac{y^4 + y^3}{(y^2 + y + 1)^2} + y^{1/2} \right)$$

$$= \frac{1}{x+y}\left(\frac{(x^2+x)(x^4+x^3)}{(x^2+x+1)^4} + \frac{(x^2+x)(y^4+y^3)}{(x^2+x+1)^2(y^2+y+1)^2}\right.$$

$$+(x+y)^{1/2}\frac{x^2+x}{(x^2+x+1)^2} + \frac{(y^2+y)(y^4+y^3)}{(y^2+y+1)^4}$$

$$+\frac{(x^4+x^3)(y^2+y)}{(x^2+x+1)^2(y^2+y+1)^2} + (x+y)^{1/2}\frac{y^2+y}{(y^2+y+1)^2}$$

$$\left.+(x+y)^{1/2}\frac{x^4+x^3}{(x^2+x+1)^2} + (x+y)^{1/2}\frac{y^4+y^3}{(y^2+y+1)^2} + (x+y)\right).$$

We can express this last line as

$$A+B+1$$

where

$$A = \frac{1}{x+y}\left(\frac{(x^2+x)(x^4+x^3)}{(x^2+x+1)^4} + \frac{(x^2+x)(y^4+y^3)}{(x^2+x+1)^2(y^2+y+1)^2}\right.$$

$$\left.+\frac{(y^2+y)(y^4+y^3)}{(y^2+y+1)^4} + \frac{(x^4+x^3)(y^2+y)}{(x^2+x+1)^2(y^2+y+1)^2}\right),$$

and

$$B = \frac{1}{(x+y)^{1/2}}\left(\frac{x^2+x}{(x^2+x+1)^2} + \frac{y^2+y}{(y^2+y+1)^2}\right.$$

$$\left.+\frac{x^4+x^3}{(x^2+x+1)^2} + \frac{y^4+y^3}{(y^2+y+1)^2}\right).$$

Hence, we have 'reduced' the problem to showing that $A + B$ has trace zero, as trace$(1) = 1$ for e odd. Since all elements of trace zero are of the form $X + X^2$, this is equivalent to showing that $A + B = X + X^2$ for some expression X. Since trace is additive, we have

$$\text{trace}(A+B) = \text{trace}(A+B) + \text{trace}(B+B^2) = \text{trace}(A+B^2).$$

(By showing $A + B^2$ has trace zero, instead of $A + B$, we can eliminate the $x^{1/2}$ terms from the latter.)

Now

$$A+B^2 = \frac{1}{x+y}\left(\frac{(x^2+x)(x^4+x^3)}{(x^2+x+1)^4} + \frac{(x^2+x)(y^4+y^3)}{(x^2+x+1)^2(y^2+y+1)^2}\right.$$

$$+\frac{(y^2+y)(y^4+y^3)}{(y^2+y+1)^4}+\frac{(x^4+x^3)(y^2+y)}{(x^2+x+1)^2(y^2+y+1)^2}$$

$$+\frac{x^4+x^2}{(x^2+x+1)^4}+\frac{y^4+y^2}{(y^2+y+1)^4}+\frac{x^8+x^6}{(x^2+x+1)^4}$$

$$+\left.\frac{y^8+y^6}{(y^2+y+1)^4}\right)$$

$$=\frac{1}{x+y}\left(\frac{x^8+x^2}{(x^2+x+1)^4}+\frac{y^8+y^2}{(y^2+y+1)^4}\right.$$

$$\left.+\frac{(x^2+x)(y^4+y^3)+(x^4+x^3)(y^2+y)}{(x^2+x+1)^2(y^2+y+1)^2}\right).$$

Since $x^8+x^2=(x^4+x^2)(x^2+x+1)^2$ all the terms can be placed over a common denominator, giving:

$$\frac{1}{x+y}\left(\frac{(x^4+x^2)(y^2+y+1)^2+(y^4+y^2)(x^2+x+1)^2}{(x^2+x+1)^2(y^2+y+1)^2}\right.$$

$$\left.+\frac{(x^2+x)(y^4+y^3)+(x^4+x^3)(y^2+y)}{(x^2+x+1)^2(y^2+y+1)^2}\right).$$

We expand, group, noting that $x^3+y^3=(x+y)(x^2+xy+y^2)$, and divide by $x+y$ to obtain:

$$\frac{(x+y)^3+x+y+x^2y^2(x+y)+x^2y^2+xy(x^2+xy+y^2)+xy(x+y)}{(x^2+x+1)^2(y^2+y+1)^2}.$$

With some cancellation we continue with:

$$\frac{x^3+y^3+x+y+x^3y^2+x^2y^3+x^3y+xy^3}{(x^2+x+1)^2(y^2+y+1)^2}$$

$$=\frac{(x+y)(x^2+x+1)(y^2+y+1)+(x+y)^2}{(x^2+x+1)^2(y^2+y+1)^2}$$

$$=\frac{x+y}{(x^2+x+1)(y^2+y+1)}+\frac{(x+y)^2}{(x^2+x+1)^2(y^2+y+1)^2}$$

which is of the form $X+X^2$ where

$$X = \frac{x+y}{(x^2+x+1)(y^2+y+1)}. \qquad \square$$

7.2. THE CASE $q = 4^e$, WHERE e IS ODD

THEOREM 4. *Let* $q = 4^e, e$ *odd, with* $\omega \in \mathrm{GF}(q)$ *satisfying* $\omega^2 + \omega + 1 = 0$. *Let*

$$f(x) = \frac{x^2(x^2+\omega x+\omega)}{(x^2+\omega x+1)^2} + \omega^2 x^{1/2} \quad \textit{and}$$

$$g(x) = \frac{\omega x(x^2+x+\omega^2)}{(x^2+\omega x+1)^2} + \omega^2 x^{1/2}.$$

Then

$$\mathbf{S}' = \left\{ A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & \omega g(t) \end{pmatrix} \middle| t \in \mathrm{GF}(q) \right\}$$

is a q-clan.

Proof. The proof is similar to that of Theorem 3. For brevity we denote f and g by

$$f(x) = \frac{N_f(x)}{D(x)^2} + \omega^2 x^{1/2} \quad \text{and} \quad g(x) = \frac{N_g(x)}{D(x)^2} + \omega^2 x^{1/2},$$

where

$$D(x) = x^2 + \omega x + 1, N_f(x) = x^2(x^2+\omega x+\omega), \quad \text{and}$$

$$N_g(x) = \omega x(x^2+x+\omega^2).$$

We show that $\mathrm{trace}(\omega(f(x)+f(y))(g(x)+g(y))/(x+y)) = 1$ for all $x \neq y$. From now on we assume $x \neq y$, so:

$$\frac{\omega}{x+y}(f(x)+f(y))(g(x)+g(y))$$

$$= \frac{\omega}{x+y}\left(\frac{N_f(x)}{D(x)^2} + \omega^2 x^{1/2} + \frac{N_f(y)}{D(y)^2} + \omega^2 y^{1/2}\right)$$

$$\times \left(\frac{N_g(x)}{D(x)^2} + \omega^2 x^{1/2} + \frac{N_g(y)}{D(y)^2} + \omega^2 y^{1/2}\right)$$

$$= \frac{\omega}{x+y}\left(\frac{N_f(x)N_g(x)}{D(x)^4} + \frac{N_f(x)N_g(y)+N_f(y)N_g(x)}{D(x)^2D(y)^2} + \frac{N_f(y)N_g(y)}{D(y)^4}\right)$$

$$+ \frac{1}{(x+y)^{1/2}}\left(\frac{N_f(x)}{D(x)^2} + \frac{N_f(y)}{D(y)^2} + \frac{N_g(x)}{D(x)^2} + \frac{N_g(y)}{D(y)^2}\right) + \omega^2.$$

We can express the last line as

$$A + B + \omega^2$$

where

$$A = \frac{\omega}{x+y}\left(\frac{N_f(x)N_g(x)}{D(x)^4} + \frac{N_f(x)N_g(y) + N_f(y)N_g(x)}{D(x)^2D(y)^2} + \frac{N_f(y)N_g(y)}{D(y)^4}\right),$$

and

$$B = \frac{1}{(x+y)^{1/2}}\left(\frac{N_f(x)}{D(x)^2} + \frac{N_f(y)}{D(y)^2} + \frac{N_g(x)}{D(x)^2} + \frac{N_g(y)}{D(y)^2}\right).$$

As ω^2 has trace 1 in GF(q), $q = 4^e$, for any e odd, we have reduced the problem to showing that $A + B$ has trace zero. As trace$(A + B)$ = trace$(A + B^2)$ this is equivalent to showing that $A + B^2$ has trace zero:

$$\begin{aligned} A + B^2 \;=\; \frac{1}{x+y}\Bigg(&\frac{\omega N_f(x)N_g(x) + N_f(x)^2 + N_g(x)^2}{D(x)^4} \\ &+ \frac{\omega N_f(y)N_g(y) + N_f(y)^2 + N_g(y)^2}{D(y)^4} \\ &+ \frac{\omega N_f(x)N_g(y) + \omega N_f(y)N_g(x)}{D(x)^2D(y)^2}\Bigg). \end{aligned}$$

Now $\omega N_f(x)N_g(x) + N_f(x)^2 + N_g(x)^2$ simplifies to $x^2(x^2 + \omega^2 x + 1)(x^2 + \omega x + 1)^2 = x^2(x^2 + \omega^2 x + 1)D(x)^2$. Then the expression, placed over a common denominator $D(x)^2D(y)^2$, becomes:

$$\begin{aligned} \frac{1}{x+y}\Bigg(&\frac{x^2(x^2+\omega^2 x+1)(y^2+\omega y+1)^2 + y^2(y^2+\omega^2 y+1)(x^2+\omega x+1)^2}{D(x)^2D(y)^2} \\ &+\frac{\omega^2 x^2 y(x^2+\omega x+\omega)(y^2+y+\omega^2) + \omega^2 y^2 x(y^2+\omega y+\omega)(x^2+x+\omega^2)}{D(x)^2D(y)^2}\Bigg) \end{aligned}$$

By expanding the terms we get

$$\begin{aligned} \frac{1}{x+y}\Bigg(&\frac{\omega^2x^4y^3 + \omega^2x^4y^2 + \omega x^4 y + \omega^2 x^3 y + \omega^2 x^2 y}{D(x)^2D(y)^2} \\ &+\frac{\omega^2x^3y^4 + \omega^2x^2y^4 + \omega x y^4 + \omega^2 x y^3 + \omega^2 x y^2}{D(x)^2D(y)^2}\Bigg), \end{aligned}$$

and then grouping to obtain

$$\frac{1}{x+y}\left(\frac{\omega^2x^3y^3(x+y)+\omega^2x^2y^2(x^2+y^2)+\omega xy(x^3+y^3)}{D(x)^2D(y)^2}\right.$$
$$\left.+\frac{\omega^2xy(x^2+y^2)+\omega^2xy(x+y)}{D(x)^2D(y)^2}\right).$$

Divide by $x+y$ and simplify to get:

$$\frac{x^3y^2+x^2y^3+\omega x^3y+\omega x^2y+\omega xy^3+\omega xy^2+x^3+\omega^2x^2+x+y^3+\omega^2y^2+y}{(x^2+\omega x+1)^2(y^2+\omega y+1)^2}$$
$$=\frac{(x+y)(x^2+\omega x+1)(y^2+\omega y+1)+x^2+y^2}{(x^2+\omega x+1)^2(y^2+\omega y+1)^2}$$
$$=\frac{x+y}{(x^2+\omega x+1)(y^2+\omega y+1)}+\frac{(x+y)^2}{(x^2+\omega x+1)^2(y^2+\omega y+1)^2},$$

which is of the form $X+X^2$. □

7.3. EXISTENCE OF SUBIACO q-CLANS

THEOREM 5. *Let* $d \in \mathrm{GF}(q)$, *q even, such that* $d^2+d+1 \neq 0$ *and* $\mathrm{trace}(1/d)=1$. *Let*

$$a = \frac{d^2+d^5+d^{1/2}}{d(1+d+d^2)},$$
$$f(x) = \frac{d^2(x^4+x)+d^2(1+d+d^2)(x^3+x^2)}{(x^2+dx+1)^2}+x^{1/2},$$

and

$$g(x)=\frac{d^4x^4+d^3(1+d^2+d^4)x^3+d^3(1+d^2)x}{(d^2+d^5+d^{1/2})(x^2+dx+1)^2}+\frac{d^{1/2}}{d^2+d^5+d^{1/2}}x^{1/2}.$$

Then

$$\mathbf{S}=\mathbf{S}_d=\left\{A_t=\begin{pmatrix} f(t) & t^{1/2}\\ 0 & ag(t)\end{pmatrix}\middle|\, t\in \mathrm{GF}(q)\right\}$$

is a q-clan.

Proof. This proof is similar to the proof of Theorem 3 and the proof of Theorem 4. We start by showing $\text{trace}(a) = 1$ for all q even:

$$\begin{aligned}\text{trace}(a) &= \text{trace}(a^2)\\ &= \text{trace}\left(\frac{d^9+d^3+1}{d(d^2+d+1)^2}\right)\\ &= \text{trace}\left(\frac{1}{d}\right) + \text{trace}\left(d + \frac{d^4}{d^2+d+1}\right) + \text{trace}\left(d^2 + \frac{d^8}{(d^2+d+1)^2}\right)\end{aligned}$$

So $\text{trace}(a) = \text{trace}(1/d) = 1$.

In this form the equations will become quite unwieldy, so initially we will simplify f and g to

$$f(x) = \frac{N_f(x)}{D(x)^2} + x^{1/2}$$

and

$$g(x) = \frac{1}{d^2+d^5+d^{1/2}}\frac{N_g(x)}{D(x)^2} + \frac{d^{1/2}}{d^2+d^5+d^{1/2}}x^{1/2}.$$

We now show for $\text{GF}(q), q = 2^e$, that the matrices

$$\begin{pmatrix} f(t) & t^{1/2} \\ 0 & ag(t) \end{pmatrix}, t \in \text{GF}(q),$$

form a q-clan. We do this by showing that $\mathcal{T}_a(f, g)$ is true. That is we show $a(f(x)+f(y))(g(x)+g(y))/(x+y)$ has trace 1 for all $x \neq y$. From now on we assume that $x \neq y$, so:

$$\begin{aligned}&\frac{(f(x)+f(y))(ag(x)+ag(y))}{x+y}\\ &= \frac{1}{x+y}\left(\frac{N_f(x)}{D(x)^2} + x^{1/2} + \frac{N_f(y)}{D(y)^2} + y^{1/2}\right)\left(\frac{N_g(x)}{d(1+d+d^2)D(x)^2}\right.\\ &\quad \left. + \frac{d^{1/2}}{d(1+d+d^2)}x^{1/2} + \frac{N_g(y)}{d(1+d+d^2)D(y)^2} + \frac{d^{1/2}}{d(1+d+d^2)}y^{1/2}\right)\\ &= \frac{1}{x+y}\left(\frac{N_f(x)N_g(x)}{d(1+d+d^2)D(x)^4} + \frac{N_f(x)N_g(y)}{d(1+d+d^2)D(x)^2D(y)^2}\right.\end{aligned}$$

$$+\frac{(x+y)^{1/2}}{d^{1/2}(1+d+d^2)}\frac{N_f(x)}{D(x)^2}+\frac{N_f(y)N_g(y)}{d(1+d+d^2)D(y)^4}$$

$$+\frac{N_f(y)N_g(x)}{d(1+d+d^2)D(x)^2D(y)^2}+\frac{(x+y)^{1/2}}{d^{1/2}(1+d+d^2)}\frac{N_f(y)}{D(y)^2}$$

$$+\frac{(x+y)^{1/2}}{d(1+d+d^2)}\frac{N_g(x)}{D(x)^2}+\frac{(x+y)^{1/2}}{d(1+d+d^2)}\frac{N_g(y)}{D(y)^2}+\frac{x+y}{d^{1/2}(1+d+d^2)}\Bigg).$$

If we let $E_1,\dots,E_8$ correspond to the first eight terms inside the brackets of the above expression, we can express the last line as

$$\frac{1}{x+y}\Bigg(E_1+E_2+E_3+E_4+E_5+E_6$$
$$+E_7+E_8+\frac{x+y}{d^{1/2}(1+d+d^2)}\Bigg).$$

We will do a further substitution on the above line to obtain

$$A+B+\frac{1}{d^{1/2}(1+d+d^2)}$$

where

$$A=\frac{1}{x+y}(E_1+E_2+E_4+E_5)$$

and

$$B=\frac{1}{x+y}(E_3+E_6+E_7+E_8).$$

Now

$$\begin{aligned}\operatorname{trace}\left(\frac{1}{d^{1/2}(1+d+d^2)}\right) &= \operatorname{trace}\left(\frac{1}{d(1+d^2+d^4)}\right)\\ &= \operatorname{trace}\left(\frac{1}{d}\right)+\operatorname{trace}\left(\frac{d+d^3}{1+d^2+d^4}\right)\\ &= \operatorname{trace}\left(\frac{1}{d}\right)+\operatorname{trace}\left(\frac{d}{1+d+d^2}\right)\\ &\quad+\operatorname{trace}\left(\frac{d^2}{1+d^2+d^4}\right)\\ &= \operatorname{trace}\left(\frac{1}{d}\right)=1.\end{aligned}$$

So it is now left to show that $A + B$ has trace zero. Since $\operatorname{trace}(A + B) = \operatorname{trace}(A + B^2)$, this is equivalent to showing that $A + B^2$ has trace zero.

$$\begin{aligned} A + B^2 &= \frac{1}{x+y}\left(E_1 + E_2 + E_4 + E_5 + \frac{N_f(x)^2}{d(1+d^2+d^4)D(x)^4}\right. \\ &\quad + \frac{N_f(y)^2}{d(1+d^2+d^4)D(y)^4} + \frac{N_g(x)^2}{d^2(1+d^2+d^4)D(x)^4} \\ &\quad \left. + \frac{N_g(y)^2}{d^2(1+d^2+d^4)D(y)^4}\right) \\ &= \frac{1}{x+y}\left(E_2 + E_5 + \frac{d(1+d+d^2)N_f(x)N_g(x) + dN_f(x)^2 + N_g(x)^2}{d^2(1+d^2+d^4)D(x)^4}\right. \\ &\quad \left. + \frac{d(1+d+d^2)N_f(y)N_g(y) + dN_f(y)^2 + N_g(y)^2}{d^2(1+d^2+d^4)D(y)^4}\right). \end{aligned}$$

After much simplification we obtain:

$$\begin{aligned} &\frac{1}{x+y}\left(E_2 + E_5\right. \\ &\quad + \frac{d^5(1+d+d^2)x^8 + d^6(1+d^6)x^7 + d^5(1+d^6)x^6 + d^8(1+d^6)x^5}{d^2(1+d^2+d^4)D(x)^4} \\ &\quad + \frac{d^5(1+d^6)x^4 + d^6(1+d^6)x^3 + d^5(1+d+d^2)^2x^2}{d^2(1+d^2+d^4)D(x)^4} \\ &\quad + \frac{d^5(1+d+d^2)y^8 + d^6(1+d^6)y^7 + d^5(1+d^6)y^6 + d^8(1+d^6)y^5}{d^2(1+d^2+d^4)D(y)^4} \\ &\quad \left. + \frac{d^5(1+d^6)y^4 + d^6(1+d^6)y^3 + d^5(1+d+d^2)^2y^2}{d^2(1+d^2+d^4)D(y)^4}\right). \end{aligned}$$

Using $1 + d^6 = (1 + d^2 + d^4)(1 + d^2)$ with more simplification, dividing by $D(x)^2$ (or $D(y)^2$), and then placing over a common denominator, we obtain:

$$\begin{aligned} &\frac{1}{x+y}\left(E_2 + E_5 + \frac{d^3x^8 + d^4(1+d^2)x^7 + d^3(1+d^2)x^6 + d^6(1+d^2)x^5}{D(x)^4}\right. \\ &\quad + \frac{d^3(1+d^2)x^4 + d^4(1+d^2)x^3 + d^3x^2}{D(x)^4} \end{aligned}$$

$$+\frac{d^3y^8+d^4(1+d^2)y^7+d^3(1+d^2)y^6+d^6(1+d^2)y^5}{D(y)^4}$$

$$\left.+\frac{d^3(1+d^2)y^4+d^4(1+d^2)y^3+d^3y^2}{D(y)^4}\right)$$

$$=\frac{1}{x+y}\left(E_2+E_5+\frac{d^3(x^4+d(1+d^2)x^3+x^2)D(x)^2}{D(x)^4}\right.$$

$$\left.+\frac{d^3(y^4+d(1+d^2)y^3+y^2)D(y)^2}{D(y)^4}\right)$$

$$=\frac{1}{x+y}\left(\frac{N_f(x)N_g(y)}{d(1+d+d^2)D(x)^2D(y)^2}+\frac{N_f(y)N_g(x)}{d(1+d+d^2)D(x)^2D(y)^2}\right.$$

$$\left.+\frac{d^3(x^4+d(1+d^2)x^3+x^2)}{D(x)^2}+\frac{d^3(y^4+d(1+d^2)y^3+y^2)}{D(y)^2}\right)$$

$$=\frac{1}{x+y}\left(\frac{d^5(1+d^3)(1+d)x^4y^3+d^6(1+d+d^2)x^4y^2+d^5(1+d+d^2)x^4y}{d(1+d+d^2)D(x)^2D(y)^2}\right.$$

$$+\frac{d^5(1+d^3)(1+d)x^3y^4+d^6(1+d+d^2)x^2y^4+d^5(1+d+d^2)xy^4}{d(1+d+d^2)D(x)^2D(y)^2}$$

$$+\frac{d^5(1+d+d^2)^3x^3y^2+d^6(1+d+d^2)x^3y+d^5(1+d)(1+d^3)xy^2}{d(1+d+d^2)D(x)^2D(y)^2}$$

$$+\frac{d^5(1+d+d^2)^3x^2y^3+d^6(1+d+d^2)xy^3+d^5(1+d^3)(1+d)x^2y}{d(1+d+d^2)D(x)^2D(y)^2}$$

$$\left.+\frac{d^3(x^4+d(1+d^2)x^3+x^2)D(y)^2}{D(x)^2D(y)^2}+\frac{d^3(y^4+d(1+d^2)y^3+y^2)D(x)^2}{D(x)^2D(y)^2}\right).$$

After some substantial simplification, dividing by $d(1+d+d^2)$ where appropriate and grouping, we obtain:

$$\frac{1}{x+y}\left(\frac{d^3x^2y^2(x^2+y^2)+d^4x^2y^2(x+y)+d^4xy(x^3+y^3)+d^5xy(x^2+y^2)}{D(x)^2D(y)^2}\right.$$

$$\left.+\frac{d^4(1+d^2)xy(x+y)+d^3(x^4+y^4)+d^3(x^2+y^2)+d^4(1+d^2)(x^3+y^3)}{D(x)^2D(y)^2}\right).$$

We divide by $x + y$ and collect terms to obtain:

$$\frac{d^4x^3y + d^4xy^3 + (d^5 + d^3)x^2y + (d^5 + d^3)xy^2 + d^3x^3y^2 + d^3x^2y^3}{D(x)^2D(y)^2}$$

$$+\frac{d^3x^3 + d^3y^3 + d^4(1 + d^2)x^2 + d^4(1 + d^2)y^2 + d^3x + d^3y}{D(x)^2D(y)^2}$$

$$= \frac{(d^3x + d^3y)(x^2 + dx + 1)(y^2 + dy + 1) + d^6x^2 + d^6y^2}{(x^2 + dx + 1)^2(y^2 + dy + 1)^2}$$

$$= \frac{d^3x + d^3y}{(x^2 + dx + 1)(y^2 + dy + 1)} + \frac{d^6x^2 + d^6y^2}{(x^2 + dx + 1)^2(y^2 + dy + 1)^2},$$

which is of the form $X + X^2$. □

We call **S** the *Subiaco* q-clan. We call the ovals of H(**S**) the *Subiaco* ovals and the resulting hyperovals the *Subiaco* hyperovals. We also call the flocks $\mathcal{F}(\mathbf{S})$ the *Subiaco* flocks, GQ(**S**) the *Subiaco* elation generalized quadrangles, and $\pi(\mathbf{S})$ the *Subiaco* translation planes.

The q-clan $\mathbf{S}'$ is a Subiaco q-clan for $q = 4^e$, e odd (see Section 8.1). Hence the herd of ovals H($\mathbf{S}'$), the flocks of the quadratic cone $\mathcal{F}(\mathbf{S}')$, the elation generalized quadrangles GQ($\mathbf{S}'$), and the translation planes $\pi(\mathbf{S}')$ from the q-clan $\mathbf{S}'$ are all Subiaco, for $q = 4^e$, e odd. For $q = 2^e$, where e is odd, we can let $d = 1$, hence we find $\mathbf{S}'' = \mathbf{S}_1$. This gives a family of o-polynomials of the herd H($\mathbf{S}_1$) over GF(2) for $q = 2^e$, e odd.

The construction of $\mathbf{S}''$ for $q = 2^e, e$ odd was the first q-clan to be found. This was followed by the construction of $\mathbf{S}'$ for $q = 4^e, e$ odd. From these two constructions it was possible to generalize to construct **S**.

8. Concluding Remarks

8.1. THE SUBIACO q-CLANS

In [18, 4.4] it is shown that if d and d' are elements of GF(q) with trace($1/d$) = trace($1/d'$) = 1, for $q = 2^e$, then $\mathbf{S}_d$ is equivalent to $\mathbf{S}_{d'}$. In [18, 2] it is shown that **S** is equivalent to $\mathbf{S}'$, for $q = 2^e, e \equiv 2\,(\mathrm{mod}\,4), e \neq 2$. In [1], [17], [18] the automorphism group of GQ(**S**) is calculated. For $q = 2, 4$, GQ(**S**) $\cong$ H($3, q^2$). For $q = 8$, GQ(**S**) $\cong$ GQ($\mathbf{C}_2$). For $q = 16$, GQ(**S**) $\cong$ GQ($\mathbf{C}_4$) by results of [4]. For $q \geq 32$, GQ(**S**) is new (although for $q = 32, 64, 128, 256$, they appear in computer results of [22]). This can be seen from the automorphism groups. Alternatively, no previously known q-clans **C** gave rise to a generalized quadrangle GQ(**C**) with subquadrangle on (∞) and $(\mathbf{0}, 0, \mathbf{0})$ isomorphic to $T_2(\mathcal{O})$, for $\mathcal{O}$ a Subiaco oval. This follows from the results on the Subiaco hyperovals that follow in Section 8.4 for

$q \geq 64$. For $q = 32$, note that while the Subiaco hyperovals are Payne hyperovals, the *ovals* of the Subiaco herd, $H(\mathbf{S})$, are not equivalent to the ovals of the Payne herd, $H(\mathbf{C}_3)$.

As $H(\mathbf{S})$ contains no ovals that give rise to regular hyperovals, by [6], [11] we have all the ovals being from Lunelli–Sce hyperovals for $q = 16$. Since the Subiaco 16-clan is equivalent to $\mathbf{C}_4$, it follows that $H(\mathbf{C}_4)$ consists of 17 Lunelli–Sce ovals.

8.2. THE SUBIACO FLOCKS

In [1], [17], [18] it is shown that each Subiaco q-clan, $\mathbf{S}$, gives rise to exactly one Subiaco flock of a quadratic cone in $PG(3, q)$, up to isomorphism, by showing that the automorphism group of $GQ(\mathbf{S})$ is transitive on the lines on (∞). This also determines the stabilizer in $P\Gamma L(4, q)$ of the Subiaco flock in $PG(3, q)$.

8.3. THE SUBIACO PLANES

In [1, VII] the automorphism groups of the Subiaco planes $\pi(\mathbf{S})$ are studied.

8.4. THE SUBIACO HYPEROVALS

In [18, Cor. 5.4] it is shown that all Subiaco hyperovals in $PG(2, q)$ are equivalent for $q = 2^e, e \not\equiv 2 \pmod 4$. Also in [18, 6.1, 6.4] it is shown that there are two orbits in $PG(2, q)$, for $q = 2^e, e \equiv 2 \pmod 4$.

For $q = 2, 4, 8$, the Subiaco hyperovals are regular. For $q = 16$, they are Lunelli–Sce hyperovals [10]. For $q = 32$, they are Payne hyperovals. For $q = 64$, they are the hyperovals discovered by Penttila and Pinneri [20], with groups of orders 15 and 60. For $q = 128, 256$, they are the hyperovals discovered by Penttila and Royle [21].

In [12] it is shown that the stabilizer in $P\Gamma L(3, q)$ of a Subiaco hyperoval in $PG(2, q)$ is cyclic of order $2e$, for $q = 2^e, e \not\equiv 2 \pmod 4$. In [18, 6.13] the stabilizers in $P\Gamma L(3, q)$ of the Subiaco hyperovals in $PG(2, q)$ for $q = 2^e, e \equiv 2 \pmod 4$ are computed (one is $C_5 \rtimes C_{2e}$, the other is $C_5 \rtimes C_{e/2}$).

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