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ON COMPLETENESS IN AFFINE DIFFERENTIAL GEOMETRY

In affine differential geometry there are at least three notions of completeness for nondegenerate hypersurfaces M of the affine space $\mathbb{R}^{n+1}$, namely:

(1) affine-metric completeness, that is, completeness of the Levi–Civita connection of the affine metric of M (whether it is positive definite or not);

(2) Euclidean completeness, that is, completeness of the Riemannian metric on M induced from a Euclidean metric in $\mathbb{R}^{n+1}$;

(3) completeness of the canonical equiaffine connection on M (see Section 1 for this notion).

In [6, §7] Schneider, among others, has studied conditions (1) and (2) and has given an example of a surface in $\mathbb{R}^3$ which is Euclidean complete but not affine-metric complete. Calabi's work in [1], [2] shows the importance of condition (1) in some global problems. In the present paper, we consider one more completeness property:

(2′) Lorentzian completeness, that is, completeness of the metric (assumed nondegenerate) induced on M from a flat Lorentzian metric in $\mathbb{R}^{n+1}$.

It was shown in [4] that if M is a spacelike hypersurface in $\mathbb{R}^{n+1}$ with Lorentzian metric $\Sigma_{k=1}^{n} dx_k^2 - dx_{n+1}^2$ and if the induced metric on M is complete, then the metric induced on M from the Euclidean metric $\Sigma_{k=1}^{n+1} dx_k^2$ is also complete. In this sense, we may say that (2′) implies (2) at least for a spacelike hypersurface.

We wish to propose a more systematic study of these completeness conditions, but the purpose of this paper is to give an example of a spacelike surface M in $\mathbb{R}^3$ with metric $dx^2 + dy^2 - dz^2$ (that is, the Lorentz–Minkowski space L^3) whose induced metric is complete but whose affine metric is not complete.

In order to clarify our approach to affine differential geometry we shall start with a brief introduction to the subject which emphasizes the notion of equiaffine structure. An *equiaffine structure* on a differentiable manifold is a pair (∇, θ), where ∇ is a linear connection with zero torsion and θ is a volume element which is parallel relative to ∇. This approach was first given in my talk at the Conference on Differential Geometry, Münster, June 1982 [5].

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1. Basic theory for hypersurfaces

Let $\mathbb{R}^{n+1}$ be an $(n+1)$-dimensional affine space with a volume element given by the determinant: $\det(e_1,\ldots,e_n) = 1$, where $\{e_1,\ldots,e_n\}$ is the standard basis of the underlying vector space for $\mathbb{R}^{n+1}$. We denote by D the standard linear connection in $\mathbb{R}^{n+1}$ relative to which the volume element '*det*' is parallel.

To deal with a more general situation, let us consider an $(n+1)$-dimensional manifold $\tilde{M}$ with a certain equiaffine structure (D, ω), namely, a linear connection D with zero torsion and a volume element ω which is parallel relative to D. Let M be a hypersurface, namely, an n-manifold with an immersion f into $\tilde{M}$. For a local theory, we think of M as imbedded and suppress f in all basic formulas we write. Let ξ be a transversal field of tangent vectors on M so that for each x in M, the tangent space $T_x(\tilde{M})$ is the direct sum of the tangent space $T_x(M)$ and the span of ξ. For tangent vector fields X and Y on M, we decompose $D_X Y$ at each point x in the form

(1) $$D_X Y = \nabla_X Y + h(X, Y)\xi,$$

where $\nabla_X Y$ is the component tangent to M and $h(X, Y)\xi$ is the component in the direction of ξ. It is quite routine to check that

(2) $$(X, Y) \to \nabla_X Y$$

defines a linear connection on M with zero torsion and that

(3) $$(X, Y) \to h(X, Y)$$

defines a bilinear symmetric form on each tangent space of M, called the second fundamental form. Note that both the connection ∇ and the form h depend on the choice of ξ. In addition to (1), we may also decompose $D_X\xi$ in the form

(4) $$D_X\xi = -S(X) + \tau(X)\xi,$$

where $S(X)$ is the component tangent to M and $\tau(X)\xi$ is the component in the direction of ξ. We see that S is a $(1,1)$ tensor and τ is a 1-form. We shall also define a volume element v on M by

(5) $$\theta(X_1,\ldots,X_n) = \omega(X_1,\ldots,X_n,\xi),$$

for any tangent vectors $X_1,\ldots,X_n$ on M. It is easy to check that

(6) $$\nabla_X\theta = \tau(X)\theta.$$

Our approach is the following. Assuming nondegeneracy of M (as explained below) we first show that there is a choice of ξ for which the form τ vanishes identically so that the volume element θ is parallel relative to the connection ∇.

We then impose one further condition which will determine ξ, ∇ and θ uniquely. The resulting pair (∇, θ) is the *canonical equiaffine structure* on M.

Now for our purpose, we begin with

LEMMA 1. *Let $\bar{\xi} = Z + r\xi$ be another choice of a transversal vector field, where Z is tangent to M and $r > 0$ is a differentiable function. Then we have the relationships*

(i) $h = r\bar{h}$

(ii) $\nabla_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y)Z$

(iii) $\bar{\tau}(X) = \tau(X) + Xr/r + h(X, Z)/r$

between ∇, h, and τ for ξ and $\bar{\nabla}, \bar{h}$, and $\bar{\tau}$ for $\bar{\xi}$.

Proof. Straightforward.

From (i) it follows that the condition that h is nondegenerate is independent of the choice of ξ. In this case, we say that M is *nondegenerate*. Now we have

LEMMA 2. *If M is nondegenerate, then we can choose ξ so that $\tau = 0$ (and thus θ is parallel relative to ξ).*

Proof. For $r = 1$, we can find Z such that $h(X, Z) = -\tau(X)$ for every tangent vector X.

REMARK. If Σ denotes the set of transversal vector fields for which $\tau = 0$, then the map $\xi \in \Sigma \to \theta$ is injective. For, if $\xi, \bar{\xi} \in \Sigma$, then in the notation of Lemma 1, $\theta = \bar{\theta}$ implies $r = 1$. From $\tau = \bar{\tau} = 0$, we have $h(X, Z) = 0$ for all X, so that $Z = 0$.

Now we impose a further condition on ξ. For h corresponding to ξ, let v be the volume element on M defined by

$$v(X_1, \ldots, X_n) = \sqrt{|\det[h(X_i, X_j)]|}, \tag{6}$$

where $\{X_1, \ldots, X_n\}$ is any basis in the tangent space. Let us consider the condition

$$v = \theta. \tag{C}$$

By choosing a basis $\{X_1, \ldots, X_n\}$ such that $\theta(X_1, \ldots, X_n) = 1$, let

$$h_{ij} = h(X_i, X_j) \quad \text{and} \quad H = \det[h_{ij}]. \tag{7}$$

Then $v(X_1, \ldots, X_n) = \sqrt{|H|}$ and hence $v = \sqrt{|H|}\theta$; condition (C) is thus equivalent to $|H| = 1$.

LEMMA 3. *Let $\xi, \bar{\xi} \in \Sigma$, and write H and $\bar{H}$ for the values for ξ and $\bar{\xi}$ defined in (7). If $\bar{\xi} = Z + r\xi$ as in Lemma 1, then*

$$h = r\bar{h} \quad \textit{and} \quad H = r^{n+2}\bar{H}.$$

Proof. Straightforward.

In view of Lemma 3, we see that

$$\hat{h} = \frac{h}{|H|^{1/(n+2)}} \tag{8}$$

is independent of the choice of ξ. (8) is called the *affine metric* for the nondegenerate hypersurface M. If $\xi \in \Sigma$ satisfies condition (C), then $|H| = 1$ so that $\hat{h} = h$ in (8). Thus the volume element $\theta = v$ for ξ coincides with the volume element $\hat{\theta}$ for the affine metric $\hat{h}$. The uniqueness part in the following theorem follows from the remark just after Lemma 2.

THEOREM 1. *If M is a nondegenerate hypersurface in $\tilde{M}$, we can choose a unique transversal vector field $\xi \in \Sigma$ satisfying condition* (C).

Proof. By Lemma 2, we choose $\xi \in \Sigma$ and compute $H = H_\xi$. Take $r = |H|^{1/(n+2)}$ and then choose a tangent vector field Z such that $\bar{\xi} = Z + r\xi$ is in Σ again. Then $\bar{H} = H_{\bar{\xi}}$ is given by $H/r = H/|H|$ so that $|\bar{H}| = 1$. Thus $\bar{\xi} \in \Sigma$ satisfies condition (C).

The transversal vector field ξ established in Theorem 1 is called the *affine normal* for the nondegenerate hypersurface M. For this ξ, the second fundamental form h coincides with the affine metric $\hat{h}$, and the volume element θ coincides with the volume element $\hat{\theta}$ of the affine metric $\hat{h}$. The linear connection ∇ arising from the affine normal is called the *canonical affine connection* on M. The affine metric $\hat{h}$ is nondegenerate. The Levi–Civita connection on M for the metric $\hat{h}$ will be called the *affine metric connection.*

When $\tilde{M} = \mathbb{R}^{n+1}$ with its equiaffine structure $(D, \det)$, we obtain the canonical equiaffine structure (∇, ω) on any nondegenerate hypersurface M in $\mathbb{R}^{n+1}$. This is indeed the object of study in classical affine differential geometry.

2. An example

We shall give an example of a spacelike surface in the Lorentz–Minkowski space L^3 whose induced metric is complete but whose affine metric is not complete. In fact, this surface is one of the surfaces constructed in [3] in the following way.

Let f be a mapping of $\mathbb{R}^2$ into L^3 with metric $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$:

$$(u, \phi) \in \mathbb{R}^2 \to f(u, \phi) = (x, y, z) \in L^3,$$

where

$$x = \int_0^u \sqrt{1 + \mathrm{e}^{2t}}\, \mathrm{d}t, \qquad y = \mathrm{e}^u \operatorname{sh} \phi, \qquad z = \mathrm{e}^u \operatorname{ch} \phi. \tag{9}$$

Then f is an imbedding of the entire (u, ϕ)-plane $\mathbb{R}^2$ into L^3 and the induced metric on $\mathbb{R}^2$

$$(10) \qquad ds^2 = du^2 + e^{2u} d\phi^2$$

is positive definite. This metric is complete, since the transformation $(u, \phi) \to (X, Y)$, where $X = \phi$ and $Y = e^{-u}$ takes it into the Poincaré metric $(dX^2 + dY^2)/Y^2$ in the upper half-plane $Y > 0$, which is known to be complete. It also follows that (10) has constant Gaussian curvature -1. We denote by M_0 this spacelike surface $f: \mathbb{R}^2 \to L^3$.

In order to view M_0 from the affine point of view, we take a unit timelike normal vector field ξ and the corresponding second fundamental form h. It is known, in the theory of submanifolds of a Lorentzian manifold, that the Gaussian curvature K, which is -1 for our surface M_0, is related to h by the Gauss equation

$$-K = h(X_1, X_1)h(X_2, X_2) - h(X_1, X_2)^2,$$

where $\{X_1, X_2\}$ is an orthonormal basis (relative to the metric (10)) in the tangent space. This means that from the affine point of view, the quantity H defined in (7) for ξ is equal to 1. Thus the affine metric of M_0 coincides with h. We know from [3] that

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = e^u/\sqrt{1 + e^{2u}}, \qquad h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \phi}\right) = 0,$$

$$h\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right) = e^u \sqrt{1 + e^{2u}}.$$

Thus h may be written in the form

$$(11) \qquad d\sigma^2 = (e^u/\sqrt{1 + e^{2u}})du^2 + (e^u \sqrt{1 + e^{2u}})d\phi^2.$$

This affine metric is elliptic. In order to show that it is not complete, we recall the following well-known basic fact.

LEMMA 4. *Suppose that* $d\sigma^2$ *and* $d\tau^2$ *are two Riemannian metrics on a differentiable manifold such that* $d\sigma^2 \leqslant d\tau^2$.

(i) *If* $\{x_n\}$ *is a Cauchy sequence relative to* $d\tau^2$, *it is also relative to* $d\sigma^2$.

(ii) *If* $d\sigma^2$ *is complete, so is* $d\tau^2$.

To apply this lemma, let

$$d\tau^2 = \sqrt{1 + e^{2u}} d\sigma^2 = e^u du^2 + e^u(1 + e^{2u})d\phi^2$$

and observe that

$$d\sigma^2 \leqslant \sqrt{1 + e^{2u}} d\sigma^2 = d\tau^2.$$

We shall show that the metric $d\tau^2$ is not complete. This implies that $d\sigma^2$ is not complete.

Consider the curve C given by $u = -t$, $\phi = 0$, where $0 \leqslant t < \infty$. The tangent vector $(du/dt, d\phi/dt) = (-1, 0)$ has a length (relative to $d\tau^2$) equal to $e^{-t/2}$. So the arclength of C is

$$\int_0^\infty e^{-t/2}\, dt = 2.$$

Obviously, the curve C has no limit point as $t \to \infty$. This proves that $d\tau^2$ is not complete.

REMARK 1. The canonical affine connection ∇ of M_0 coincides with the Levi–Civita connection of the metric ds^2. Hence it is complete. Thus M_0 is also an example showing that condition (3) in the introduction does not imply condition (1).

REMARK 2. It was also shown in [3] that for each $a > 0$, $a \neq 1$, there is a surface M_a in L^3 which is a nonstandard imbedding of the hyperbolic plane into L^3. We can show that each of these surfaces is affine-metric complete in the following way.

The surface M_a is defined by

$$x = \int_0^u \sqrt{1 + a^2 \operatorname{sh}^2 t}\, dt, \qquad y = a \operatorname{ch} u \operatorname{sh} \phi, \qquad z = a \operatorname{ch} u \operatorname{ch} \phi.$$

The induced metric on M is

$$ds^2 = du^2 + a^2 \operatorname{ch}^2 u\, d\phi^2,$$

and the affine metric (which coincides with the second fundamental form of M_a as a spacelike surface of L^3) is given by

$$d\sigma^2 = \left(\frac{a \operatorname{ch} u}{\sqrt{1 + a^2 \operatorname{sh}^2 u}}\right) du^2 + \sqrt{1 + a^2 \operatorname{sh}^2 u}\, a \operatorname{ch} u\, d\phi^2.$$

Case 1. $a < 1$. We have

$$1 + a^2 \operatorname{sh}^2 u < 1 + \operatorname{sh}^2 u = \operatorname{ch}^2 u \quad \text{so} \quad \left(\frac{a \operatorname{ch} u}{\sqrt{1 + a^2 \operatorname{sh}^2 u}}\right) > a$$

from which we have

$$d\sigma^2 > a\, du^2 + a\, d\phi^2.$$

Since the metric on the right-hand side is complete, so is $d\sigma^2$ by Lemma 4.

Case 2. $a > 1$. We have

$$a \operatorname{ch} u / \sqrt{1 + a^2 \operatorname{sh}^2 u} > 1$$

so that

$$d\sigma^2 > du^2 + a d\phi^2.$$

Again, $d\sigma^2$ is complete.

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