MAGNETIC INSTABILITY OF CORONAL ARCADES AS THE ORIGIN OF TWO-RIBBON FLARES

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Abstract. The generally accepted scenario for the events leading up to a two-ribbon flare is that a magnetic arcade (supporting a plage filament) responds to the slow photospheric motions of its footpoints by *evolving passively* through a series of (largely) force-free equilibria. At some critical amount of shear the configuration *becomes unstable and erupts* outwards. Subsequently, the field *closes back down* in the manner modelled by Kopp and Pneuman (1976); but the main problem has been to explain the eruptive instability.

The present paper analyses the magnetohydrodynamic stability of several possible arcade configurations, including the *dominant* stabilizing effect of line-tying at the photospheric footpoints. One low-lying force-free structure is found to be stable regardless of the shear; also some of the arcades that lie on the upper branch of the equilibrium curves are shown to be stable. However, another force-free configuration appears more likely to represent the preflare structure. It consists of a large flux tube, anchored at its ends and surrounded by an arcade, so that the field transverse to the arcade axis contains a magnetic island. Such a configuration is found to become unstable when either the length of the structure, the twist of the flux tube, or the height of the island becomes too great; the higher the tube is situated, the smaller is the twist required for instability.

1. Introduction

The great variety of observed flares can be divided essentially into two basic kinds (Priest, 1976). These are: (a) *Small loop flares*,* which may be connected with new flux emerging from below the photosphere (Heyvaerts *et al.*, 1977) and (b) *large two-ribbon flares*, which may arise from a large-scale magnetohydrodynamic instability. In this paper we look at the latter class and discuss some of the previous ideas on the subject.

Work on explaining the two-ribbon flare is itself split into three different approaches. One method attempts to show the *existence of multiple equilibria* satisfying the force-free equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \tag{1.1}$$

and the possibility of non-equilibrium when some parameter becomes too large. The problem is certainly non-trivial and the general approach taken so far is to assume that the basic state depends only on two space coordinates, the third coordinate being along the length of the arcade (Low, 1977a, b; Birn *et al.*, 1978a, b; Heyvaerts *et al.*, 1979). Thus, in cylindrical coordinates, the equilibrium magnetic field may be written as

$$\mathbf{B} = \left(\frac{1}{r}\frac{\partial A}{\partial \theta}, -\frac{\partial A}{\partial r}, B_z(A)\right),$$

* Or compact flares.

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where A is a function of r and θ alone. Then (1.1) reduces to the non-linear equation for A,

$$\nabla^2 A + \frac{\mathrm{d}(\frac{1}{2}B_z^2(A))}{\mathrm{d}A} = 0\,. \tag{1.2}$$

The problem is to solve (1.2) subject to the relevant boundary conditions. One would like to impose the normal component of magnetic field, B_n, at the photosphere and the amount by which the footpoints are sheared. Thus shear is here defined to be the difference of the z-component of the two footpoints; it is given by integrating along a particular line of force the equation

$$\frac{\mathrm{d}z}{B_z(A)} = \frac{r\,\mathrm{d}\theta}{B_\theta} = -\frac{r\,\mathrm{d}\theta}{\partial A/\partial r}$$

for a field line, with the result

$$z = \int_{-\pi/2}^{\pi/2} \left[\frac{rB_z}{\partial A/\partial r}\right]_{A=\mathrm{const.}} \mathrm{d}\theta\,. \tag{1.3}$$

The Equations (1.2) and (1.3) then form an extremely complicated, non-linear, integrodifferential problem (Heyvaerts *et al.*, 1979). For this situation, in which *both the shear and B_n are imposed*, analytic progress has been made by Priest and Milne (1979), involving the presence of magnetic bubbles; although the example used was rather artificial, it does indicate that such multiple solutions can exist. In one case, the energy of a second equilibrium was found to be less than that of a first equilibrium when the shear reaches a critical value and so it was suggested that this may correspond with the onset of instability.

Most authors, on the other hand, have so far considered only the *simpler* problem of prescribing the z-component of the magnetic field, $B_z(A)$, rather than the shear (Low, 1977a; Birn *et al.*, 1978a; Heyvaerts *et al.*, 1979).

Then (1.2) reduces to

$$-\nabla^2 A = \lambda F(A)\,, \tag{1.4}$$

where $F(A)$ is some prescribed function. $B_z(A)$ is increased by raising the value of λ but this does not always result in an increased shear, as emphasized by Jockers (1978) and Priest and Milne (1980).

On qualitative grounds, Van Tend and Kuperus (1978) show that non-equilibrium occurs if the filament current or its height exceeds some critical values; they suggest that the filament erupts to find another equilibrium.

A second approach to two-ribbon flares is to study the *magnetohydrodynamic stability* of a basic equilibrium directly and this is most easily done with the energy principle of Bernstein *et al.* (1958). This method has been used by Low (1977b) and Birn *et al.* (1978b) to show that one of two possible equilibria is always stable. Also

Hood and Priest (1979a) treated the stability of a flux tube anchored in the photosphere. They demonstrated that the *dominant stabilizing mechanism for coronal loops is photospheric line-tying* and found that a flux tube becomes *kink unstable* when it is twisted too much. This was suggested as the *basic cause for the eruption of a plage filament at the start of a two-ribbon flare.* Their analysis is extended in Section 4 to a more realistic preflare configuration, containing the extra stabilizing effect of an overlying arcade.

It should be noted that the first and second approaches complement one another. Although it is preferable to demonstrate instability directly by the second method, it can in practice only be used for relatively simple configurations. By contrast, the first method can deal with more complex equilibria and then makes the hypothesis that an instability occurs from one equilibrium to another; but, of course, a full stability analysis may subsequently demonstrate instability at a lower threshold. Furthermore, although the second approach may show that an equilibrium is linearly stable, the first may imply the possibility of nonlinear instability.

The third suggestion for producing two-ribbon flares involves instabilities due to current sheets and will not be discussed in this paper at all. Any interested reader is referred to Priest (1976), Tur and Priest (1976), Heyvaerts *et al.* (1977) and references therein.

In Section 2 we discuss the energy principle and derive necessary and sufficient conditions for the stability of a cylindrically symmetric arcade to a wide class of perturbations. This principle is used in Section 3 to investigate two particular arcades. The next section investigates the stability of a flux tube contained within an arcade and the last section summarizes the results.

2. Energy Principle

In order to investigate the stability of a given magnetic field configuration, the linearized magnetohydrodynamic equations are first obtained by perturbing the basic state. The basic state is then said to be *stable* to infinitesimal disturbances if all the perturbations decay (or at least do not grow) with time. The two main approaches used in deciding the stability are the *normal mode analysis* and the *energy principle* derived by Bernstein *et al.* (1958). The two methods produce identical results but differ in complexity. The normal mode analysis provides a full solution to the problem and gives the growth rate, ω, explicitly. However, when the basic state is not uniform the solution of the differential equations can be very difficult. The advantage of the energy principle is that it is easier to treat more complex basic equilibria and quite often one may easily find upper bounds on their stability.

The change in the potential energy for a *force-free* magnetic field due to a perturbation $\boldsymbol{\xi}$ is

$$\delta W = \int \{|\nabla \times (\boldsymbol{\xi} \times \mathbf{B})|^2 - \alpha (\boldsymbol{\xi} \times \mathbf{B}) \nabla \times (\boldsymbol{\xi} \times \mathbf{B})\} \, \mathrm{d}\tau , \tag{2.1}$$

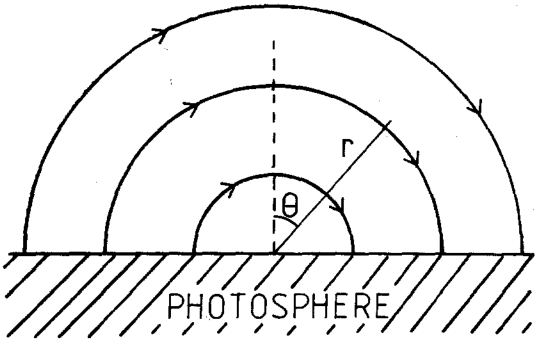

Fig. 1. A typical cylindrically symmetric arcade. When viewed along its length the field lines appear as semicircles. The displacement of the photospheric footpoints out of the plane of the figure depends on the specific field components.

where $\nabla \times \mathbf{B} = \alpha \mathbf{B}$; and the stability is then governed by the sign of δW: $\delta W > 0$ for all $\boldsymbol{\xi}$ implies stability, but if there exists a $\boldsymbol{\xi}$ such that $\delta W < 0$ the field is unstable.

Suppose, in particular, that the basic state is *cylindrically symmetric* and is given by

$$\mathbf{B} = (0, B_\theta(r), B_z(r)),$$

as shown schematically in Figure 1. The 'line-tying' effect of the dense photosphere on perturbations that are initiated in the corona will be modelled by making all displacements perpendicular to the field vanish at $\theta = +\frac{1}{2}\pi$. Now, if we have a complete set of basis functions that span the two-dimensional (θ, z) space, we may write any perturbation $\boldsymbol{\xi}$ as a linear combination of them, namely

$$\boldsymbol{\xi}(r, \theta, z) = \sum_{k,n} \boldsymbol{\xi}_{k,n}(r) f_k(\theta, z) h_n(\theta),$$

where the functions $h_n(\theta)$ satisfy the boundary conditions that

$$\boldsymbol{\xi}(r, +\tfrac{1}{2}\pi, z) = 0.$$

For simplicity, we choose individual components of the series in turn and minimize δW with respect to them. The stability conditions obtained are, thus, only necessary and sufficient to this restricted class (see Appendix A). In other words, we choose, as a trial perturbation,

$$\begin{aligned}\boldsymbol{\xi}(r, \theta, z) = [&\xi_r(r) \cos(m\theta + kz), -\xi_0(r)(B_z/B) \sin(m\theta + kz),\\ &\xi_0(r)(B_\theta/B) \sin(m\theta + kz)] \cos h\theta,\end{aligned} \tag{2.2}$$

where m is arbitrary and non-zero (i.e. not restricted to be an integer); also

$$k = i2\pi/L, \qquad i = 1, 2, \ldots,$$

$$h = 2n + 1, \qquad n = 0, 1, 2, \ldots,$$

$$B(r) = (B_\theta^2(r) + B_z^2(r))^{1/2},$$

and L is the wavelength in the axial direction.

Minimizing δW with respect to ξ_0 (see Hood and Priest, 1979a), we find

$$\xi_0 = \frac{r\{(krB_\theta - mB_z)\xi_r'(r) - (krB_\theta + mB_z)\xi_r(r)/r\}}{B(m^2+h^2+k^2r^2)}, \tag{2.3}$$

where $\xi_r'(r) = \mathrm{d}\xi_r/\mathrm{d}r$; a dash denotes differentiation with respect to r. Then, after some algebra, δW reduces to an integral over r alone, namely,

$$\delta W = \tfrac{1}{2}L \int_0^\infty (F\xi_r'^2 + 2H\xi_r\xi_r' + G\xi_r^2)\,\mathrm{d}r, \tag{2.4}$$

where

$$\begin{aligned} F &= \frac{r((krB_z + mB_\theta)^2 + h^2B^2)}{m^2+h^2+k^2r^2}, \\ H &= \frac{h^2(B_z^2 - B_\theta^2) + (k^2r^2B_z^2 - m^2B_\theta^2)}{m^2+h^2+k^2r^2}, \\ G &= \frac{1}{r}\left[h^2B_\theta^2 + (mB_\theta + krB_z)^2 - 2rB_\theta B_\theta' - 2B_\theta^2 + \frac{h^2B^2 + (krB_z - mB_\theta)^2}{m^2+h^2+k^2r^2}\right]. \end{aligned} \tag{2.5}$$

Minimizing (2.4) with respect to ξ_r leads to the familiar Euler–Lagrange equation

$$(F\xi_r')' = (G - H')\xi_r\,; \tag{2.6}$$

and the stability of the basic state is then decided by determining the zeros of the solution to (2.6) (Newcomb, 1960; Hood and Priest, 1979a, b). If ξ_r has no zeros then the initial equilibrium is stable, but if ξ_r has at least one zero then it is unstable. It is easily shown that a sufficient condition for stability is

$$G - H' > 0 \quad \text{for} \quad all\, r. \tag{2.7}$$

When the arcade is *not cylindrically symmetric* a slightly different approach is needed. If the basic magnetic field is a function of both r and θ, we may only Fourier analyse $\boldsymbol{\xi}$ in its z-variation,

$$\boldsymbol{\xi}(r, \theta, z) = \sum_n (\boldsymbol{\xi}_{1k}(r, \theta)\cos kz + \boldsymbol{\xi}_{2k}(r, \theta)\sin kz), \tag{2.8}$$

where $k = 2\pi n/L$. Substituting (2.8) into (2.1) and integrating over a period in z gives

$$\begin{aligned} \delta W = &\int \{|\nabla\times\mathbf{E}_1|^2 + |\nabla\times\mathbf{E}_2|^2 - \alpha\mathbf{E}_1\cdot\nabla\times\mathbf{E}_1 - \alpha\mathbf{E}_2\cdot\nabla\times\mathbf{E}_2\}\, r\,\mathrm{d}r\,\mathrm{d}\theta + \\ &+ k^2\int \{|\hat{\mathbf{e}}_2\times\mathbf{E}_1|^2 + |\hat{\mathbf{e}}_2\times\mathbf{E}_2|^2|\, r\,\mathrm{d}r\,\mathrm{d}\theta + \\ &+ k\int 2\hat{\mathbf{e}}_2\cdot\{\mathbf{E}_1\times(\nabla\times\mathbf{E}_2) - \mathbf{E}_2\times(\nabla\times\mathbf{E}_1) + \alpha(\mathbf{E}_1\times\mathbf{E}_2)\}\, r\,\mathrm{d}r\,\mathrm{d}\theta, \end{aligned} \tag{2.9}$$

where

$$\mathbf{E}_1 = \boldsymbol{\xi}_{1k} \times \mathbf{B} \quad \text{and} \quad \mathbf{E}_2 = -(\boldsymbol{\xi}_{2k} \times \mathbf{B}) .$$

To decide on stability, we minimize (2.9), but the result forms a complicated, though linear, system of *partial* differential equations.

3. Arcade Stability

Consider first a simple *cylindrically symmetric arcade*, whose field components,

$$\begin{aligned} B_z^2 &= f(r) + \tfrac{1}{2} r \, \mathrm{d}f/\mathrm{d}r , \\ B_\theta^2 &= -\tfrac{1}{2} r \, \mathrm{d}f/\mathrm{d}r , \end{aligned} \tag{3.1}$$

satisfy the force-free condition (Lüst and Schlüter, 1954). The function $f(r)$ is arbitrary apart from the restrictions that both B_z^2 and B_θ^2 remain positive for all r. Note that we may introduce an arbitrary constant into $f(r)$ without affecting B_θ and so the normal component at the photosphere. This constant may then be varied to describe the shearing motion of the footpoints. (However, one drawback is that not all such fields start from a potential field. The only cylindrical potential field is

$$\mathbf{B} = (0, 1/r, B_0) ,$$

where B_0 is a constant. Increasing B_0 leaves the field potential and thus this field is stable to all perturbations, as is seen by putting $\alpha = 0$ in (2.1).)

3.1. Cylindrically symmetric field with axis on the photosphere

As an illustration, we assume the equilibrium magnetic field is given by

$$\begin{aligned} B_\theta &= B_0 (r/b)/(1 + r^2 b^2) , \\ B_z &= B_0 (a^2 + 1/(1 + r^2/b^2)^2)^{1/2} . \end{aligned} \tag{3.2}$$

Before attempting to solve (2.6) numerically we must expand about the singular point at $r = 0$ in order to find the behaviour of the solution there. The result is

$$\xi_r \sim r \quad \text{for} \quad h \neq 0 ,$$

which is independent of the values of h, m, and k. If $h = 0$ so that there is no line-tying and if $m = 1$, the solution that is small, in the Newcomb sense (Newcomb, 1960), is

$$\xi_r \sim 1 .$$

Before attempting to search for instability, it was decided to seek perturbations that violate the sufficient condition (2.7), for stability. However, for all values of m, a, and k that were tried, the function $(G - H')$ is found to be positive. This implies that the field (3.2) is *stable* to the attempted perturbations. This result is in marked contrast to the analysis of coronal loops (Hood and Priest, 1979a), where instability was easily found. It is therefore useful to study the effect of line-tying on the functions F, G, and H for general cylindrically symmetric fields as follows.

For the coronal loop (Hood and Priest, 1979a), line-tying was governed by the parameter, $\pi b/2L$, and its effect could be diminished by increasing the aspect ratio, L/b. In the present case of a coronal arcade, h is a fixed integer and so we need to find some other way to reduce the effect of line-tying. Putting $k = qm$, $h = nm$ and $r = x/q$ in (2.5) we find

$$F = \frac{x((xB_z + B_\theta)^2 + n^2B^2)}{1+n^2+x^2},$$

$$H = \frac{x^2B_z^2 - B_\theta^2 + n^2(B_z^2 - B_\theta^2)}{1+n^2+x^2},$$

$$G = \frac{1}{x}\left\{m^2((B_\theta + xB_z)^2 + n^2B_\theta^2) - 2xB_\theta B'_\theta - 2B_\theta^2 + \frac{n^2B^2 + (xB_z - B_\theta)^2}{1+n^2+x^2}\right\}.$$

Since h is fixed, it can be seen that the effect of line-tying can be reduced by choosing large values of m (small n). Unfortunately, these approximate the function $(G - H')$ by the positive definite term $m^2(B_\theta + xB_z)^2/x$. So the difficulty in finding an unstable perturbation is now understood: for small values of m (normally the least stable) line-tying dominates, whereas for large values of m, although line-tying is almost negligible, the field is again stable. The only hope for obtaining instability of (3.2) appears to be with some intermediate value of m but this has not been yet successful.

Another cylindrical arcade was examined. It has the generating function

$$f = a^2 + \tfrac{1}{2}e^{-2r^2/b^2}$$

and the magnetic field components

$$B_\theta = (r/b)\,e^{-r^2/b^2},$$

$$B_z = (a^2 + (\tfrac{1}{2} - r^2/b^2)\,e^{-2r^2/b^2})^{1/2},$$

where a^2 is a constant larger than or equal to $\frac{1}{2}e^{-2}$.

Again for all the values of m, a, and k that we tried, it was found that this field satisfies the sufficient condition for stability and so is stable.

3.2. Cylindrically symmetric field with axis below the photosphere

In the second paper of a series, Low (1977b) investigated the stability of a magnetic field in cartesian coordinates

$$\mathbf{B} = \frac{B_0(2(y/b + (1-\mu^2)/(1+\mu^2)), -2x/b, 4\mu/(1+\mu^2))}{x^2/b^2 + y^2/b^2 + 2(y/b)(1-\mu^2)/(1+\mu^2) + 1}, \tag{3.3}$$

with μ a parameter used to describe the shear. If, instead, we use cylindrical coordinates, with

$$r^2 = x^2 + (y + (1-\mu^2)/(1+\mu^2))^2,$$

$$\theta = \tan^{-1}\{x/(y + (1-\mu^2)/(1+\mu^2))\},$$

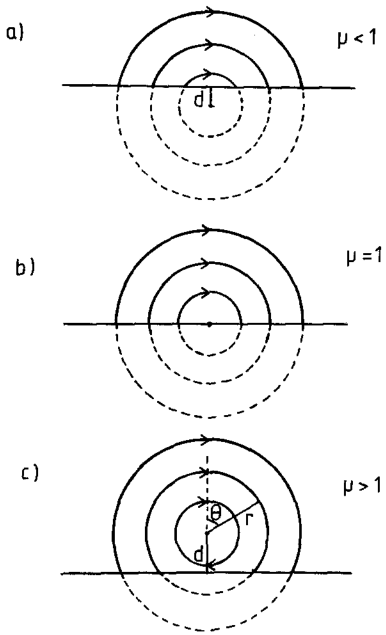

Fig. 2. The field lines for the cylindrical arcade, $\bar{\mathbf{B}} = -(0, -\bar{r}, \lambda^{1/2})/(\lambda + \bar{r}^2)$ whose axis is situated a distance, $\bar{d} = (1-\mu^2)/(1+\mu^2)$, below the photosphere. (a), (b), and (c) show the rising motion as the parameter, μ, is increased.

and then change to dimensionless variables

$$\bar{r} = r/b\,, \qquad \bar{\mathbf{B}} = 2\mathbf{B}/B_0\,, \qquad \lambda^{1/2} = 2\mu/(1+\mu^2)\,,$$

we obtain

$$\bar{\mathbf{B}} = (0, -\bar{r}, \lambda^{1/2})/(\bar{r}^2 + \lambda)\,.$$

The field is then just cylindrically symmetric with its axis situated a distance, $d = b(1-\mu^2)/(1+\mu^2)$, below the photosphere (x-axis). The variation of the position of the axis as μ increases is shown schematically in Figure 2, with a magnetic island appearing when $\mu > 1$. Unfortunately, the line-tying conditions are rather complicated except for the special case of $\mu = 1$ when they make the perturbation vanish at $\theta = \pm\frac{1}{2}\pi$. Low suggests the value $\mu = 1$ as the onset of the instability, but this was by no means verified.

Low finds that for $\mu \leq 1$ the field is stable. To test for stability, he uses a perturbation of the form

$$\boldsymbol{\xi} = (P, Q, 0)\,,$$

where P and Q are arbitrary functions of r and θ but *independent* of z. He suggests that it is the line-tying of footpoints that stabilises the field; it may be easily shown that *even if line-tying is neglected* the basic state is stable to his perturbation. This indicates that the original perturbation was not general enough and that a z-dependence should have been included. However, when this is done for $\mu \leq 1$, we find the field is still stable. For the particular field governed by $\mu = 1$ the field is given by (3.2) with $a = 0$ and this has already been shown in Section 3.1 to be stable. When $\mu > 1$, so that magnetic islands exist, we shall prove in Section 4 that the field is unstable.

3.3. BIRN *et al.*'s FIELD

Birn *et al.* (1978a) discuss possible equilibrium solutions to (1.4) when the form of $B_z(A)$ is prescribed to be

$$B_z(A) = \lambda^{1/2} e^{-A} \cosh\left(\tfrac{1}{2}p\pi\right).$$

They found that for $\lambda < \lambda_{\text{crit}}$, a critical value of the parameter, there exist two possible equilibrium states that bifurcate at $\lambda = \lambda_{\text{crit}}$. They have the form

$$A = \log\left(r \cosh\left(p\theta\right)\right),$$

where p is given by

$$p = \lambda^{1/2} \cosh\left(\tfrac{1}{2}p\pi\right).$$

For $\lambda > \lambda_{\text{crit}}$, no equilibrium exists and so they suggest that a two-ribbon flare begins when λ increases beyond λ_{crit}. However, the resulting shearing displacement of the photospheric footpoints has the form

$$z = mx,$$

where

$$m = \sinh\left(\tfrac{1}{2}p\pi\right);$$

thus, if we assume that the field evolves through prescribed shearing motions, it can be seen that increasing the gradient m causes λ to increase up to the value of λ_{crit}; then any further shearing makes λ *decrease*, with the magnetic field taking on the characteristics of the upper branch solutions (see Figure 3, with $l = 0$). In other words, increasing the shear does *not* cause λ to exceed λ_{crit} and so does not lead to the onset of non-equilibrium.

Now the question arises: is the field magnetohydrodynamically unstable for a critical value of (a) shear or (b)λ? In their next paper, Birn *et al.* (1978b) address this question and attempt to answer it by considering perturbations of the form

$$\boldsymbol{\xi} = (\tilde{\boldsymbol{\xi}}(x, y) + \hat{\mathbf{e}}_z \xi_z(x, y))\, e^{ikz},$$

or equivalently in cylindrical coordinates

$$\boldsymbol{\xi} = (\tilde{\boldsymbol{\xi}}(r, \theta) + \hat{\mathbf{e}}_z \xi_z(r, \theta))\, e^{ikz}. \tag{3.5}$$

They show that the change in the potential energy may be written in the form of (2.9) with the last integral absent, which may be obtained by putting $\mathbf{E}_1 \equiv \mathbf{E}_2$ or $\mathbf{E}_2 \equiv 0$.

Following Low (1977b), we may now *without loss of generality* choose the axial component of the perturbation, ξ_z, identically zero, so that (2.9) becomes,

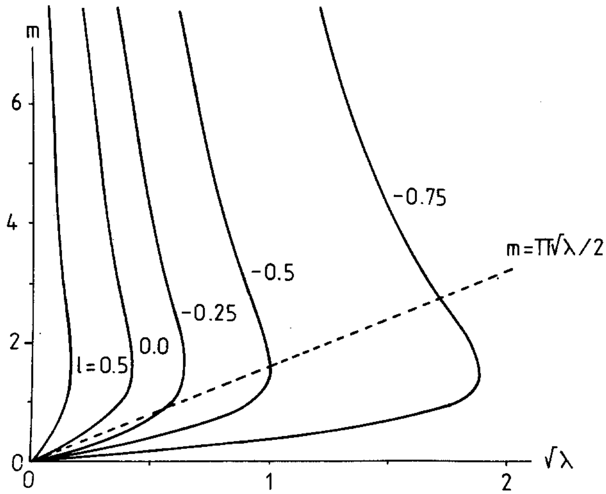

Fig. 3. The shear gradient, m, as a function of the axial parameter, λ, for families of force-free solutions, specified by the parameter l. The dashed line corresponds to the locus of cylindrically symmetric arcades and is given by $m = \frac{1}{2}\pi\lambda^{1/2}$. Note that for $-1 < l < \frac{1}{2}$ these arcades lie on the upper branch of the equilibrium curves. (From Priest and Milne, 1980.)

in Birn *et al.*'s notation,

$$\delta W = \delta F + \frac{1}{2\mu}\int \{|B_z \nabla \tilde{\xi}|^2 + k^2 |\tilde{\xi} \times \mathbf{B}_z|^2\} r \, \mathrm{d}r \, \mathrm{d}\theta , \tag{3.6}$$

where

$$\delta F = \int \{|\nabla \delta A|^2 - \pi''(A)|\delta A|^2\} r \, \mathrm{d}r \, \mathrm{d}\theta$$

and

$$\delta A = -\boldsymbol{\xi} \cdot \nabla A, \; \pi''(A) = \mathrm{d}^2(B_z^2(A)/2)/\mathrm{d}A^2 .$$

Since k^2 multiplies a positive definite term in (3.6), this seems to suggest that including any axial dependence is stabilizing. However, this is true only for *the assumed form* (3.5); more generally, one would expect that the least stable displacement is one that does not bend the field lines (or at least reduces the effect of the stabilizing magnetic tension). The above perturbation (3.5) cannot satisfy such a condition; proving stability to this perturbation *does not prove stability to all perturbations*, but if we can show instability we certainly have a useful result.

With these reservations, we may minimize δF, subject to a suitable constraint, and obtain the Euler–Lagrange equation

$$\nabla^2(\delta A) + \pi''(A)\delta A = -\mu \delta A . \tag{3.7}$$

The sign of δF is then the same as that of $\mu_{\min}$, the minimum eigenvalue of (3.7). In cylindrical coordinates (3.7) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \delta A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \delta A}{\partial \theta^2} + \frac{\mathrm{d}^2}{\mathrm{d}A^2}\left(\frac{1}{2}B_z^2(A)\right)\delta A + \mu \delta A = 0 . \tag{3.8}$$

For the second magnetic field considered by Birn *et al.* (1978a) with

$$A = \log(r \cosh(p\theta)),$$

$$B_z = p\, e^{-A},$$

(3.8) possesses separable solutions with

$$\delta A = f(r) g(\theta).$$

Solving the relevant equation for $f(r)$ with the boundary conditions

$$\delta A \quad \text{finite at} \quad r = 0,$$

$$\delta A \to 0 \quad \text{as} \quad r \to \infty,$$

we find that $f(r)$ is just a Bessel function

$$f(r) = J_a(\sqrt{\mu}\, r),$$

where a is the separation constant.

The equation for $g(\theta)$ reduces to the Sturm–Liouville equation

$$\mathrm{d}^2 g/\mathrm{d}\theta^2 + (2p^2/\cosh^2(p\theta) + a^2) g = 0, \tag{3.9}$$

with boundary conditions

$$\left.\begin{aligned} g &= 1 \quad \text{at} \quad \theta = 0 \text{ (normalization)}, \\ \mathrm{d}g/\mathrm{d}\theta &= 0 \quad \text{at} \quad \theta = 0, \end{aligned}\right\} \tag{3.10}$$

$$g = 0 \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \text{ (line-tying)}. \tag{3.11}$$

The solutions yield the value of μ as a function of a. It transpires that, for $a^2 > 0$, $\mu_{\min}$ is positive which implies that $\delta F > 0$; but when $a^2 < 0$, $\mu_{\min} < 0$ and so $\delta F < 0$.

To find the sign of a^2 and hence of $\mu_{\min}$, we solve (3.9) and (3.10) with $a^2 = 0$ and then use Sturm's comparison theorems (Ince, 1944, Chapter 10; Hood and Priest, 1979b). When $a^2 = 0$ we do not expect to satisfy the extra boundary condition (3.11) at $\theta = \frac{1}{2}\pi$. However, if $g(\frac{1}{2}\pi) > 0$, we may deduce that $a^2 > 0$, whereas if $g(\frac{1}{2}\pi) < 0$ then $a^2 < 0$. Now the solution to (3.9) and (3.10), with $a^2 = 0$, is just

$$g(\theta) = 1 - p\theta \tanh(p\theta).$$

So (3.11) is satisfied when p is equal to p_c given by

$$\tfrac{1}{2} p_c \pi \tanh(\tfrac{1}{2} p_c \pi) = 1. \tag{3.12}$$

For $p < p_c$ we find $a^2 > 0$, which implies that δF is positive. Similarly, for $p > p_c$, δF is negative. The condition (3.12) corresponds exactly to the point λ_{crit} at which the two solutions merge (Figure 3). Thus δF is positive on the lower branch and negative on the upper branch. Therefore, returning to the expression (3.6) for we see that the *lower branch is certainly stable* to perturbations (3.5) with $\mathbf{E} \equiv \mathbf{E}_2$ (as found by Birn *et al.* (1978b)). Also the upper branch will be unstable only if δF dominates the remaining terms in (3.6); if line-tying in the radial and azimuthal directions is neglected and $k=0$ then we may choose a perturbation that is incompressible ($\nabla \cdot \boldsymbol{\xi} = 0$) and so produce instability. Unfortunately, when line-tying is included, the minimising perturbation is compressible and the sign of δW is no longer given by the sign of δF. The minimising equation is complicated and so we calculated δW for several different types of perturbations. For all choices we found that δW was positive, even for large values of the shear gradient, m.

3.4. More general field

The magnetic field of Birn *et al.* is a special case of the more general solutions to the force-free equation generated by Priest and Milne (1980). They have

$$B_z(A) = \lambda^{1/2}(A)^{1+1/l}$$

and

$$A = -F_\theta(\theta)(l/r)^l .$$

The field components are

$$B_r = -\frac{r^{-1-l}}{l}\frac{\mathrm{d}F_\theta}{\mathrm{d}\theta}, \quad B_\theta = -r^{-1-l}F_\theta, \quad B_z = \pm\lambda^{1/2}r^{-1-l}\,|F_\theta|^{l+1/l}, \tag{3.13}$$

where F_θ satisfies the differential equation

$$\mathrm{d}^2F_\theta/\mathrm{d}\theta^2 + l^2F_\theta + \lambda(l+1)lF_\theta|F_\theta|^{2/l} = 0 . \tag{3.14}$$

(As $l \to 0$ we recover the solution of Birn *et al.* from (3.14).)

The fields display the same general features as before, namely there exist two solutions for $\lambda < \lambda_{\text{crit}}$ and no solution for $\lambda < \lambda_{\text{crit}}$. Thus we may expect the stability properties of our wide class of fields to be similar to those of the particular case ($l \to 0$) of Birn *et al.* (1978a, b). They showed that, for their class of perturbations (3.5), the expression δF in (3.6), for all basic states, is *positive on the lower branch and negative on the upper branch.* These results may be verified for our class of fields as follows.

Consider first the stability of the critical solution when $\lambda = \lambda_{\text{crit}}$. Expand this solution, using the fact that λ is a maximum at λ_{crit} to write

$$\lambda = \lambda_{\text{crit}}(1 - \delta^2 + \cdots),$$

$$F_\theta = F_{\text{crit}}(\theta) + \delta F_1(\theta) .$$

To order δ, (3.14) becomes

$$\mathrm{d}^2F_1/\mathrm{d}\theta^2 + (l^2 + \lambda_{\text{crit}}(l+1)(l+2)|F_{\text{crit}}|^{2/l})F_1 = 0 . \tag{3.15}$$

However, the Euler–Lagrange equation (3.7) is separable; its θ-dependence, when $\lambda = \lambda_{\text{crit}}$, is

$$\mathrm{d}^2g/\mathrm{d}\theta^2 + (a^2 + \lambda_{\text{crit}}(l+1)(l+2)|F_{\text{crit}}|^{2/l})g = 0 . \tag{3.16}$$

Since (3.15) and (3.16) are identical when $a^2 = l^2$ and the boundary conditions are the same, we conclude that a^2 must equal l and so is positive, this implies that $\delta F > 0$

and so (for this class of perturbation) the critical basic state, defined by $F_{\rm crit}(\theta)$, is stable.

Now consider the lower branch. From (3.6) the integral may be written as

$$\delta F = \int \{|\nabla \delta A|^2 - \pi''(A_{\rm crit})|\delta A|^2 + [\pi''(A_{\rm crit}) - \pi''(A)]|\delta A|^2\}\, r\, {\rm d}r\, {\rm d}\theta$$
$$= \delta F_{\rm crit} + \delta F_1\,,$$

where $A_{\rm crit}$ is the critical basic state defined by $\lambda_{\rm crit}$. We have shown above that $\delta F_{\rm crit}$ is positive and so stability of the lower branch certainly holds if δF_1 is positive. A sufficient condition for stability is therefore

$$\pi''(A_{\rm crit}) \geq \pi''(A) \qquad \text{for all} \quad r \quad \text{and} \quad \theta\,.$$

From the definition of $\pi''(A)$ in Section 3.3, this condition reduces to

$$\lambda_{\rm crit}|F_{\rm crit}|^{2/l} \geq \lambda\,|F_\theta|^{2/l}\,, \tag{3.17}$$

where both $F_{\rm crit}$ and F_θ satisfy (3.14) with the respective values of $\lambda_{\rm crit}$ and λ. From the numerical solution to (3.14), (3.17) is always found to be satisfied; analytically, this result may be established approximately by considering the first two terms of the Taylor's series for F_θ,

$$F_\theta(\theta) \simeq F_0 + (1 - F_0)4\theta^2/\pi^2\,,$$

which satisfies the boundary conditions $F_\theta(\frac{1}{2}\pi) = 1$, $F'_\theta(0) = 0$. Substitution into (3.14) determines the constant F_0 from

$$8(1 - F_0)/\pi^2 + l^2 F_0 = \lambda(-l)(1 + l)F_0^{1+2/l}\,. \tag{3.18}$$

For $\lambda < \lambda^*_{\rm crit}$, (the approximation to the actual critical value, $\lambda_{\rm crit}$), (3.18) has two solutions, whereas for $\lambda > \lambda^*_{\rm crit}$ there are no solutions. When $\lambda = \lambda^*_{\rm crit}$, (3.18) has one solution denoted by $F_{0\,\rm crit}$. If l is negative the lower branch solutions have F_0 greater than $F_{0\,\rm crit}$ and so have $F_\theta(\theta)$ greater than or equal to $F_{\rm crit}(\theta)$ for all values of θ; (3.17) is therefore satisfied and so the lower branch solutions are stable to this class of perturbation. A similar argument holds if l is positive, for then F_0 is less than $F_{0\,\rm crit}$.

One feature of our more general set of equilibria is that, for a given l, there is always one value of m for which the configuration becomes a cylindrically symmetric arcade and this has the advantage that the energy principle of Section 2 may be applied. This case of cylindrical symmetry occurs when $F_\theta = 1$ and so (3.14) gives the value of λ for which it occurs as

$$\lambda = -l/(1 + l)\,.$$

From (1.3) the shear is

$$z = mx\,,$$

where x is the horizontal photospheric distance and m, the shear gradient, is given by

$$m = \tfrac{1}{2}\pi\lambda^{1/2}\,.$$

For the particular case $l = -\frac{1}{2}$, (3.14) was solved analytically by Priest and Milne; the two solutions are

$$F_\theta(\theta) = (1 \pm (1-\lambda)^{1/2} \cos \theta)^{1/2} .$$

The lower branch solutions are given by the positive sign and the upper branch by the negative sign. $F_{\text{crit}} = 1$ when $\lambda = \lambda_{\text{crit}} = 1$ and it can easily be shown that the inequality (3.17) holds for this basic state. In this case the cylindrical arcade lies exactly at the bifurcation point, $\lambda_{\text{crit}} = 1$. According to Figure 3, reproduced from Priest and Milne, when $-\frac{1}{2} < l < 0$ the cylindrical arcade lies on the lower branch and when $-1 < l < -\frac{1}{2}$ it lies on the upper branch.

Thus we may test the stability of an equilibrium on the upper branch by taking this cylindrical field. Substituting these fields into (2.5) we find that the sufficient condition for stability, (2.7), holds for all values of l and m considered. This analysis shows that there are *certain fields which lie on the upper branch and are stable to all perturbations of the form* (2.2). Therefore, although two fields can satisfy the same boundary conditions, it is *not* safe to assume that the field with the higher energy (larger shear) will necessarily be linearly unstable and erupt to reform as the field with the lower energy.

4. Stability of a Twisted Flux Tube within an Arcade

In many cases before a two-ribbon flare occurs, the site contains a long, winding, *plage filament,* which is situated near the 'magnetic inversion line' and within a magnetic arcade. Frequently motions are observed along the length of the filament, suggesting that the magnetic field is directed along the filament and forms a (possibly twisted) flux tube that is embedded within an external arcade; the ends of the flux tube are tied in the photosphere. This configuration readily leads itself to analytic treatment and in this section we derive *sufficient* conditions for its *instability.*

We model the flux tube, within which lies the filament (Figure 4) by a cylindrical structure, line-tied at its ends and surrounded by a magnetic arcade. A vertical section through the structure is shown in Figure 2c; the weak curvature of the tube normal to the plane of Figure 2c is neglected. The axis is located at a distance, d, above the photosphere and the form we adopt for the magnetic field is simply the cylindrically symmetric field of uniform twist, namely

$$B_r = 0 , \qquad B_\theta = B_0(r/b)/(1 + r^2/b^2) , \qquad B_z = B_0/(1 + r^2/b^2) ,$$

with the distance r measured from the flux tube axis. In this analysis the form of the external field is not important. In order to model the effect of the line-tying of the dense photosphere on the arcade, the perturbations are assumed to vanish when the azimuthal coordinate $\theta = \cos^{-1}(-d/r)$ for $r \geq d$. However, by replacing this line-tying condition with the stricter condition that all perturbations vanish at $r = d$, we may obtain sufficient conditions for instability. In other words, if the stricter conditions gave instability, then the line-tied field is certainly unstable, but if it only

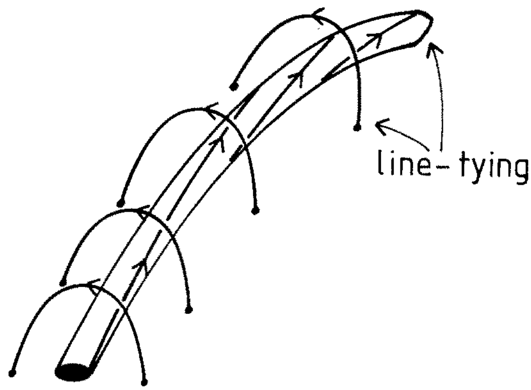

Fig. 4. A possible preflare configuration consisting of a weakly twisted flux tube anchored at its ends and located within a magnetic arcade. The plage filament is assumed to be located along the flux tube.

gives stability nothing can be deduced about the line-tied structure. However, we may obtain lower bounds on the stability as follows. If the external field is the same as the flux tube (i.e. uniform twist) and we neglect the stabilizing effect of line-tying, then when this structure is stable so is the line-tied structure.

Consider first the situation when the flux tube is not tied in directions normal to the plane of Figure 2c. The energy principle derived by Newcomb (1960) is then directly applicable, a minimization of the potential energy, δW, gives the Euler–Lagrange equation (2.5) and (2.6) with $h^2=0$. The least stable case has been shown by Newcomb to occur when $m=1$ and for this field the function, F, vanishes identically when $k=-1/b$. To analyse the stability we put $k=-1/b+\delta$ and expand the perturbation in powers of δ (for $\delta r^2 \ll 1$) as follows:

$$\xi_r(r)=\xi_0(r)+\delta\xi_1(r)+\cdots,$$
$$F=\delta^2F_0(r)+\delta^3F_1(r)+\cdots,$$
$$H=\delta H_0(r)+\delta^2H_1(r)+\cdots,$$
$$G=\delta G_0(r)+\delta^2G_1(r)+\cdots,$$

where

$$F_0=B_0^2r^3/(1+r^2/b^2)^3,\ H_0=-2(B_0^2/b)r^2/(1+r^2/b^2)^3$$

and

$$G_0=4(B_0^2/b)r(-1+r^2/b^2)/(1+r^2/b^2)^4\,.$$

The Euler–Lagrange equation has a boundary layer structure near $\delta=0$ and so we stretch the radial coordinate by putting, $r=\delta^{1/2}bx$. To lowest order, namely $0(\delta^{5/2})$, (2.6) then reduces (for $x^2 \ll \delta^{-1}$) to

$$\mathrm{d}(x^3 d\xi_0/\mathrm{d}x)/\mathrm{d}x+4x^3\xi_0=0$$

and the solution that is finite at the origin is

$$\xi_0 = AJ_1(2x)/x\,,$$

in terms of the Bessel function, J_1, of first order. This vanishes when $2x$ equals the first zero of J_1, namely 3.83. Since the solution vanishes somewhere, we may deduce from Newcomb's analysis that the *arcade containing a magnetic island* (Figure 2c) *is unstable*, even though the arcade overlying the central flux tube is line-tied. A similar result for an isolated flux tube with no arcade outside was obtained by Anzer (1968).

The above analysis has assumed that the flux tube containing the filament has an infinite length, whereas in practice its ends are anchored in the photosphere (sometimes in a sunspot). Let us now, therefore, include the *stabilizing effect of line-tying at the ends of the flux tube.* Such an effect has already been studied by Hood and Priest (1979a) when the flux tube is not surrounded by an arcade (tied at the photosphere) but extends to infinity in the radial direction. Thus in order to obtain sufficient conditions for instability we just have to modify their analysis to include the stabilizing effect of a solid boundary at a radius d.

Following Hood and Priest (1979a), for a flux tube of length $2L$ and typical width, $2b$, the functions (2.5) are given by

$$F = \frac{(B_0^2/b^2)[((kb+1)^2 + h^2b^2)r^3 + h^2r^5]}{(1+r^2/b^2)^2(1+(k^2+h^2)r^2)}\,,$$

$$G - \mathrm{d}H/\mathrm{d}r = \frac{(B_0^2/b^4)r^3(a_0 + a_1r^2/b^2 + a_2r^4/b^4)}{(1+r^2/b^2)^3(1+(k^2+h^2)r^2)^2}\,,$$

where a_0, a_1, and a_2 are the following constants:

$$a_0 = 3k^4b^4 + 2k^3b^3 + (6h^2b^2 - 1)k^2b^2 + 2h^2kb^3 + h^2b^2(3h^2b^2 + 4)\,,$$

$$a_1 = k^6b^6 + 2k^5b^5 + (3h^2b^2 + 4)k^4b^4 + (4h^2b^2 + 2)k^3b^3 + \\ + (3h^4b^4 + 11h^2b^2 - 1)k^2b^2 + 2h^2(h^2b^2 + 1)kb^3 + h^4b^4(h^2b^2 + 7)\,,$$

$$a_2 = k^6b^6 + 2k^5b^5 + (3h^2b^2 + 1)k^4b^4 + 4h^2k^3b^5 + \\ + h^2(3h^2b^2 + 1)kb^3 + h^6b^6\,,$$

and $h = \pi/2L$. We integrate (2.6) from the axis, $r = 0$, out to the bounding radius, $r = d$, and can deduce that, if ξ_r is zero before this distance is reached, then the filament is unstable (Newcomb, 1960). The results obtained by considering different twists, Φ, and heights, d, are shown in Figure 5.

Here we have plotted the twist, Φ, that is sufficient for instability, as a function of the wavenumber, k. For Φ greater than some critical value, Φ_{crit}, there exists a range of values of k for which the tube is unstable. To the right of each curve the equilibrium is definitely *unstable*. Because the conditions derived are only sufficient, we may not state that the region to the left of each curve is stable. However, the region to the left of the curve, $d = \infty$, is *stable* to perturbations of the form (2.2). This

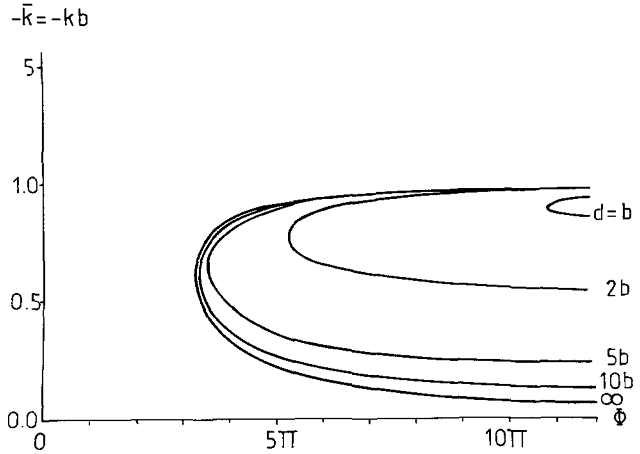

Fig. 5. Sufficient conditions for the amount of twist, $\Phi \equiv 2L/b$, required to produce instability of a flux tube embedded in an arcade. The flux tube has length L and its axis is situated at a height d above the photosphere. For Φ greater than some critical value, Φ_{crit}, (i.e. to the right of each curve), the equilibrium is unstable for a range of wavenumbers, k. (For all values of d, a sufficient condition for stability to (2.2) is that the twist, Φ, be less than that shown for $d = \infty$.)

curve gives us a lower bound on the critical twist for these perturbations and the relevant curve, for a given height, supplies the upper bound; thus, for example, when $d = 3b$, the configuration is stable to perturbations (2.2) when $\Phi < 3.3\pi$ and has certainly become unstable by the time $\Phi > 4.2\pi$. Moreover, a stronger result for stability to *all* kink ($m = 1$) perturbations can be proved, namely that when $\Phi \leqslant 2\pi$ the flux tube is certainly stable (Appendix B).

Thus the filament may become unstable in two possible ways. In the first case, if the length or twist of the flux tube is increased while its height remains fixed, the effect of line-tying is reduced so much that ultimately instability ensues; e.g. for $d = 5b$, $\Phi_{\text{crit}} = 3.6\pi$. On the other hand, if the height increases, while the length and twist remain fixed, the stabilizing effect of the arcade that overlies the filament is gradually reduced until again the structure becomes unstable, e.g. for $\Phi = 5\pi$, $d_{\text{crit}} = 2.16b$.

In general a more complicated relationship between these two possibilities hold in that the filament may rise as it is twisted. Since we have derived only sufficient conditions for instability, the onset of the eruption may take place at somewhat lower thresholds than we have obtained.

5. Conclusion

In this paper we have attempted to analyse the magnetohydrodynamic stability of force-free arcade configurations that may erupt and initiate a two-ribbon flare. We used the energy method of Bernstein *et al.* (1958) and found that the *stabilizing influence of photospheric line-tying* is so effective that it is extremely difficult to obtain instability! *Cylindrically symmetric arcades* with their axis lying on the photosphere are completely *stable* for a wide class of perturbations. The nonlinear force-free equilibria of Priest and Milne (1980) contain two branches of solutions and the *lower branch* was shown to be *stable* to this class. On each upper branch, one of the solutions is cylindrically symmetric in type and so is *stable*; thus, although a lower branch solution exists with a lower energy, the upper solution is not necessarily (linearly) unstable.

We have been led to consider more complex equilibria than a simple sheared arcade; this is partly because it seems so remarkably stable and partly because the presence of a plage filament may indicate a magnetic field that is predominantly along rather than across the long axis of the arcade. In a vertical section across the arcade these more complex equilibria contain magnetic islands; the three-dimensional configuration is that of a long magnetic *flux tube imbedded in an overlying arcade*. If line-tying at the ends of the tube is neglected, the configuration is unstable. When such line-tying is included, *instability occurs provided the length, twist or height of the flux tube is too great*; in particular, the higher the tube is situated, the smaller is the twist required for instability.

Limitations of the analysis include the fact that the conditions for instability are only sufficient; a more comprehensive stability analysis should be able to reduce the thresholds we have obtained and analyse more complex magnetic configurations. Furthermore, the filament has been regarded as a tracer of the field eruption while neglecting the departure from a force-free state that the filament may cause. This neglect is possibly justified for a plage filament, where the magnetic field strength is so much greater than in a quiescent filament. The view taken in this paper is that the filament eruption is due to an instability of the whole magnetic structure surrounding the filament. However, the alternative possibility, especially for quiescent prominences, is that the instability is driven by the filament itself; this has shown up as a characteristic of a recent prominence model (Milne *et al.*, 1979) when the shear angle is too great.

Acknowledgements

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Appendix A

To include the effects of line-tying we require that all perturbations vanish at two values of the azimuthal coordinate, i.e. $\theta = \pm\pi/2$. If we neglect this condition, we wish to be able to recover the analysis derived for cylinder of infinite length. Thus we may express any function of the three coordinates, and in particular the radial coordinate of the perturbation, $\boldsymbol{\xi}$, as

$$\xi_r(r, \theta, z) = \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \{\xi_{n,l}(r) \cos(m\theta + kz) + \\ + \eta_{n,l}(r) \sin(m\theta + kz)\} \cos(2n+1)\theta + \\ + \{\xi^*_{n,l}(r) \cos(m\theta + kz) + \eta^*_{n,l}(r) \sin(m\theta + kz)\} \sin 2n\theta ,$$

where $k = 2\pi l/L$ and m is arbitrary. To find expressions for the particular coefficients, we note that

$$\int_0^L \cos\left(m\theta + \frac{2\pi l}{L} z\right) \cos\left(m\theta + \frac{2\pi j}{L} z\right) \mathrm{d}z = \frac{L}{2} \delta_{l,j} ,$$

$$\int_0^L \sin\left(m\theta + \frac{2\pi l}{L} z\right) \sin\left(m\theta + \frac{2\pi j}{L} z\right) \mathrm{d}z = \frac{L}{2} \delta_{l,j} , \tag{A.1}$$

$$\int_0^L \sin\left(m\theta + \frac{2\pi l}{L} z\right) \cos\left(m\theta + \frac{2\pi j}{L} z\right) \mathrm{d}z = 0 .$$

Similar results exist for $\cos(2n+1)\theta$ and $\sin 2n\theta$ and it can be shown that

$$\xi_{n,l}(r) = \frac{2}{L} \int_{-\pi/2}^{\pi/2} \left\{ \int_0^L \xi_r(r, \theta, z) \cos(m\theta + kz) \, \mathrm{d}z \right\} \cos(2n+1)\theta \, \mathrm{d}\theta ;$$

similar expressions exist for $\eta_{n,l}(r)$, $\xi^*_{n,l}(r)$ and $\eta^*_{n,l}(r)$. In this manner we may obtain expressions for the azimuthal and axial components $\xi_\theta(r, \theta, z)$ and $\xi_z(r, \theta, z)$. Thus a typical component of the series for the vector $\boldsymbol{\xi}$ is

$$(\xi_{r,n,l}(r) \cos(m\theta + kz), \xi_{\theta,n,l}(r) \sin(m\theta + kz), \xi_{z,n,l}(r) \sin(m\theta + kz)) \times \\ \times \cos(2n+1)\theta + (\eta_{r,n,l}(r) \sin(m\theta + kz), \eta_{\theta,n,l}(r) \cos(m\theta + kz) , \\ \eta_{z,n,l}(r) \cos(m\theta + kz)) \cos(2n+1)\theta .$$

This we denote by

$$\boldsymbol{\xi}_{n,l} \cos(2n+1)\theta + \boldsymbol{\eta}_{n,l} \cos(2n+1)\theta , \tag{A.2}$$

so that the perturbation $\boldsymbol{\xi}$ may be written as

$$\boldsymbol{\xi} = \sum_n \sum_l \boldsymbol{\xi}_{n,l} \cos(2n+1)\theta + \boldsymbol{\eta}_{n,l} \cos(2n+1)\theta . \tag{A.3}$$

Substituting just one component of the series (A.3) into (2.1), integrating first over z then over θ and using the results (A.1), we may show that

$$\delta W(\boldsymbol{\xi}) = \delta W_{n,l}(\boldsymbol{\xi}_{n,l}) + \delta W_{n,l}(\boldsymbol{\eta}_{n,l}) ,$$

where

$$\delta W_{n,l}(\boldsymbol{\xi}_{n,l}) = \frac{\pi L}{4}\int_0^\infty r\,\mathrm{d}r\left\{\frac{\xi_r^2}{r^2}(mB_\theta + krB_z)^2 + h^2 B_\theta^2\frac{\xi_r^2}{r^2} + \right.$$

$$+ [k(\xi_\theta B_z - \xi_z B_\theta) - (\xi_r B_\theta)']^2 +$$

$$+ \frac{1}{r^2}((r\xi_r B_z)' + m(\xi_\theta B_z - \xi_z B_\theta))^2 + \frac{h^2}{r^2}(\xi_\theta B_z - \xi_z B_\theta)^2 -$$

$$- \alpha\left[-(\xi_\theta B_z - \xi_z B_\theta)\left(\frac{mB_\theta}{r} + kB_z\right)\xi_r - \xi_r B_z(k(\xi_\theta B_z - \xi_z B_\theta) - \right.$$

$$\left.\left. - (\xi_r B_\theta)') - \frac{\xi_r}{r}B_\theta((r\xi_r B_z)' + m(\xi_\theta B_z - \xi_z B_\theta))\right]\right\},$$

$$\delta W_{n,l}(\boldsymbol{\eta}_{n,l}) = \frac{\pi L}{4}\int_0^\infty r\,\mathrm{d}r\left\{\frac{\eta_r^2}{r^2}(mB_\theta + krB_z)^2 + h^2 B_\theta^2\frac{\eta_r^2}{r^2} + \right.$$

$$+ (-k(\eta_\theta B_z - \eta_z B_\theta) - (\eta_r B_\theta)')^2 +$$

$$+ \frac{1}{r^2}((r\eta_r B_z)' - m(\eta_\theta B_z - \eta_z B_\theta))^2 + \frac{h^2}{r^2}(\eta_\theta B_z - \eta_z B_\theta)^2 -$$

$$- \alpha\left[(\eta_\theta B_z - \eta_z B_\theta)\left(\frac{m}{r}B_\theta + kB_z\right)\eta_r - \right.$$

$$- \eta_r B_z(-k(\eta_\theta B_z - \eta_z B_\theta) - (\eta_r B_\theta)') -$$

$$\left.\left. - \frac{\eta_r}{r}B_\theta((r\eta_r B_z)' - m(\eta_\theta B_z - \eta_z B_\theta))\right]\right\}.$$

The important fact is that the integrals, $\delta W(\boldsymbol{\xi}_{n,l})$ and $\delta W(\boldsymbol{\eta}_{n,l})$, are in no sense coupled and we may minimize each integral separately and obtain the Euler Lagrange equation (2.6). In fact the minimum value of the two integrals is identical, as may be seen by minimizing $\delta W_{n,l}(\boldsymbol{\xi}_{n,l})$, then putting $\eta_r = \xi_r$, $\eta_\theta = -\xi_\theta$ and $\eta_z = -\xi_z$.

Now the question of stability may be answered; for, if at least one $\delta W_{n,l}(\boldsymbol{\xi}_{n,l})$ is negative, then $\delta W(\boldsymbol{\xi})$ can be made negative by choosing all the other $\boldsymbol{\xi}_{n,l}$ to vanish. If all the $\delta W_{n,l}$ are positive, $\delta W(\boldsymbol{\xi})$ need not be positive since the cross terms may be negative. However, to consider stability to the perturbations (A.2) alone, we minimize each $\delta W_{n,l}(\boldsymbol{\xi}_{n,l})$ in turn and note the resulting sign. In fact, it can be shown that $n = 0$ is the least stable of these particular perturbations. Note that for force-free fields we may choose, without loss of generality, $\boldsymbol{\xi}$ perpendicular to the magnetic field **B**, since the parallel component does not provide a contribution to the potential energy.

It is important to notice that the methods of Section 2 deal with stability only to the particular perturbations

$$\xi_{n,l} \cos(2n+1)\theta + \eta_{n,l} \cos(2n+1)\theta .$$

The full series (A3) for *all* possible perturbations has not been treated; it leads to cross product terms in δW which do not all vanish and give rise to an infinite sequence of coupled Euler–Lagrange equations which have not been solved. In Hood and Priest (1979a) a similar method was used to analyse the stability of a flux tube to particular perturbations and it was wrongly claimed that necessary and sufficient conditions for stability to *all* perturbations had been obtained. Rather, the stability conditions in both Hood and Priest (1979a) and the present paper are necessary and sufficient only for the chosen class of perturbations. This means that when instability has been proved the conditions are sufficient, whereas when stability has been established the conditions are necessary. A full analysis could be conducted by either solving the infinite sequence of Euler–Lagrange equations or working with the partial differential equations that result from minimizing (2.9).

Appendix B

It is possible to derive a simple sufficient condition for the stability of a line-tied flux tube of finite length, $2L$, to *all* possible kink ($m = 1$) perturbations. We may write any kink perturbation as

$$\boldsymbol{\xi} = \sum_n \boldsymbol{\xi}_n(r) e^{i\theta + kz} , \tag{B.1}$$

where $k = 2n\pi/(2L)$. This is the form used by Newcomb (1960) but it is important to note that the line-tying conditions, on a tube of finite length, namely

$$\boldsymbol{\xi}(r, \theta, \pm L) = 0 , \tag{B.2}$$

cannot be satisfied by each individual term of (B.1). (B.2) implies that a linear combination of the basis functions are required such that

$$\boldsymbol{\xi}(r, \theta, \pm L) = \sum_n \boldsymbol{\xi}_n(r)\, e^{i\theta + kL} = 0 .$$

Newcomb showed that, for the perturbation (B.1), the potential energy δW from (2.1) may be written as

$$\delta W = \sum_n \delta W_n , \tag{B.3}$$

where

$$\delta W_n = \int \{|\nabla \times (\boldsymbol{\xi}_n \times B)|^2 - \alpha(\boldsymbol{\xi}_n \times B) \cdot \nabla \times (\boldsymbol{\xi}_n \times \mathbf{B})\}\, \mathrm{d}\tau .$$

The total potential energy δW is positive if the minimum value of each δW_n is positive. If the minimum value of one δW_n is negative, this *does not* necessarily imply instability.

Using Newcomb's analysis, we may minimise each δW_n in turn and derive

$$\delta W_n = \int_0^\infty \{F\xi_r'^2 + (G - H')\xi_r^2\}\,\mathrm{d}r\,, \tag{B.4}$$

where a dash denotes differentiation with respect to r and the functions F, H, and G are given by (2.5) with $h = 0$, $m = 1$, and $k = 2n\pi/(2L)$. The function F is positive and $G - H'$ may be written as (Newcomb, 1960)

$$G - H' = \frac{k^2 r}{1 + k^2 r^2}(krB_z + B_\theta)^2 + \frac{2k^2 r}{(1 + k^2 r^2)^2}(k^2 r^2 B_z^2 - B_\theta^2)\,. \tag{B.5}$$

Thus a sufficient condition for stability is that (B.5) is always positive, which is true if

$$k^2 \geq (B_\theta^2/(r^2 B_z^2)\,, \qquad \text{for all } r$$

or, in other words,

$$4n^2\pi^2/(2L)^2 \geq (\Phi/2L)^2\,, \qquad \text{for all } r\,. \tag{B.6}$$

In general, Φ is a function of r with maximum value $\Phi_{\max}$, say, and so

$$\Phi_{\max} \leq 2\pi\,. \tag{B.7}$$

In particular, the uniform-twist field has $\Phi_{\max} = \Phi$.

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