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GENERALIZED HOPF MANIFOLDS

ABSTRACT. Let (M, J, g) be a Hermitian manifold with complex structure J, metric g, and Kähler form Ω. Then g is locally conformal Kähler iff $d\Omega = \omega \wedge \Omega$ for some closed and non-exact 1-form ω. Moreover, if ω is a parallel form, M is called a generalized Hopf manifold. The main results of this paper are: (a) the description of the geometric structure of the compact locally conformal Kähler-flat manifolds; (b) the description of the geometric structure of the compact generalized Hopf manifolds on which a certain canonically defined foliation is regular; (c) a description of the harmonic forms and Betti numbers of a general compact generalized Hopf manifold; (d) a method for studying analytic vector fields on generalized Hopf manifolds; (e) conditions for submanifolds of generalized Hopf manifolds to belong to the same class.

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1. INTRODUCTION

In the present paper, we are going on with our study of the special class of complex manifolds first discussed in [29], and called there *locally conformal Kähler manifolds with parallel Lee form or $\mathcal{P}.\mathcal{K}$.-manifolds*. Since the classical Hopf manifolds $H^n \approx S^1 \times S^{2n-1}$ belong to this class [29], we shall replace now the name of $\mathcal{P}.\mathcal{K}$.-manifolds by *generalized Hopf manifolds*.

Let $M^n (n \geqslant 2)$ be a complex manifold ($n = \dim_{\mathbb{C}} M$) with the complex structure tensor J. For simplicity, all the manifolds of this paper are considered connected. Then, a *locally conformal Kähler* (*l.c.K.*) *structure* on (M, J) (if any) is defined as a maximal open covering $\{U_\alpha\}$ of M, with Kähler metrics $\tilde{g}_\alpha$ on every U_α, such that over every $U_\alpha \cap U_\beta \neq \emptyset$, $\tilde{g}_\alpha$ and $\tilde{g}_\beta$ are conformally related. Because of the Kählerian character of $\tilde{g}_\alpha$ this means

$$\tilde{g}_\beta = c_{\alpha\beta}\tilde{g}_\alpha, \tag{1.1}$$

where $c_{\alpha\beta}$ are positive locally constant coefficients, and satisfy the co-cycle condition $c_{\alpha\beta}c_{\beta\gamma} = c_{\alpha\gamma}$.

The co-cycle condition above yields $\ln c_{\alpha\beta} = \sigma_\alpha - \sigma_\beta$ for some C^∞-functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$, which are defined up to an arbitrary term φ/U_α, $\varphi \in C^\infty(M)$. It follows that

$$g = e^{\sigma_\alpha}\tilde{g}_\alpha \tag{1.2}$$

is a global Hermitian metric on M, defined up to a global conformal change $g \mapsto e^\varphi g$, and called an *l.c.K. metric*.

In [27] a configuration (M, J, g) as considered above was called an *l.c.K. manifold*. An l.c.K. manifold has a unique underlying l.c.K. structure, and every l.c.K. structure defines an l.c.K. manifold up to a global conformal

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change of the metric. In this sense, these two notions are essentially equivalent.

If we have an l.c.K. structure, the forms $\{d\sigma_\alpha\}$ yield a global closed 1-form ω on $M(\omega/U_\alpha = d\sigma_\alpha)$, which is defined up to cohomology, and is fixed by a choice of a metric (1.2). Namely, let $\Omega, \tilde{\Omega}_\alpha$ be the Kähler forms of $\tilde{g}, \tilde{g}_\alpha$ respectively $(\Omega(.,.) = g(.,J.))$. Then (1.2) implies $\Omega = e^{\sigma_\alpha}\tilde{\Omega}_\alpha$, whence

$$(1.3) \qquad d\Omega = \omega \wedge \Omega,$$

and one gets [31]

$$(1.4) \qquad \omega = [1/(n-1)]i(\Omega)d\Omega = [1/(n-1)]\delta\Omega \circ J,$$

where $i(\Omega)$, δ are with respect to g.

The form ω is called the *Lee form* of g, and it can be defined by (1.4) for every Hermitian metric. Moreover, if $n = 2$, ω always satisfies (1.3), and for $n \geqslant 3$, if (1.3) holds, ω is closed. Always, if ω is closed and (1.3) holds, g is an l.c.K. metric [27].

It is easy to understand from the above considerations that an l.c.K. metric *is globally c.K.* iff ω is an exact form [27]. In this paper, *we make the convention that l.c.K. means locally c.K. and not globally c.K.* (Globally c.K. is not viewed as a particular case of locally c.K. but as an opposite case.)

Now, it is rather natural to give the following definition: a *generalized Hopf manifold* (g.H.m.) is an l.c.K. manifold (M, J, g) whose Lee form is parallel, i.e.

$$(1.5) \qquad \nabla\omega = 0(\omega \neq 0),$$

where ∇ is the Riemannian connection of g. Let us note that, since ω is closed, (1.5) is equivalent with the condition that the *Lee vector field* $B = \# \omega$ is a Killing vector field.

The manifolds considered are just the $\mathcal{P}.\mathcal{K}$.-manifolds of [29], and the name g.H.m. was chosen now since the Hopf manifolds [14]

$$H^n = (\mathbb{C}^n \backslash \{0\})/\Delta_\lambda$$

(where $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$, and Δ_λ is generated by $z \mapsto \lambda z$, $z \in \mathbb{C}^n$), with the metric

$$(1.6) \qquad g_0 = \left(4 \Big/ \sum_{h=1}^{n} z^h \bar{z}^h\right) \sum_{k=1}^{n} dz^k\, d\bar{z}^k$$

are l.c.K. and satisfy (1.5). (See the proof in [29]. The insertion of the factor 4 is a matter of convenience only.)

REMARK. In the case of a compact g.H.m., the l.c.K. structure defines a unique metric satisfying (1.5) and with $|\omega| = 1$. Indeed, if $g \mapsto e^f g$, $\omega \mapsto \omega + df$, and if we ask (1.5) to be satisfied for both metrics we get by usual computations

([9, p. 115], and a contraction by $-g^{ij}$)

$$\Delta f - (n-1)g(\omega, \mathrm{d}f) = (n-1)|\mathrm{d}f|^2,$$

where Δ is the Laplacian of g. Hence, by a well-known theorem of Hopf [10], $f = \text{const}$. The condition $|\omega| = 1$ will subsequently fix the value of this constant.

Of course, $|\omega| = 1$ can also be imposed in the non-compact case and *hereafter we shall assume*

$$(1.7) \qquad |\omega| = 1$$

for all the g.H.m.

Until now we described the object of our study. The motivation of its interest follows from our previous results on l.c.K. manifolds [27], [29–32], and it is related in particular with the fact that compact l.c.K. manifolds have no Kähler metrics [32]. Hence we are studying some compact complex *non-Kähler* manifolds where only a reduced information of a differential-geometric nature is available. We hope that the results of this paper will provide further motivation of our study.

Very shortly, these results are: (a) the description of the geometric structure of the compact l.c. (K. flat) manifolds; (b) the description of the geometric structure of the compact g.H.m. on which a certain canonically defined foliation is regular; (c) a description of the harmonic forms and Betti numbers of a general compact g.H.m.; (d) a method for studying analytic vector fields on g.H.m.; (e) conditions for submanifolds of g.H.m. to be again g.H.m.

Of course, we shall use our previous results on g.H.m. ($\mathscr{P}.\mathscr{K}.$-manifolds) given in [29].

2. Locally conformally Kähler-flat manifolds

Let us begin by establishing the following result which is obviously of a more general interest.

THEOREM 2.1. *Let (M, g) be a compact l.c.K. manifold, and assume that its underlying l.c.K. structure consists of Kähler metrics with non-negative Ricci curvature. Then, there is a global function $\varphi > 0$ on M such that $(M, \varphi g)$ is a g.H.m.*

Proof. This result is a consequence of a more general theorem of P. Gauduchon [7]. Namely, the torsion form of [7] is $-(n-1)\omega$, and the *vanishing eccentricity theorem* of [7] provides a metric $\varphi g (\varphi > 0)$ whose Lee form is co-closed, and therefore harmonic.

The main steps in the proof of this result can be described as follows.

If the metric g is changed to $e^{\psi}g$, ω changes to $\omega + d\psi$ which is co-closed for $e^{\psi}g$ iff

$$(*)\qquad \Delta\psi - (n-1)[i(\omega)d\psi + |d\psi|^2] + \delta\omega = 0.$$

Now, a straightforward existence proof of a function ψ satisfying (*) would be enough, but we have no such proof since (*) is a non-linear equation. Consider the transformation $\psi = [1/(n-1)]\ln\lambda$. Then (*) becomes

$$(**)\qquad \Delta\lambda - (n-1)i(\omega)d\lambda + (n-1)\lambda\,\delta\omega = 0,$$

which is a linear equation, but we shall have to ask for a strictly positive solution $\lambda > 0$. Following [7], we shall notice that the elliptic operator L^* in the left-hand side of (**) is the adjoint of the elliptic operator of the Hopf type [10] $L = \Delta + (n-1)i(\omega)d$. Both L and L^* have the index of Δ, and since $\ker L = \mathbb{R}$ [10], it follows $\dim\ker L^* = 1$, whence all the solutions of (**) are of the form $\lambda = k\lambda_0$, where $k\in\mathbb{R}$, and $\lambda_0 \not\equiv 0$. Moreover, since $C^{\infty}(M) = \ker L \oplus \operatorname{im} L^* = \ker L^* \oplus \operatorname{im} L$, and since $1 \notin \operatorname{im} L$ [10], we have $\langle\lambda_0, 1\rangle \neq 0$ ($\langle\ ,\ \rangle$ denotes global scalar product on M), and λ_0 can be normed by $\langle\lambda_0, 1\rangle = \mathrm{Volume}(M)$. Furthermore, $\lambda_0 \geqslant 0$, and $\lambda_0 > 0$ are ensured by Lemmas 1 and 2 of [7].

Thus, in order to prove Theorem 2.1 it remains to prove that the Lee form of φg is, in fact, parallel. Or, by redenoting φg as g, we shall prove that if (M, g) satisfies our hypotheses, and if, moreover, ω is harmonic, then ω has vanishing covariant derivative.

Using the notation of Section 1, the local Kähler metrics conformal to g are $\tilde{g}_{\alpha} = e^{-\sigma_{\alpha}}g$ ($\omega = d\sigma_{\alpha}$), whence we can easily compute the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}_{\alpha}$, and find [27]

$$(2.1)\qquad \tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(Y)X + \tfrac{1}{2}g(X, Y)B,$$

where again $B = \#\omega$. The connection $\tilde{\nabla}$ is global since the $c_{\alpha\beta}$ of (1.1) are locally constant. In [27] we called it the *Weyl connection* of (M, g), and proved that $\tilde{\nabla}J = 0$ characterizes the l.c.K. manifolds.

Furthermore, (2.1) and $\delta\omega = 0$ yield the following relation between the corresponding Ricci tensors $\tilde{R}$ and R

$$(2.2)\qquad \tilde{R} = R - \frac{n-1}{2}(|\omega|^2 g - \omega\otimes\omega) + (n-1)\nabla\omega,$$

whence

$$(2.3)\qquad \int_M R(B, B)*1 = \int_M \tilde{R}(B, B)*1 - (n-1)\int_M (\nabla_B\omega)(B)*1.$$

Now, let us note that

$$d(|\omega|^2)(B) = B(|\omega|^2) = B(\omega(B)) = (\nabla_B\omega)(B) + \omega(\nabla_B B),$$

and that the metric character of ∇ yields $Bg(B,B) = 2g(\nabla_B B, B)$; i.e.,

$$B(|\omega|^2) = 2\omega(\nabla_B B).$$

As a consequence, we obtain

$$d(|\omega|^2)(B) = 2(\nabla_B \omega)(B),$$

and, by integrating over M we get

$$\int_M (\nabla_B \omega)(B) * 1 = \tfrac{1}{2}\langle d(|\omega|^2), \omega\rangle = \tfrac{1}{2}\langle |\omega|^2, \delta\omega\rangle = 0.$$

This relation, together with (2.3) and with the hypothesis $\tilde{R}(B,B) \geqslant 0$, implies $\int_M R(B,B) * 1 \geqslant 0$. Since ω is harmonic, the latter inequality implies by a well-known theorem [9, p. 87] that $\nabla\omega = 0$. Q.E.D.

REMARKS. (1) It is worth emphasizing the fact stated at the beginning of the proof above; namely, on every compact l.c.K., and up to a global conformal change of the metric, one can assume without loss of generality that the Lee form ω is harmonic.

(2) If the Ricci curvatures of the metrics $\tilde{g}_\alpha$ would be positive, the given proof would imply $\omega = 0$, i.e., that the original metric g was globally conformal Kähler – a case which we do not want to consider.

Now, the class of manifolds in which we are interested in this section is that of the l.c.K. manifolds, whose l.c.K. structure consists of flat Kähler metrics. Such a manifold will be called an *l.c. Kähler-flat* manifold or, in short, an *l.c.*K_0*. manifold*. (Do not forget that this is not considered to include the globally c.K_0. case.) Let us also mention that one can prove by easy tensor computations that the l.c.K_0. manifolds are precisely the manifolds (M,g) such that g is simultaneously l.c.K. and l.c. Riemannian flat.

Moreover, we shall be interested only in compact l.c.K_0. manifolds, hence, by Theorem 2.1, we can assume, without loss of generality from the topological and analytical viewpoints, that our manifolds (M,g) are compact, l.c.K_0., and g.H.m. In [29], we also discussed l.c.K_0. manifolds which are simultaneously g.H.m., under the name of $\mathscr{P}_0.\mathscr{K}$.-manifolds. For our present discussion, we shall need the results of Theorem 3.8 and 3.9 of [29, p. 277], which we are reformulating here as

THEOREM 2.2. *Let M be a compact l.c.K_0. manifold. Then the universal covering space of M is $\mathbb{C}^n \backslash \{0\}$, and up to a global conformal change of the metric, M is a g.H.m. with the metric induced by* (1.6). *Every such manifold M has the same Betti numbers as the Hopf manifold H^n of the same complex dimension n.*

We shall continue to denote by g_0 the metric defined by Formula (1.6) on any M as in Theorem 2.2. Clearly, every such M is of the form

$$M = (\mathbb{C}^n \backslash \{0\})/G, \tag{2.4}$$

where G is the corresponding group of covering transformations, and the transformations of G leave the metric (1.6) invariant. This also implies that the transformations of G are conformal with respect to the flat Kähler metric $\Sigma_{k=1}^{n} \mathrm{d}z^k \,\mathrm{d}\bar{z}^k$ of $\mathbb{C}^n\backslash\{0\}$.

REMARK. Every l.c.K. manifold M can be written as

$$M = \tilde{M}/G, \tag{2.5}$$

where $\tilde{M}$ is a simply connected Kähler manifold, and G is a covering transformation group whose elements are conformal for the respective Kähler metric of $\tilde{M}$. Indeed, if M is like that, and $\tilde{M}$ has the Kähler metric k, then k and all its transformed by G project to an l.c.K. structure of M. Conversely, if M is l.c.K. with metric g, the lift $\tilde{g}$ of g to $\tilde{M}$ has an exact Lee form, and $\tilde{g} = \varphi k (\varphi > 0)$ for some Kähler metric k, which is transformed conformally by G.

Now, let us come back to the l.c.K$_0$. manifold (2.4), and go on with the analysis of the covering group G. Clearly, every $\gamma \in G$ preserves the lift of the Weyl connection (2.1) to $\mathbb{C}^n\backslash\{0\}$, i.e., it preserves the Levi–Civita connection of $\Sigma_{k=1}^{n} \mathrm{d}z^k \,\mathrm{d}\bar{z}^k$. Hence γ is an affine transformation of $\mathbb{C}^n$, which is also conformal, and preserves the origin. Consequently, G consists of transformations γ of the form

$$\tilde{z}^k = \rho \sum_{p=1}^{n} u_p^k z^p, \tag{2.6}$$

where $\rho > 0$, and (u_p^k) is a unitary matrix. If necessary, we shall denote $\rho = \rho(\gamma)$, and call it the *module* of γ.

Furthermore, following [15] we shall say that every compact complex manifold covered by $\mathbb{C}^n\backslash\{0\}$ is a *Hopf–Kodaira manifold*, and (2.4) proves that the compact l.c.K$_0$. manifolds are of this type. Accordingly, we shall use in a convenient manner the results of [15] and [12] in order to describe the structure of the covering group G.

An analytic mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ is called a *contraction* if $f^k(B) \to 0$ for the unit ball B, and $k \to \infty$. The mapping γ of (2.6) is a contraction iff $\rho(\gamma) < 1$, and one has [15, II, p. 695, 12].

LEMMA 2.3. *Any contraction $\gamma \in G$ generates an infinite cyclic group $\{\gamma\}$ of finite index in G. There is a contraction $\gamma_0 \in G$ such that $\rho(\gamma_0)$ is maximal < 1.*

Proof. If $\gamma \in G$ is a contraction $\rho(\gamma^k) = (\rho(\gamma))^k$ are distinct numbers, hence $\{\gamma\}$ is infinite. Then, it also holds that $(\mathbb{C}^n\backslash\{0\})/\{\gamma\}$ is compact. Since this is clearly a covering space of M, we get that $G/\{\gamma\}$ is a finite set. Moreover, the compactness of M ensures that a contraction γ exists. Since the modules of the elements of a class $[g]_{\{\gamma\}} (g \in G)$ are of the form $\rho(g)(\rho(\gamma))^k$, we can find in this class a contraction whose module is the closest possible to 1. Finally, since we have a finite number of such classes, we can chose the γ_0 needed. Q.E.D.

Now, again following [12], let us consider

(2.7) $$H = \{\gamma \in G/\rho(\gamma) = 1\}.$$

which is a normal subgroup of G, and prove

THEOREM 2.4. *H is a finite subgroup of G, which commutes with γ_0, and one has*

(2.8) $$G = \{h\gamma_0^k / h \in H, k \in \mathbb{Z}\}.$$

Proof. H commutes with γ_0 since it is a normal subgroup of G. If a class of $G/\{\gamma_0\}$ has an element $h \in H$, all the elements of the class have modules $\rho(h)\rho^k(\gamma_0) = \rho^k(\gamma_0) \neq 1$ for $k \neq 0$, and h is unique in this class. Hence H is finite. Moreover, let us take a class $\{g\gamma_0^k\}$, and let λ be an element of maximal module < 1 in this class. Then, since $\rho(\lambda\gamma_0^{-1}) > \rho(\lambda)$, we get $\rho(\lambda\gamma_0^{-1}) \geqslant 1$ and $\rho(\lambda) \geqslant \rho(\gamma_0)$. Because of the choice of γ_0 and λ this yields $\rho(\lambda) = \rho(\gamma_0)$, i.e., $\lambda\gamma_0^{-1} \in H$. In other words every class of $G/\{\gamma_0\}$ has a unique representative in H, which proves (2.8). Q.E.D.

Conversely, let us start with a finite unitary group H, and with a transformation γ_0 which commutes with H, and has the form

(2.9) $$\tilde{z}^k = \rho_0 \exp(2\pi i\lambda_k) z^k \qquad \text{(not summed)}$$

(with respect to a conveniently chosen coordinate system z^k of $\mathbb{C}^n$). Let us construct the group G of (2.8), and $M = (\mathbb{C}^n \backslash \{0\})/G$. Then, if M is a compact manifold, M is a compact l.c.K$_0$. manifold. Hence, we have proven

THEOREM 2.5. *The formulas* (2.4) *and* (2.8) *yield all the compact l.c.K$_0$. manifolds.*

REMARK. In [12], all the Kodaira–Hopf surfaces are determined, which includes, of course, all the compact l.c.K$_0$. manifolds of complex dimension $n = 2$.

Now, let us also generalize Kato's results about the differentiable fibre bundle structure of the Hopf–Kodaira surfaces [12] to compact l.c.K$_0$. manifolds.

THEOREM 2.6. *A compact l.c.K$_0$. manifold is a locally trivial differentiable fibre bundle with base space S^1, fibre S^{2n-1}/H, and structure group $\{h_0\}$, where H is a finite unitary group, and h_0 is a unitary transformation commuting with H.*

Proof. In this theorem, as usual, S^1 is a circle, and S^{2n-1} is the unit $(2n-1)$-dimensional sphere. We shall discuss the l.c.K$_0$. manifold given by (2.4), (2.8).

Let us consider the well-known diffeomorphism $\mathbb{C}^n\backslash\{0\} \approx S^{2n-1} \times \mathbb{R}$ given by

$$(2.10) \qquad (z^k) \mapsto (z^k/|z|, r = \ln|z|).$$

Then a transformation $h\gamma_0^k$ similar to that in (2.8) acts by hh_0^k on the component S^{2n-1}, where h_0 is the unitary component of γ_0, and by $\tilde{r} = r + k\ln\rho_0$ (ρ_0 = module of γ_0) on the component $\mathbb{R}$.

Now, by known results about covering spaces (e.g., [28, p. 115]) we obtain the diffeomorphism

$$(2.11) \qquad M = (\mathbb{C}^n\backslash\{0\})/G = [(\mathbb{C}^n\backslash\{0\})/H]/(G/H) \approx [(S^{2n-1}/H) \times \mathbb{R}]/\{\gamma_0\},$$

and this yields the commutative diagram

$$(2.12) \qquad \begin{array}{ccc} (S^{2n-1}/H) \times \mathbb{R} & \xrightarrow{pr} & \mathbb{R} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[q]{} & S^1 = \mathbb{R}/\Gamma \end{array}$$

where Γ is the translations group $\tilde{r} = r + k\ln\rho_0$, π' is the corresponding covering map $\mathbb{R} \to S^1$, and π is the covering map of (2.11).

From (2.12), it is easy to understand that q is precisely the fibring whose existence is stated by Theorem 2.6. Q.E.D.

REMARKS. (1) A fibre bundle as described in Theorem 2.6 has a compact l.c.K$_0$. manifold as its total space only if the transition functions of the given bundle are in accordance with the diagram (2.12).

(2) From (2.12), and from the analysis made in [29], it follows that the fibres of q are precisely the leaves of the *canonical foliation* $\mathscr{F}$ defined on M by the integral manifolds of the Lee form ω.

(3) Similarly, the geometric trajectories of the Lee vector field $B = \#\omega$ are the images by π of subsets of the form $\{x\} \times \mathbb{R}$, where $x \in S^{2n-1}/H$. Clearly, if we replace x by $h_0^k(x)$ we get the same trajectory. If $\{h_0\}$ is infinite, $\{h_0^k(x)\}$ can be an infinite set which will have an accumulation point x_0 in the compact manifold S^{2n-1}/H. In this case, the corresponding trajectory of B enters infinitely many times in any neighbourhood of $\pi(x_0)$, and the foliation defined by these trajectories is *non-regular* [20]. But, if the group $\{h_0\}$ is finite, the manifold M is *regular* is the sense of [29].

We shall end this section by two other remarks of a different nature.

The flat Kähler metric of $\mathbb{C}^n$ has coefficients defined by $\partial^2 F/\partial z^h\,\partial\bar{z}^k$, where $F = \Sigma_{h=1}^n z^h\bar{z}^h$, and it is naturally imbedded in the one-parameter family of Kähler metrics with coefficients $[1/(1+t)]\partial^2(F^{1+t})/\partial z^h\partial\bar{z}^k (t > -1)$. The transformations (2.6) are conformal with respect to all these Kähler metrics, and this leads easily to the following family of l.c.K. metrics existing on every compact l.c.K$_0$. manifold M

$$(2.13)\qquad g_t = 4\frac{\left(\sum_{k=1}^n z^k\bar{z}^k\right)^t\left(\sum_{j=1}^n dz^j\,d\bar{z}^j\right) + t\left(\sum_{k=1}^n z^k\bar{z}^k\right)^{t-1}\left(\sum_{h=1}^n \bar{z}^h\,dz^h\right)\left(\sum_{j=1}^n z^j\,d\bar{z}^j\right)}{\left(\sum_{k=1}^n z^k\bar{z}^k\right)^{1+t}}$$

$(t > -1)$. (This explains the notation g_0 of (1.6).)

The Lee form of g_t is

$$(2.14)\qquad \omega_t = -(1+t)\frac{\sum_{j=1}^n (z^j\,d\bar{z}^j + \bar{z}^j\,dz^j)}{\sum_{k=1}^n z^k\bar{z}^k},$$

and this form is not exact for $t \neq -1$ [29]. (But for $t = -1$, (2.13) yields only a non-negative Hermitian form.) Hence, we have an example where the Lee chomology class of an l.c.K. structure can be deformed continuously but not reaching zero.

However, let us note that, for $t \neq 0$, the Lee form of g_t is not parallel. Indeed, from (2.13), we get the local coefficients

$$(2.15)\qquad (g_t)_{i\bar{j}} = \frac{2}{\sum_{h=1}^n z^h\bar{z}^h}\left\{\delta_{ij} + t\frac{\bar{z}^i z^j}{\sum_{h=1}^n z^h\bar{z}^h}\right\},$$

whence

$$(2.16)\qquad (g_t)^{\bar{i}j} = \frac{\sum_{h=1}^n z^h\bar{z}^h}{2}\left\{\delta^{ij} - \frac{t}{1+t}\,\frac{\bar{z}^i z^j}{\sum_{h=1}^n z^h\bar{z}^h}\right\},$$

and, consequently, the Lee vector field B_t of g_t is

$$(2.17)\qquad B_t = -\frac{1}{2}\left(z^j\frac{\partial}{\partial z^j} + \bar{z}^j\frac{\partial}{\partial \bar{z}^j}\right),$$

and it is interesting that it does not depend on t. Moreover, up to corresponding factors, (2.15) and (2.16) also yield the local coefficients of the local Kähler metrics conformal to g_t, and using known formulas (e.g., [9]) we can compute the connection coefficients of the Weyl connection (2.1) of g_t. The essentials among them will be

$$(2.18)\qquad (\tilde{\Gamma}_t)^k_{ij} = \frac{t}{\sum_{h=1}^n z^h\bar{z}^h}\left\{\delta^k_i\bar{z}^j + \delta^k_j\bar{z}^i - \frac{z^k\bar{z}^i\bar{z}^j}{\sum_{h=1}^n z^h\bar{z}^h}\right\}.$$

Finally, by using (2.1), it is easy to see that B_t is parallel iff $(\tilde{\nabla}_t)_X B = -\frac{1}{2}X$, which reduces in our case to $(\tilde{\nabla}_t)_i\omega^j = -\frac{1}{2}\delta^j_i$. But (2.17) and (2.18) yield

$$(2.19)\qquad (\tilde{\nabla}_t)_i\omega^j = -\frac{1+t}{2}\delta^j_i,$$

and, hence, for $t \neq 0$, B_t is not g_t-parallel. Q.E.D.

Finally, our last remark consists of

PROPOSITION 2.7. *A compact (connected) complex surface S admits a quaternionic structure [13] iff it is an l.c.K$_0$. manifold with a trivial canonical line bundle.*

Proof. According to [13], a quaternionic structure on S is a complex atlas with the transition functions of the form

$$(2.20)\qquad \tilde{z}^1 = \alpha z^1 + \beta z^2 + \lambda,\ \tilde{z}^2 = -\bar{\beta}z^1 + \bar{\alpha}z^2 + \mu,$$

where $\alpha, \beta, \lambda, \mu \in \mathbb{C}$, and the bar denotes complex conjugation. Now, (2.20) implies

$$\mathrm{d}\tilde{z}^1\,\mathrm{d}\bar{\tilde{z}}^1 + \mathrm{d}\tilde{z}^2\,\mathrm{d}\bar{\tilde{z}}^2 = (\alpha\bar{\alpha} + \beta\bar{\beta})(\mathrm{d}z^1\,\mathrm{d}\bar{z}^1 + \mathrm{d}z^2\,\mathrm{d}\bar{z}^2),$$

and we see that the local metrics $\Sigma_{k=1}^2 \mathrm{d}z^k\,\mathrm{d}\bar{z}^k$ yield an l.c.K$_0$: structure on S. Moreover, the transition functions of the canonical line bundle $\kappa(S)$ which correspond to (2.20) are positive real constants, whence $\kappa(S)$ is trivial.

Conversely, if S has an l.c.K$_0$. structure, its local Kähler metrics can be considered as $\Sigma_{k=1}^2 \mathrm{d}z^k_\sigma\,\mathrm{d}\bar{z}^k_\sigma$ with respect to some atlas $\{U_\sigma\,;z^k_\sigma\}$ of S, and the transition functions of this atlas on $U_\sigma \cap U_\tau$ can be put under the form

$$(2.21)\qquad \begin{aligned} z^1_\tau &= c_{\sigma\tau}\,\mathrm{e}^{i\theta_{\sigma\tau}}(a_{\sigma\tau}z^1_\sigma + b_{\sigma\tau}z^2_\sigma) + p_{\sigma\tau},\\ z^2_\tau &= c_{\sigma\tau}\,\mathrm{e}^{i\theta_{\sigma\tau}}(-\bar{b}_{\sigma\tau}z^1_\sigma + \bar{a}_{\sigma\tau}z^2_\sigma) + q_{\sigma\tau},\end{aligned}$$

where $c_{\sigma\tau} > 0$, $\theta_{\sigma\tau} \in \mathbb{R}$, and $a_{\sigma\tau}\bar{a}_{\sigma\tau} + b_{\sigma\tau}\bar{b}_{\sigma\tau} = 1$.

The corresponding transition functions of the canonical bundle $\kappa(S)$ are $c^2_{\sigma\tau}\,\mathrm{e}^{2i\theta_{\sigma\tau}}$, and $\kappa(S)$ is trivial iff $\theta_{\sigma\tau} = \varphi_\tau - \varphi_\sigma$ for some numbers $\varphi_\sigma \in \mathbb{R}$. Then, by defining on each U_σ new coordinates $\zeta^k_\sigma = \mathrm{e}^{i\varphi_\sigma}z^k_\sigma$, we get a new atlas for which the relations (2.21) are replaced by relations of the type (2.20). Therefore S has a quaternionic structure. Q.E.D.

Moreover, one can see in a similar manner that if the surface S above is represented by (2.4), (2.8) then the convering group G of (2.8) must consist of transformations whose unitary component belongs to the special unitary group, and conversely.

Proposition 2.7 was introduced here since, in connection with the previous results of this Section, it could provide a shorter solution of the problem of the determination of all the compact quaternionic 4-manifolds treated in [13].

3. The vertical foliation, strong regularity

In this section we shall discuss an important foliation which exists on a general g.H.m. M^n, and describe the geometric structure of M in the case where this foliation is regular.

Let us recall that we had the Lee form ω and the Lee vector field $B = \# \omega$ on every g.H.m., and $|\omega| = |B| = 1$ by the normation condition (1.7). Let us also introduce

$$\theta = -\omega \circ J = [1/(n-1)]\, \delta\Omega, \qquad A = JB = \# \theta, \tag{3.1}$$

whence again $|\theta| = |A| = 1$. In [29], ω, θ, B, A were respectively denoted as u, v, U, V (here we use the notation of [32] but with the sign convention of [29]), and we proved that $[A, B] = 0$. Accordingly, there is an action of the additive group $\mathbb{R}^2$ on M, whose orbits are tangent to A, B, and define a foliation $\mathscr{E}$ on M. This will be called the *vertical foliation* of M.

THEOREM 3.1. *The vertical foliation $\mathscr{E}$ is a complex analytic* 1*-dimensional foliation of M with complex parallelizable leaves which are totally geodesic and locally flat submanifolds of M. The metric of M is a bundle-like transversally Kählerian metric with respect to $\mathscr{E}$.*

Proof. Since by [29] B is an analytic vector field it follows easily that $\mathscr{E}$ is a complex analytic foliation with complex 1-dimensional leaves whose holomorphic tangent spaces are generated by $B - iA (i = \sqrt{-1})$ and, therefore, these leaves are complex parallelizable. Accordingly, M has local complex coordinates $\{z^\alpha\} (\alpha = 1, \dots, n)$ such that $\mathscr{E}$ is given by

$$\mathrm{d}z^a = 0 \quad (a = 1, \dots, n-1), \tag{3.2}$$

and

$$B - iA = \lambda(z^\alpha) \frac{\partial}{\partial z^n}, \tag{3.3}$$

where $\lambda(z^\alpha)$ is a local and nowhere zero analytic function.

Furthermore, using the complex connection $\tilde{\nabla}$ of (2.1) as an intermediary where necessary, we can derive easily

$$\nabla_A A = \nabla_A B = \nabla_B A = \nabla_B B = 0, \tag{3.4}$$

whence it follows that the leaves of $\mathscr{E}$ are totally geodesic and locally flat submanifolds of M.

Now let us note that $\omega + i\theta$ is a form of the complex type (1, 0), and, since $(\omega + i\theta)(B - iA) = 2$, (3.3) yields

$$\omega + i\theta = \mu_a(z^\alpha, \bar{z}^\alpha)\mathrm{d}z^a + \frac{2}{\lambda}\mathrm{d}z^n, \tag{3.5}$$

where the Latin and Greek indices will always be considered with the range mentioned when we first met them. Hence $\{dz^a, \omega + i\theta\}$ are $(1, 0)$ cobases on M, and, since $\omega = 0, \theta = 0$ obviously define the orthogonal distribution $\mathscr{E}^\perp$, and $|B| = 1$, we see that the metric of M can be expressed as

$$(3.6) \qquad ds^2 = 2g_{a\bar{b}}\, dz^a \otimes d\bar{z}^b + (\omega + i\theta) \otimes (\omega - i\theta).$$

The bundle-like character of the metric (3.6) means that the coefficients $g_{a\bar{b}}$ depend only on the z^a, $\bar{z}^a$ [21], which is equivalent to

$$(3.7) \qquad (L_X g)(Y, Z) = 0$$

for every $X \in \mathscr{E}$, $Y, Z \in \mathscr{E}^\perp$, where L_X denotes the Lie derivative. But (3.7) is an easy consequence of the fact that A, B are Killing vector fields [29], and this proves the corresponding assertion of Theorem 3.1.

Finally, the Kähler form of the metric (3.6) is

$$(3.8) \qquad \Omega = -ig_{a\bar{b}}\, dz^a \wedge d\bar{z}^b - \omega \wedge \theta.$$

But, by Proposition 2.4 and Formula (2.6) of [29], one also has $\Omega = d\theta - \omega \wedge \theta$, and hence

$$(3.9) \qquad d\theta = -ig_{a\bar{b}}\, dz^a \wedge d\bar{z}^b.$$

This implies that the transversal part of the Kähler form Ω is closed, which is precisely what we meant in the last conclusion of Theorem 3.1. Q.E.D.

REMARK. The description of the structure of a g.H.m. given in Theorem 3.1 leads to a nice reformulation of the definition of this class of manifolds, in the spirit of the theory of the contact manifolds. Namely, let us define a $(1, 0)$-*contact structure* on a complex manifold M^n as a 1-form ζ_0 of the complex type $(1, 0)$ such that

$$(*) \qquad \zeta_0 \wedge \bar{\zeta}_0 \wedge (d_{\bar{z}}\zeta_0)^{n-1} \neq 0$$

at every point of M. Notice that this structure defines a *fundamental complex vector field* Z_0 of type $(1, 0)$ by

$$\zeta_0(Z_0) = 1, \; i(Z_0)\, d_{\bar{z}}\zeta_0 = 0.$$

Now, the $(1, 0)$-contact structure ζ_0 will be called *Hermitian* if

$$(**) \qquad \Omega_0 = -\sqrt{-1}\,(d\zeta_0 + \tfrac{1}{2}\zeta_0 \wedge \bar{\zeta}_0)$$

is a real positive $(1, 1)$-form. As a matter of fact, because of the type, (**) yields $d_z\zeta_0 = 0$, and implies (*). Therefore (**) is the only condition to be stated in the definition of a Hermitian $(1, 0)$-contact structure.

It is obvious that a Hermitian $(1, 0)$-contact manifold has a well-defined Hermitian metric g_0 whose Kähler form is Ω_0, and it is easy to show that g_0 is an l.c.K. metric with the Lee form Re ζ_0, and the Lee vector field $B = Z_0 + \bar{Z}_0$. Moreover, if B is a Killing field for g_0 or, equivalently, if B is an analytic field which preserves the (1, 0)-contact form ζ_0, M will be a g.H.m. Conversely, if M is a g.H.m., and we take $\zeta_0 = \omega + i\theta$, Ω_0 of (**) is the Kähler form of M, as proven by Formulas (3.8), (3.9).

Let us also note some interesting corollaries of Theorem 3.1.

COROLLARY 3.2. *All the Chern numbers of a compact g.H.m. vanish.*

Proof. This follows from the existence of the non-singular complex analytic vector field (3.3) on M, and from a known theorem of Bott [2].

COROLLARY 3.3. *Let M^2 be a compact complex g.H. surface. Then, with the usual notation for the invariants of a complex surface* [15] *one has*

$$\begin{aligned}(3.10)\qquad & c_1^2[M] = c_2[M] = 0, p_g = \tfrac{1}{2}(b_1 - 1), q = \tfrac{1}{2}(b_1 + 1),\\ & b_2 = 2(b_1 - 1) = 4p_g, b^+ = b^- = b_1 - 1.\end{aligned}$$

Proof. These relations are simple consequences of: (a) Corollary 3.2 above, (b) Proposition 2.3 of [32] stating that the first Betti number of our M^2 is odd, (c) the Noether Formula and other relations exhibited in [15]. (Note that $c_2[M] = 0$ is the vanishing of the Euler–Poincaré characteristic of M.)

The vertical foliation $\mathscr{E}$ plays an essential role for a g.H.m., and we shall show this by the results of this and of the next section.

THEOREM 3.4. *Let M^n be a compact g.H.m. If all the leaves of $\mathscr{E}$ are proper, M^n is the total space of an analytic V-submersion onto a Kählerian Satake V-manifold* [22] *of complex dimension $n - 1$, and the fibres of this submersion are complex 1-dimensional toruses.*

Proof. The proper leaves hypothesis means that the leaves of $\mathscr{E}$ are embedded in M, and since they are totally geodesic submanifolds (and M is compact), these leaves are also complete, whence closed and compact. Moreover, since the leaves of $\mathscr{E}$ are complex parallelizable, they will, in our case, be complex 1-toruses. Then, the statements of Theorem 3.4 follow from known results of foliation theory [8], [21].

For instance, if the vertical foliation $\mathscr{E}$ is hyperbolic [6], Theorem 3.4 applies because of the results of [6].

An even stronger property of $\mathscr{E}$ is *regularity* [20], and if it holds, the g.H.m. M will be called *strongly regular.* In the sequel, we shall see that there are enough known results in the literature to allow us a complete description of the compact strongly regular g.H.m.

REMARK. It is obvious that a strongly regular g.H.m. is regular in the sense of Definition 4.1 of [29] but the converse may not be true. For instance, the product $\Sigma \times S^1$, where Σ is a generalized Brieskorn manifold with a non-regular Sasakian structure as constructed by Abe [1], carries a regular but not strongly regular g.H.m. structure which is given by Theorem 4.1 of [29].

Now, let M^n be a compact (connected by our general convention of Section 1) strongly regular g.H.m., for which we shall continue to use the notation from the beginning of this section. Then, the space $N = M/\mathscr{E}$ of the leaves of $\mathscr{E}$ is a compact Kähler manifold of complex dimension $n-1$, with the metric induced by the first term of (3.6). On the other hand, as shown by Proposition 2.7 of [29], M is a regular f-manifold in the sense of [4] (in fact, a g.H.m. is a good example of an S-structure of Blair [3], [4], and, therefore, by Theorem 1 of [4], M is a differentiable principal T^1_c-bundle over N. (T^1_c denotes the complex 1-dimensional torus.) It is also simple to see that $\omega + i\theta$ is a connection form on this principal bundle, and up to a factor $i = \sqrt{-1}$, its curvature projects to the Kähler form of N.

Moreover, if M is strongly regular both the trajectories of B and A yield regular foliations. By Theorem 3 of [4], and Theorem 4.1 of [29], M fibres over the manifold P of the trajectories of B, and P is a regular Sasakian manifold with the structure vector field induced by A. The Lee form ω defines a flat connection of this bundle. Then, P fibres as a Boothby–Wang fibration [5] over N viewed as the manifold of the trajectories of A in P, while θ defines on P a connection with curvature $d\theta$. Thereby, we get a commutative diagram

(3.11)
$$\begin{tikzcd} M \arrow[rr, "q"] \arrow[dr, "p"'] & & P \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

where π and q are principal circle bundles, and p is a principal T^1_c-bundle.

Furthermore, since $d\theta$ is the curvature of a principal circle bundle, it represents an integral cohomology class of N, whence by (3.9) N is a compact Hodge manifold, and there is an embedding

(3.12 $\quad \imath : N \to \mathbb{C}P^k$

for a convenient dimension k. Now, let us consider the classical Hopf fibration

(3.13) $\quad \alpha : S^{2k+1} \to \mathbb{C}P^k,$

which is an S^1-principal bundle endowed with a connection λ whose curvature projects to the Kähler form Λ of the Fubini metric of $\mathbb{C}P^k$. Since the Kähler form of N is also given by $\imath^*\Lambda$, we see that $\pi : P \to N$ has a connection with the same curvature form on N as the induced bundle $\imath^*(\alpha)$. In other words, the Chern classes of π and $\imath^*(\alpha)$ differ by a torsion element of $H^2(N, \mathbb{Z})$ only.

Conversely, if we start with a compact Hodge manifold N, and a Boothby–

Wang fibration $\pi : P \to N$ with the same real Chern class as $\iota^*(\alpha)$, we have the Sasakian manifold P. Then, we can use the construction of Theorem 4.1 of [29], and get a corresponding compact g.H.m. M with the two vector fields B, A. This manifold M is locally trivial over N. Indeed, let $U \subset N$ be a convex open neighbourhood with $P/U \approx U \times S^1$. Then M/U is a flat S^1-principal bundle over P/U. But

$$H^2(P/U, \mathbb{Z}) = H^2(U \times S^1, \mathbb{Z}) = H^2(S^1, \mathbb{Z}) = 0,$$

hence P/U is trivial over M/U, and $M/U \approx U \times S^1 \times S^1$. Finally, we see that $T_c^1 \approx S^1 \times S^1$ acts freely on M by $\exp(\tau A) \times \exp(tB)$ such as to produce the fibration $p = \pi \circ g$ of (3.11), which proves that the obtained g.H.m. M is strongly regular.

By this analysis, we have proven

THEOREM 3.5. *The class of the compact strongly regular g.H.m. M is equal to the class of the differentiable T_c^1-principal fibre bundles over the compact Hodge manifolds N, where M can be obtained as a flat principal circle bundle over a principal circle bundle P over N whose Chern class differs only by a torsion element from the Chern class of the induced Hopf fibration* $\iota^*(\alpha)$.

COROLLARY 3.6. *The Betti numbers of a compact connected g.H.m. M and of its Kählerian basis N are related by*

$$\begin{aligned} & b_h(M) = b_h(N) + b_{h-1}(N) - b_{h-2}(N) - b_{h-3}(N) \quad (0 \leqslant h \leqslant n-1), \\ (3.14) \qquad & b_h(M) = b_{h-2}(N) + b_{h-1}(N) - b_h(N) - b_{h+1}(N) \quad (n+1 \leqslant h \leqslant 2n), \\ & b_n(M) = 2(b_{n-1}(N) - b_{n-3}(N)). \end{aligned}$$

Proof. The relations follow from the structure of M given in Theorem 3.5 by using twice the Gysin exact sequence, and then using Poincaré duality and the vanishing of the Euler–Poincaré characteristic $\chi(M)$.

This corollary will be given an interesting generalization in the next section.

REMARK. Theorem 3.5 determines all the compact strongly regular g.H.m., and the *induced Hopf fibrations* [32] form an interesting subclass. It is worthwhile noticing that induced Hopf fibrations can provide examples of compact g.H.m. which do not admit any l.c.K_0 structure. Indeed, let T be the induced Hopf fibration over an irreducible algebraic curve of genus g. Then, Corollary 3.6 yields

$$b_0(T) = 1, \quad b_1(T) = 2g + 1, \quad b_2(T) = 4g,$$

and using (3.10) we get $p_g(T) = g$. Therefore, if $g \geqslant 1$, T belongs to the Kodaira class VI [15], while, by Theorem 2.2, a compact l.c.K_0 complex surface must be in the class VII of [15]. Q.E.D.

4. Harmonic forms and holomorphic vector fields

This section is dealing with general compact g.H.m., and we shall use the notation of Section 3 and particularly the local coordinates z^α of (3.2), (3.6). The elements (functions, forms, etc.) which depend only on the leaves of $\mathscr{E}$, i.e., depend locally on $z^a, \bar{z}^a$ $(a = 1, \ldots, n-1)$ only, will be called *$\mathscr{E}$-foliate.*

It is well known in foliation theory [22], [26] that the decomposition $TM = \mathscr{E} \oplus \mathscr{E}^\perp$ yields a decomposition of the forms of M into sums of bihomogeneous forms of *type* (p, q), where p denotes the *transversal degree* and q the *leaf degree.* Accordingly, there is a decomposition

$$\text{(4.1)} \qquad \mathrm{d} = \mathrm{d}' + \mathrm{d}'' + \partial,$$

where d' is of type $(1, 0)$, d'' is of type $(0, 1)$, and ∂ is of type $(2, -1)$, and a differential form φ on M is $\mathscr{E}$-foliate iff it is of type $(p, 0)$ $(0 \leqslant p \leqslant 2n-2)$ and $\mathrm{d}''\varphi = 0$. (Then, locally, φ contains only $z^a, \bar{z}^a, \mathrm{d}z^a, \mathrm{d}\bar{z}^a$.)

The Hodge $*$-operator of (M, g) acts homogeneously and (4.1) implies a decomposition of the corresponding adjoint operators [26]

$$\text{(4.2)} \qquad \delta = (-1)^{p+q} *^{-1} \mathrm{d}* = \delta' + \delta'' + \tilde{\partial},$$

where (p, q) is the type of the form acted on, and the type of the terms are, respectively, $(-1, 0)$, $(0, -1)$, $(-2, 1)$.

Now, any r-form λ of M has a unique decomposition

$$\text{(4.3)} \qquad \lambda = \alpha + \omega \wedge \beta,$$

where with respect to the cobases $(\mathrm{d}z^a, \mathrm{d}\bar{z}^a, \omega, \theta)$, α, β do not contain ω any more. Since $\nabla\omega = 0$, (4.3) implies [17, p. 159]

$$\text{(4.4)} \qquad \Delta\lambda = \Delta\alpha + \omega \wedge \Delta\beta,$$

and we see that λ is harmonic if α, β are such.

Conversely, $\Delta\lambda = 0$ and (4.4) yield $\omega \wedge \Delta\alpha = 0$, and, by applying $i(B)$ which commutes with Δ [17, p. 159], we get $\Delta\alpha = 0$, then similarly $\Delta\beta = 0$. That is, λ is harmonic iff α and β are harmonic forms.

Now if λ is harmonic, and $0 \leqslant \deg\lambda = r \leqslant n-1$, one has by a result of [11] (similar to a known result for Sasakian manifolds) $i(A)\lambda = 0$, whence $i(A)\alpha = i(A)\beta = 0$, and, since $\mathrm{d}\alpha = \mathrm{d}\beta = 0$, it also follows $L_A\alpha = L_A\beta = 0$. By (4.3) we also have $i(B)\alpha = i(B)\beta = 0$, which implies similarly $L_B\alpha = L_B\beta = 0$. All these facts together mean that α, β are $\mathscr{E}$-foliate forms.

Furthermore, for an $\mathscr{E}$-foliate r-form α we get by type comparison that $\mathrm{d}\alpha = 0$ means $\mathrm{d}'\alpha = 0$, and $\delta\alpha = 0$ means

$$\text{(4.5)} \qquad \delta'\alpha = 0, \quad \tilde{\partial}\alpha = 0.$$

If $*'$ denotes the Hodge $*$ of the transversal part of the metric g of M given by (3.6), we have easily

$$(4.6)\qquad *\alpha = \theta \wedge \omega \wedge *'\alpha.$$

We have also $d'\omega = 0$ and $d'\theta = 0$, which follows from (3.9). Thereby we get

$$(4.7)\qquad \delta'\alpha = (-1)^r *^{-1} d' * \alpha = (-1)^r *'^{-1} d' *' \alpha,$$

and we see that δ' is the codifferential with respect to the transversal part of the metric g. Hence, if α is harmonic it satisfies $d'\alpha = \delta'\alpha = 0$, i.e., α is *transversally harmonic*.

Finally, concerning the condition $\tilde{\partial}\alpha = 0$, it is clearly equivalent to

$$(4.8)\qquad \langle \tilde{\partial}\alpha, \beta \rangle = 0$$

for every form β of type $(r-2, 1)$, and where $\langle\ ,\ \rangle$ denotes the global scalar product on M. But, because of the type we have

$$\beta = \theta \wedge \beta_1 + \omega \wedge \beta_2,$$

where $i(A)\beta_h = i(B)\beta_h = 0$ $(h = 1, 2)$, whence

$$\beta_1 = i(A)\beta, \quad \partial\beta = d\theta \wedge \beta_1 = d\theta \wedge i(A)\beta,$$

and therefore

$$(4.9)\qquad \langle \tilde{\partial}\alpha, \beta \rangle = \langle \alpha, \partial\beta \rangle = \langle \alpha, e(d\theta)i(A)\beta \rangle = \langle e(\theta)i(d\theta)\alpha, \beta \rangle,$$

where the operators e, i denote exterior and interior products respectively.

It follows that $\tilde{\partial}\alpha = 0$ is equivalent to $e(\theta)i(d\theta)\alpha = 0$, or, in view of the expression (3.9) of $d\theta$, to

$$(4.10)\qquad i(d\theta)\alpha = 0,$$

and, because of (3.8), (4.10) means that α is *transversally effective*, i.e., effective [9], [17] with respect to the first term of the metric (3.6).

Hence, we have proven

THEOREM 4.1. *Let M^n be a compact g.H.m. Then, an r-form λ of M with $0 \leqslant r \leqslant n-1$ is harmonic iff $\lambda = \alpha + \omega \wedge \beta$, where α, β are transversally harmonic and transversally effective foliate forms.*

Now, let us consider the linear spaces $\mathcal{H}^h(M, \mathcal{E})$ of the $\mathcal{E}$-foliate transversally harmonic h-forms. Clearly, the operator $*'$ considered previously sends isomorphically $\mathcal{H}^h$ onto $\mathcal{H}^{2n-2-h}$, which means that these spaces satisfy the Poincaré duality. Let us also denote

$$(4.11)\qquad e_h(M, \mathcal{E}) = \dim \mathcal{H}^h(M, \mathcal{E}).$$

Then we can prove

THEOREM 4.2. *Let M^n be a compact g.H.m. Then, the numbers e_h of*

(4.11) *are finite, and the Betti numbers* $b_h(M)$ *are given by*

$$
\begin{aligned}
&b_h = e_h + e_{h-1} - e_{h-2} - e_{h-3} \quad (0 \leqslant h \leqslant n-1),\\
(4.12)\qquad &b_h = e_{h-2} + e_{h-1} - e_h - e_{h+1} \quad (n+1 \leqslant h \leqslant 2n),\\
&b_n = 2(e_{n-1} - e_{n-3}).
\end{aligned}
$$

Proof. Denoting by S^h the space of the $\mathscr{E}$-foliate transversally harmonic and effective h-forms with $0 \leqslant h \leqslant n-1$, it is clear from Theorem 4.1 that $s_h = \dim S^h < +\infty$. Now, for $\eta \in \mathscr{H}^h(M, \mathscr{E})$ $(0 \leqslant h \leqslant n-1)$, one has the following decomposition well known in Kähler geometry [9, p. 180]

$$(4.13)\qquad \eta = \sum_{k=0}^{[h/2]} (\mathrm{d}\theta)^k \wedge \xi_{h-2k}$$

(see (3.8) and (3.9)), where $\xi_{h-2k} \in S^{h-2k}$. Formula (4.13) implies

$$(4.14)\qquad e_h = \sum_{k=0}^{[h/2]} s_{h-2k} < +\infty \quad (0 \leqslant h \leqslant n-1).$$

Then $e_h = e_{2n-2-h}$ (Poincaré duality) implies $e_h < +\infty$ for $n \leqslant h \leqslant 2n-2$, and, finally, we have $e_h = 0$ for $h < 0$ or $h > 2n-2$.

Furthermore, (4.14) gives

$$(4.15)\qquad s_h = e_h - e_{h-2} \quad (0 \leqslant h \leqslant n-1),$$

and the decomposition of Theorem 4.1 yields

$$(4.16)\qquad b_h = s_h + s_{h-1} \quad (0 \leqslant h \leqslant n-1),$$

whence we deduce the first relation (4.12). The second relation follows from the first by Poincaré duality, and the third follows by using $\chi(M) = 0$. Q.E.D.

REMARK. In the case of a strongly regular g.H.m., the relations (4.12) are the same as (3.14).

It is also interesting to solve (4.12) with respect to e_h, and it suffices to do this for $0 \leqslant h \leqslant n-1$. From (4.12), it follows

$$(4.17)\qquad b_h - b_{h-1} = e_h - 2e_{h-2} + e_{h-4} \quad (0 \leqslant h \leqslant n-1),$$

whence one can prove by induction

$$(4.18)\qquad e_h = (-1)^h \sum_{i=0}^{[h/2]} ([h/2] - i + 1)(b_{2i} - b_{2i-(-1)^h}),$$

for $0 \leqslant h \leqslant n-1$.

It follows that e_h are topological invariants, and do not depend on the g.H.m. structure provided that one such structure exists on M.

Now, it is clear that the numbers e_h behave like the Betti numbers of an $(n-1)$-dimensional Kähler manifold, and therefore, as in [9], [17], we get

THEOREM 4.3. *If M^n is a compact g.H.m., and if $e_h (0 \leqslant h \leqslant n-1)$ are the linear combinations of the Betti numbers of M defined by* (4.18), *one has:*
(i) $e_h \neq 0$ *for h even,* (ii) e_h *is even for h odd,* (iii) $e_{h-2} \leqslant e_h$

For instance, from (4.18), and $b_0 = 1$, we get

$$(4.19) \qquad e_1 = b_1 - 1 = \text{even}, e_2 = b_2 - b_1 + 2 \geqslant 1 (n \geqslant 3),$$
$$e_3 = b_3 - b_2 + 2b_1 - 2 = \text{even} (n \geqslant 4), \text{etc.},$$

whence respectively: b_1 is odd (which is another proof of a theorem of Kashiwada and Sato [11], $b_2 = 0$ implies $b_1 = 1$ for $n \geqslant 3$; $b_3 - b_2$ is even and $b_3 \geqslant b_2 - b_1 + 1$ (since $e_1 \leqslant e_3$) for $n \geqslant 4$, etc.

REMARK. (1) The fact that the operators d′, δ' behave on foliate forms like in the case of a Kähler manifold can be used to transpose other Kählerian results also. For instance the reader can easily establish:

PROPOSITION 4.4. *Let M^n be a compact g.H.m., and α an $\mathscr{E}$-foliate transversally effective form of the complex type $(p, 0)$. Then the following three properties are equivalent:* (i) *α is closed;* (ii) *α is holomorphic;* (iii) *α is harmonic.*

REMARK. (2) The theory developed until now in this section is analogous to the theory of the C-harmonic forms on a contact manifold [19], [24], but with a more natural approach.

Now, let us go over to another important subject, *holomorphic* (*analytic*) *vector fields* on a compact g.H.m. We shall see that the vertical foliation $\mathscr{E}$ plays again a basic role.

With respect to the coordinates z^α of (3.2), (3.6), $\{\partial/\partial z^a, B - iA\}$ is a local basis for the holomorphic tangent bundle of M, and

$$(4.20) \qquad Z = \zeta^a \frac{\partial}{\partial z^a} + f(B - iA)$$

is an analytic vector field iff ζ^a, f are analytic functions which, whenever a coordinate change

$$(4.21) \qquad \tilde{z}^a = \tilde{z}^a(z^b), \quad \tilde{z}^n = \tilde{z}^n(z^a, z^n)$$

occurs, satisfy the transition relations

$$(4.22) \qquad \tilde{\zeta}^a = \frac{\partial \tilde{z}^a}{\partial z^b}\zeta^b, \quad \tilde{f} = f + \frac{1}{\lambda}\left(\frac{\partial \tilde{z}^n}{\partial z^a}\middle/\frac{\partial \tilde{z}^n}{\partial z^n}\right)\zeta^a$$

where λ is the analytic local function defined by Formula (3.3).

Now, let us regard $\mathscr{E}^\perp$ as the quotient bundle $TM/\mathscr{E}$, and denote by

$\pi : TM \to \mathscr{E}^\perp$ the natural projection, as well as the projection between the corresponding (1, 0)-bundles. The images by π will sometimes also be denoted by brackets. Then, if Z is the holomorphic vector field (4.20), $\pi(Z) = [Z]$ is a holomorphic section of $\mathscr{E}^\perp$.

Conversely, let us start with a holomorphic section of $\mathscr{E}^\perp$

$$\tilde{Z} = \zeta^a \left[\frac{\partial}{\partial z^a} \right], \tag{4.23}$$

and ask whether $\tilde{Z} = \pi(Z)$ for some holomorphic field (4.20)? Clearly, there are always differentiable fields (4.20) satisfying $\tilde{Z} = \pi(Z)$, and two such fields z_h $(h = 1, 2)$ must have the same components ζ^a while $f_2 - f_1 = \varphi$, where, because of (4.22), φ is a global function on M. Hence, a holomorphic Z with $\pi(Z) = \tilde{Z}$ is obtainable iff this relation holds for (4.20), and there is a global function φ on M such that $f + \varphi$ are analytic local functions, i.e., $d_{\bar{z}} \varphi = - d_{\bar{z}} f$.

Furthermore since by (4.22) $d_{\bar{z}} \tilde{f} = d_{\bar{z}} f$, the local forms $\{ - d_{\bar{z}} f \}$ define a global (0, 1)-form $\kappa(\tilde{Z})$. If Z is changed but $\pi(Z)$ remains the same, $\kappa(\tilde{Z})$ will change to $\kappa(\tilde{Z}) + d_{\bar{z}} \psi$ for some function ψ. Therefore, the $d_{\bar{z}}$-cohomology class $[\kappa(\tilde{Z})]$, which, by the Dolbeault Theorem, can be seen as a class in $H^1(M, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions of M, is well defined. It is obvious that a function φ satisfying $d_{\bar{z}} \varphi = - d_{\bar{z}} f$ exists iff $[\kappa(\tilde{Z})] = 0$. This proves

THEOREM 4.5. *Let M be a (not necessarily compact) g.H.m., and $\tilde{Z}$ be a holomorphic section of the transverse bundle of the vertical foliation $\mathscr{E}$. Then there is a well-defined cohomology class $[\kappa(\tilde{Z})] \in H^1(M, \mathcal{O})$, and $\tilde{Z} = \pi(Z)$ for some analytic vector field Z of M iff $[\kappa(\tilde{Z})] = 0$.*

REMARK. If Z_h $(h = 1, 2)$ are two holomorphic solutions of $\pi(Z) = \tilde{Z}$, the function $\varphi = f_2 - f_1$ is a global holomorphic function on M, hence it is a constant if M is compact.

Theorem 4.5 is important because it reduces the study of analytic vector fields to the study of analytic sections of $\mathscr{E}^\perp$. For instance, by applying it to $\tilde{Z} = 0$, we get

COROLLARY 4.6. *If M is a compact g.H.m., then $c(B - iA)$ $(c \in \mathbb{C})$ are the only analytic vector fields of M which belong to $\mathscr{E}$, and, if $\mathscr{E}^\perp$ has no nonvanishing holomorphic sections, these are the only analytic vector fields of M at all.*

Clearly, if we want to refer to real vectors, we should replace in Corollary 4.6 $c(B - iA)$ by $kA, k'B, k, k' \in \mathbb{R}$.

Finally, let us indicate a possibility for studying analytic sections of $\mathscr{E}^\perp$. Let (4.23) be a C^∞-section of $\mathscr{E}^\perp$, and $g_{a\bar{b}}$ be the coefficients of the metric (3.6). Then, we get an associated (0, 1)-form by

$$\zeta(\tilde{Z}) = \zeta = \zeta_{\bar{b}} \, d\bar{z}^b = (g_{a\bar{b}} \zeta^a) \, d\bar{z}^b \tag{4.24}$$

Now, by [26], the metric (3.6) has a *second connection* with respect to the foliation $\mathscr{E}$, and we shall denote by $\nabla^{\perp}$ the connection which this second connection induces in $\mathscr{E}^{\perp}$. Using the formulas given in [26] for the connection coefficients of $\nabla^{\perp}$, one can see that (4.23) is analytic iff

$$(4.25) \qquad \nabla^{\perp}_{\bar{c}}\zeta^{a} = 0, \quad \nabla^{\perp}_{n}\zeta^{a} = 0,$$

and because of the bundle-like character of the metric, this is equivalent to

$$(4.26) \qquad \nabla^{\perp}_{\bar{c}}\zeta_{\bar{b}} = 0, \quad \nabla^{\perp}_{\bar{n}}\zeta_{\bar{b}} = 0.$$

Let us restrict ourselves to the particular but important case of the analytic $\mathscr{E}$-foliate vector fields defined by the condition that their $\mathscr{E}^{\perp}$-projection is analytic and locally constant along the leaves of $\mathscr{E}$. For instance, it is easy to see that if X is a real vector field which is analytic ($L_X J = 0$), and satisfies $L_X B = 0$, then X is analytic $\mathscr{E}$-foliate. In particular, if X is analytic and Killing it is analytic $\mathscr{E}$-foliate. The necessary and sufficient condition for the analytic vector field X to be foliate is $L_X B = \alpha A + \beta B$ for some coefficients α, β.

In this case, the second condition (4.26) is automatically satisfied, and the first is the Kählerian condition of analyticity with respect to the Kähler metric given by the first term of (3.6) [18]. We can therefore do the same computations as for Kähler manifolds [18, p. 17–19], [9, p. 250–251] (while making an essential use of the relations (4.7)), and get

THEOREM 4.7. *If M is a compact g.H.m., a foliate section $\tilde{Z}$ of $\mathscr{E}^{\perp}$ is analytic iff*

$$(4.27) \qquad (\Delta'\zeta)_{\bar{b}} = 2R_{a\bar{b}}\zeta^{a},$$

where Δ' is the Laplacian, and R is the Ricci curvature of the $\mathscr{E}$-transversal part of the metric g of M.

Formula (4.27) is a second-order condition, and one could try to exploit it, as in the Kählerian case. For instance, as in [9], [18], one gets.

COROLLARY 4.8. *If M is a compact g.H.m. with a negatively defined transversal Ricci tensor, then the only analytic Killing vector fields of M are kA, $k'B$, where $k, k' \in \mathbb{R}$.*

5. Analytic submanifolds of g.H.m.

It is clear from Formula (1.3) that an analytic submanifold of a general l.c.K. manifold inherits either a locally or a globally conformal Kähler structure. Therefore it is natural to inquire what happens in the case where the ambient manifold is a g.H.m. The aim of this section is to answer this question by

THEOREM 5.1. *Let M be a g.H.m., and $j: M' \to M$ an immersed analytic submanifold. Then M' has an induced g.H.m. structure iff it satisfies one of the following two conditions:* (a) *M' is minimal;* (b) *M' is tangent to the Lee vector field B of M.*

Proof. We shall denote all the elements of M' by a prime. It is clear that the induced metric g' of M' is either l.c.K., or g.c.K., with the Lee form $\omega' = j^* \omega$, and the Lee vector field B' given by the decomposition

$$(5.1) \qquad B = B' + B^{\perp}$$

of B into a tangent and a normal component with respect to M'.

Now, by using the Gauss–Weingarten equations of M' [14], we obtain

$$(5.2) \qquad \nabla_{X'} B' = \nabla'_{X'} B' + \alpha(X', B'), \quad \nabla_{X'} B^{\perp} = -A_{B^{\perp}}(X') + D_{X'} B^{\perp},$$

where X' is tangent to M', α is the second fundamental form of M', D is the induced connection of the normal bundle of M', and for any Y' tangent to M' one has

$$(5.3) \qquad g(A_{B^{\perp}}(X'), Y') = g(\alpha(X', Y'), B^{\perp}).$$

If the induced structure of M' is that of a g.H.m., we have $\nabla'_{X'} B' = 0$. By the first equation (5.2), this is equivalent to $\nabla_{X'} B' \perp M'$, and then, by (5.1), it is equivalent to $\nabla_{X'} B^{\perp} \perp M'$, and, by the second equation (5.2), with $A_{B^{\perp}}(X') = 0$. Finally, by (5.3), our condition is equivalent to $\alpha(X', Y') \perp B^{\perp}$, and, consequently, the mean curvature vector of M'

$$(5.4) \qquad \mu = \frac{1}{2h} \operatorname{tr} \alpha \quad (h = \dim_{\mathbb{C}} M')$$

is orthogonal to $B^{\perp}$.

On the other hand, let us denote by $\tilde{\alpha}$ the (local) second fundamental form of M' with respect to the local Kähler metrics of the l.c.K. structure of M. Using (2.1), it is easy to get

$$(5.5) \qquad \tilde{\alpha}(X', Y') = \alpha(X', Y') + \tfrac{1}{2} g(X', Y') B^{\perp},$$

whence the corresponding mean curvature vectors are related by

$$(5.6) \qquad \tilde{\mu} = \mu + \tfrac{1}{2} B^{\perp}.$$

But it is well known that an analytic submanifold of a Kähler manifold is minimal. Hence $\tilde{\mu} = 0$, and $\mu = -\frac{1}{2} B^{\perp}$.

The two results about μ proven above imply $\mu = 0$ and $B^{\perp} = 0$, i.e., if M' is a g.H.m., M' is minimal and tangent to B.

Conversely, if M' is minimal, (5.6) yields $B^{\perp} = 0$, and the same is trivially true if M' is tangent to B. Therefore, $\alpha(X', Y') \perp B^{\perp}$, and we already know that this is equivalent with $\nabla'_{X'} B' = 0$. Moreover, we cannot have $B' = 0$,

since $B' = 0$ and $B^{\perp} = 0$ imply $B = 0$, which is false. Therefore M' is a g.H.m.
Q.E.D.

COROLLARY 5.2. *A generalized Hopf submanifold M' of a regular g.H.m. M is also regular.*

Indeed, regularity means the regularity of the foliation of the trajectories of B [29].

In connection with Theorem 5.1, the classical Hopf manifold H^n (see Section 1) is again a source of interesting examples. Namely, it is known (e.g., [1]) that every sequence $(q^1, \dots, q^n)$ of real positive numbers defines a natural action of $\mathbb{C}$ on $\mathbb{C}^n$ by means of the formula

$$(5.7) \qquad t(z^j) = ([\exp(2\pi q^j t)]z^j) \quad (t \in \mathbb{C}).$$

Similarly, if we choose h such sequences q^j_α ($j = 1, \dots, n$; $\alpha = 1, \dots, h$), we can define an action of the additive group $\mathbb{C}^h$ on $\mathbb{C}^n$ by means of the formula

$$(5.8) \qquad (t^\alpha)(z^j) = \left(\left[\exp\left(2\pi \sum_{\alpha=1}^{h} q^j_\alpha t^\alpha\right)\right] z^j\right) \quad ((t^\alpha) \in \mathbb{C}^h).$$

Obviously, this induces an action of $\mathbb{C}^h$ on the Hopf manifold H^n, whose orbits are analytic submanifolds of dimensions $k \leqslant h$ of M.

Now, by (2.17) the Lee vector field B of the metric (1.6) is

$$(5.9) \qquad B = -\frac{1}{2}\left(z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j}\right),$$

and the (1, 0)-components of a vector tangent to an orbit L of (5.8) at a point (z^j) are the values of the 1-forms

$$(5.10) \qquad \mathrm{d}z^j = \left(2\pi \sum_{\alpha=1}^{h} q^j_\alpha \,\mathrm{d}t^\alpha\right) z^j$$

on that vector. Hence B is tangent to L iff the system of linear equations

$$(5.11) \qquad \sum_{\alpha=1}^{h} q^j_\alpha \xi^\alpha = 1$$

is compatible with respect to ξ^α, or equivalently, iff there are homotheties among the transformations (5.8).

Therefore, for some choices of (q^j_α) L will be a g.H. submanifold, and for other choices it will not be such. (For instance, if the matrix (q^j_α) has the maximal rank h, (5.11) is compatible, and L is a g.H. submanifold). This will provide us with new examples of l.c.K. manifolds L which are not g.H.m.

Other interesting examples of this type have been given recently by F. Tricerri [25], and they consist of a great part of the Inoue surfaces endowed with suitable l.c.K. metrics. In [25] it is also proven that the blowing up and blowing down procedures [16] lead from l.c.K. manifolds to l.c.K. manifolds. (In fact, it is simpler to perform the proof on the l.c.K. structures as defined in Section 1 of this paper.) Hence, if we blow up a point of a compact g.H.m. M we get a l.c.K. manifold $\tilde{M}$ which is no longer g.H.m. because it no longer has a vanishing Euler–Poincaré characteristic.

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