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NEAR-REGULARITY CONDITIONS FOR DESIGNS

1. INTRODUCTION

For finite incidence structures (whose elements are called points and blocks), Dembowski [2], p.5, defines the regularity conditions $(R \cdot m)$ and $(\bar{R} \cdot n)$, for positive integers m and n, as follows:

$(R \cdot m)$ Any m blocks are incident with v_m points, for some positive integer v_m.

$(\bar{R} \cdot n)$ Any n points are incident with b_n blocks, for some positive integer b_n.

A *tactical configuration* is an incidence structure satisfying $(R \cdot 1)$ and $(\bar{R} \cdot 1)$. We shall always use v, b for the numbers of points and blocks, and $k = v_1$, $r = b_1$. In a tactical configuration, $(R \cdot m)$ implies $(R \cdot m')$ for $m' \leqslant m$, and $(\bar{R} \cdot n)$ implies $(\bar{R} \cdot n')$ for $n' \leqslant n$.

It is a remarkable fact that the only tactical configurations satisfying $(R \cdot 2)$ and $(\bar{R} \cdot 3)$ are the 'degenerate' ones in which either $v = k$, $b = r$, or $v - 1 = k = r = b - 1$ ([2] p.5). In addition, the only known tactical configurations satisfying $(\bar{R} \cdot 6)$ are those in which every set of k points is incident with a block. This suggests the general question: which conditions of this type can be satisfied non-trivially?

In this paper, Dembowski's conditions are refined with 'near-regularity conditions' stating, for example, that the number of points incident with any m blocks takes only a few values. On this finer mesh of conditions, the borderline between possibility and impossibility can be charted, and configurations adjacent to this borderline studied.

A *t-design* is a tactical configuration satisfying $(\bar{R} \cdot t)$; a *design* is a 2-design. (In a design, the integer b_2 is denoted by λ.) A *symmetric design* (called a projective design in [2]) is a design which also satisfies $(R \cdot 2)$. (Symmetric designs are interesting as a 'borderline case'.) A design is *quasi-symmetric* if the number of points incident with two blocks takes just two values. The nth *multiple* of an incidence structure is obtained by replacing each block by a set of n blocks all incident with the same set of points as the original block. A t-design will be called non-trivial if $k \leqslant v - t$. (In a trivial t-design, any set of k points is incident with a block.)

Fisher's inequality states that a design with $k \leqslant v - 1$ satisfies $b \geqslant v$; furthermore, equality holds if and only if the design is symmetric. Recently this has

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been generalised by Wilson and Ray-Chaudhuri as follows: A $2s$-design with $k \leqslant v-s$ satisfies $b \geqslant \binom{v}{s}$; furthermore, if equality holds, the design is called *tight*, and has the property that the number of points incident with two blocks takes just s distinct values. (This result was communicated to me by J. Doyen; part of it was reported in [7]. Part was proved independently by Bannai.) Here I investigate the reverse problem. Suppose D is a $(2s\text{-}2)$-design in which the number of points incident with two blocks takes just s values. Then the dual of D is a partial design with class number s. (This generalises a result of Goethals and Seidel [3].) Furthermore, either $b \leqslant \binom{v}{s}$ or D is a multiple of a tight $(2s\text{-}2)$-design. As a consequence, the property that the number of points incident with two blocks takes just s values is equivalent to tightness in a $2s$-design, and cannot be satisfied in a non-trivial $(2s+1)$-design. The last fact generalises Dembowski's result on the incompatibility of $(R\cdot 2)$ and $(\bar{R}\cdot 3)$. In particular, a quasi-symmetric design with more than $\frac{1}{2}v(v-1)$ blocks is a multiple of a symmetric design; a quasi-symmetric 4-design has $\frac{1}{2}v(v-1)$ blocks and is tight; and there is no non-trivial quasi-symmetric 5-design.[1]

The final section of the paper contains a discussion of the two kinds of configuration adjacent to the exclused $(R\cdot 2)\&(\bar{R}\cdot 3)$. The first is the class of symmetric designs in which the number of blocks incident with three points takes just two values. Designs in which one of these values attains the obvious upper or lower bound are considered, and the Hadamard designs in the class are determined. The second class consists of quasi-symmetric 3-designs. Apart from the facts about quasi-symmetric 4- and 5-designs already mentioned, only one observation on this class is recorded. (Some of these results were announced at a conference in Oxford in July 1972.)

An incidence structure has *repeated blocks* if two different blocks are incident with the same set of points. (If an incidence structure has no repeated blocks, a block can be identified with the set of points incident with it.) If p is a point of the incidence structure D, the *derived* (resp. *residual*) structure with respect to p is defined to have the points different from p and the blocks incident (resp. non-incident) with p, with incidence inherited from D. (See [3] p.610; this terminology is not standard.) A *Steiner system* $\mathscr{S}(t, k, v)$ is a t-design with $b_t = 1$.

[1] P. Delsarte has obtained similar results.

2. t-DESIGNS AND PARTIAL DESIGNS

LEMMA 1. *The determinant*

$$\begin{vmatrix} \binom{x_0}{0} & \binom{x_1}{0} & \cdots & \binom{x_h}{0} \\ \binom{x_0}{1} & \binom{x_1}{1} & \cdots & \binom{x_h}{1} \\ \vdots & \vdots & & \vdots \\ \binom{x_0}{h} & \binom{x_1}{h} & \cdots & \binom{x_h}{h} \end{vmatrix}$$

is equal to zero if and only if $x_i = x_j$ *for some* i, j *with* $0 \leqslant i < j \leqslant h$.

Proof. Multiplying by $0!\,1!\ldots h!$ and performing row operations transforms the given determinant into a Vandermonde determinant.

LEMMA 2. *Let D be a $(s-1)$-design in which the number of points incident with two blocks takes just the s distinct values $x_0, \ldots, x_{s-1} (s > 1)$, of which x_0 is the greatest.*

(i) *The number n_i of blocks having x_i points in common with a given block depends only on i.*

(ii) *If D has repeated blocks (that is, if $x_0 = k$) then it is the (n_0+1)th multiple of a $(s-1)$-design D' in which the number of points incident with two blocks takes just the $s-1$ values $x_1, \ldots, x_{s-1}$.*

Proof. (i) Given a block b, count in two ways the number of choices of j points incident with b and another block c incident with all these points:

$$\sum_{i=0}^{s-1} n_i \binom{x_i}{j} = \binom{k}{j} (b_j - 1), \qquad 0 \leqslant j \leqslant s-1.$$

By lemma 1, these equations determine $n_0, \ldots, n_{s-1}$ uniquely.

(ii) If $x_0 = k$ then each block is repeated $n_0 + 1$ times.

An *association scheme with class number s* on a set P is a partition of the set of (unordered) pairs of elements of P into s classes $C_1, \ldots, C_s$ with the properties

(i) given $p \in P$, the number of $q \in P$ with $\{p, q\} \in C_k$ depends only on k;

(ii) given $\{p, q\} \in C_k$, the number of $r \in P$ with $\{p, r\} \in C_i$, $\{r, q\} \in C_j$ depends only on (i, j, k).

(An association scheme is a particular case of what Higman [4] calls a *coherent configuration*: it is a 'homogeneous coherent configuration with trivial pairing'. An association scheme with class number 2 is essentially the same thing as a complementary pair of *strongly regular graphs* [3].)

Let A_i be the basis matrix corresponding to the class C_i. The rows and columns of A_i are indexed by the elements of P, and the (p, q) entry is 1 if $\{p, q\} \in C_i$, and 0 otherwise. The real vector space spanned by $I, A_i, \ldots, A_s$ (where I is the unit matrix) is an algebra, called the *centralizer algebra* of the association scheme (see [4]), and is commutative since each A_i is symmetric. So the vector space $V = \mathbb{R}P$ with basis vectors indexed by elements of P (the natural module for the centralizer algebra) can be expressed as $V = V_0 \oplus V_1 \oplus \cdots \oplus V_s$, where each V_i is an eigenspace for every matrix in $\langle I, A_1, \ldots, A_s \rangle_{\mathbb{R}}$.

If P is the set of j-element subsets of a set containing at least $2j$ points, then the classes $C_i = \{\{p, q\} \mid |p \cap q| = i\}$, $0 \leqslant i \leqslant j-1$, form an association scheme on P.

A *partial design with class number s* is a tactical configuration together with an association scheme with class number s on the points, with the property that the number of blocks incident with two points p and q depends only on the class containing $\{p, q\}$. (A partial design with class number 1 is simply a design.) Goethals and Seidel ([3] Theorem 3.1) proved that the dual of a quasi-symmetric design is a partial design with class number 2. The first part of the main theorem below is a generalisation of their result; the proof owes much to their argument.

THEOREM 1. *Let D be a $(2s-2)$-design in which the number of points incident with two blocks takes just the s distinct values $x_0, \ldots, x_{s-1} (s > 1)$.*

(i) *The dual of D is a partial design with class number s.*

(ii) *Either $b \leqslant \binom{v}{s}$ or D is a multiple of a tight $(2s-2)$-design.*

Proof. (i) Note that $2s-2 \leqslant k \leqslant v-s+1$; so by the result of Wilson and Ray-Chaudhuri, $b > \binom{v}{s-1}$. Let P and B be the sets of points and blocks of D.

There is a natural partition of the set of pairs of blocks into s classes. We must show that this is an association scheme. By Lemma 2, part (i) of the definition is satisfied. Let $A_0, \ldots, A_{s-1}$ be the basis matrices; it is sufficient to show that $\langle I, A_0, \ldots, A_{s-1} \rangle_{\mathbb{R}} = S$ is an algebra. (See [4] p.11.)

Let M_j be the matrix with rows indexed by the j-element sets of points and columns by blocks, with $(\{p_1, \ldots, p_j\}, b)$ entry 1 if $p_1, \ldots, p_j$ are incident with b, 0 otherwise.

Then M_0 is a single row of ones; and, for $0 \leqslant j \leqslant s-1$ (indeed, for all positive j),

$$M_j^T M_j = \sum_{i=0}^{s-1} \binom{x_i}{j} A_i + \binom{k}{j} I.$$

By Lemma 1, these equations can be solved to express $A_0, \ldots, A_{s-1}$ in terms of $M_0^T M_0, \ldots, M_{s-1}^T M_{s-1}$, and I. So

$$S = \langle I, M_0^T M_0, \ldots, M_{s-1}^T M_{s-1} \rangle_{\mathbb{R}}.$$

Let $V = \mathbb{R}B$. We shall show that there is an expression

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s,$$

where each V_i is an eigenspace for every matrix in S. Since $\dim(S) = s+1$, it follows that S is the set of all matrices that act as scalars on each V_i. This set is clearly an algebra.

Let $P_{(j)}$ denote the set of j-element subsets of P, carrying the association scheme described earlier; let R_j be the centralizer algebra of this association scheme, and $W_j = \mathbb{R}P_{(j)}$. For $i \leqslant j$, let B_{ij} be the matrix with rows indexed by $P_{(i)}$ and columns by $P_{(j)}$, with $(\{p_1, \ldots, p_i\}, \{q_1, \ldots, q_j\})$ entry 1 if $\{p_1, \ldots, p_i\} \subseteq \{q_1, \ldots, q_j\}$, 0 otherwise. Note that

$$B_{ij} B_{jh} = \binom{h-i}{j-i} B_{ih},$$

$$B_{ij} M_j = \binom{k-i}{j-i} M_i,$$

$$B_{ij} B_{ij}^T \in R_i, \qquad B_{ij}^T B_{ij} \in R_j.$$

A result of Kantor asserts that, for $i < j \leqslant \frac{1}{2}(v+1)$, the rank of B_{ij} is $|P_{(i)}|$; in particular, $B_{ij} B_{ij}^T$ is non-singular.

Suppose $W_i = W_{i,0} \oplus \cdots \oplus W_{i,i}$, where each $W_{i,j}$ is an eigenspace for each element of R_i. Let $W_i' = W_i B_{i,i+1} \subset W_{i+1}$ and $R_i' = B_{i,i+1}^T R_i B_{i,i+1} \subset R_{i+1}$. $W_{i+1} = W_i' \oplus U_{i+1}$, where $U_{i+1} = \{w \in W_{i+1} \mid w B_{i,i+1}^T = 0\} = (W_i')^{\perp}$, since $B_{i,i+1} B_{i,i+1}^T$ is non-singular. R_i' is a subspace of R_{i+1} of codimension 1. It has eigenspaces $W_{i,j} B_{i,i+1} (0 \leqslant j \leqslant i)$ and U_{i+1}; indeed, every element of R_i' has eigenvalue 0 on U_{i+1}. It follows that, with $W_{i+1,j} = W_{i,j} B_{i,i+1}$ $(0 \leqslant j \leqslant i)$ and $W_{i+1,i+1} = U_{i+1}$,

$$W_{i+1} = W_{i+1,0} \oplus \cdots \oplus W_{i+1,i+1},$$

where each $W_{i+1,j}$ is an eigenspace for R_{i+1}. Then we have, by induction and choice of notation,

$$W_{i,j} B_{ih} = W_{h,j} \quad \text{for} \quad 0 \leqslant j \leqslant i < h < s.$$

Since D is a $(2s-2)$-design, $M_j M_j^T \in R_j$ for $0 \leqslant j \leqslant s-1$, and so $W_{j,h}$ is an eigenspace for $M_j M_j^T$, and $W_{j,h} M_j$ is an eigenspace for $M_j^T M_j$. Now

$$W_{j,h} M_j = W_{j,h} B_{j,s-1} M_{s-1} = W_{s-1,h} M_{s-1}$$

So the subspaces $V_h = W_{s-1,h} M_{s-1} (0 \leqslant h \leqslant s-1)$ are eigenspaces for each $M_j^T M_j$. Since $b > \binom{v}{s-1}$, $V_0 \oplus \cdots \oplus V_{s-1} = W_{s-1} M_{s-1}$ is a proper subspace of V; and $V_s = (W_{s-1} M_{s-1})^{\perp}$ is a zero eigenspace for all $M_j^T M_j$ (since it is contained in all $(W_jM_j)^\perp)$. Thus

$$V = V_0 \oplus \cdots \oplus V_{s-1} \oplus V_s,$$

where each V_h is an eigenspace for each $M_j^T M_j$, and clearly also for I. So the result is proved.

(ii) Suppose $b > \binom{v}{s}$. For real numbers $\varepsilon_0, \ldots, \varepsilon_{s-1}$, define a matrix $M_s(\varepsilon_0, \ldots, \varepsilon_{s-1})$ with rows indexed by $P_{(s)}$ and columns by B as follows: the $(\{p_1, \ldots, p_s\}, b)$ entry is 1 if b is incident with all of $p_1, \ldots, p_s$, and ε_h if b is incident with just h of these points $(0 \leqslant h \leqslant s-1)$. $M_s(0, \ldots, 0)$ is the matrix we have already called M_s. We find that

$$M_s^T M_s(\varepsilon_0, \ldots, \varepsilon_{s-1}) = \sum_{i=0}^{s-1} \left\{ \binom{x_i}{s} + \sum_{h=0}^{s-1} \binom{x_i}{h} \binom{k-x_i}{s-h} \varepsilon_h \right\} A_i + \binom{k}{s} I,$$

so $M_s^T M_s(\varepsilon_0, \ldots, \varepsilon_{s-1}) \in S$. Also, since $b > \binom{v}{s}$, this matrix is singular for all choices of $\varepsilon_0, \ldots, \varepsilon_{s-1}$; so it has zero eigenvalue on a particular one of the eigenspaces of S, say V^*, for all such choices. Let $\alpha_0, \ldots, \alpha_{s-1}$ be the eigenvalues of $A_0, \ldots, A_{s-1}$ on V^*.

Putting $\varepsilon_0 = \cdots = \varepsilon_{s-1} = 0$, we have

$$0 = \sum_{i=0}^{s-1} \binom{x_i}{s} \alpha_i + \binom{k}{s}.$$

Equating the coefficient of ε_h to zero,

$$0 = \sum_{i=0}^{s-1} \binom{x_i}{h} \binom{k-x_i}{s-h} \alpha_i, \qquad h = 0, \ldots, s-1.$$

Now $\binom{k-x}{s-h}$ is a polynomial of degree $s-h$ in x, and so is a linear combination of

$$1, \frac{x-h}{h+1}, \frac{(x-h)(x-h-1)}{(h+1)(h+2)}, \ldots, \frac{(x-h)(x-h-1)\ldots(x-s+1)}{(h+1)(h+2)\ldots s};$$

say

$$\binom{k-x}{s-h} = \beta_0 + \beta_1 \frac{x-h}{h+1} + \cdots,$$

Putting $x=h$ shows that $\beta_0 \neq 0$. Multiplying by $\binom{x}{h}$,

$$\binom{x}{h}\binom{k-x}{s-h} = \sum_{t=0}^{s-h} \beta_t \binom{x}{h+t}.$$

Then

$$0 = \sum_{i=0}^{s-1} \binom{x_i}{h}\binom{k-x_i}{s-h} \alpha_i + \binom{k}{h}\binom{k-k}{s-h}$$

$$= \sum_{t=0}^{s-h} \beta_t \left\{ \sum_{i=0}^{s-1} \binom{x_i}{h+t} \alpha_i + \binom{k}{h+t} \right\}.$$

Assuming

$$\sum_{i=0}^{s-1} \binom{x_i}{h'} \alpha_i + \binom{k}{h'} = 0 \quad \text{for} \quad h < h' \leqslant s,$$

it follows that

$$\sum_{i=0}^{s-1} \binom{x_i}{h} \alpha_i + \binom{k}{h} = 0.$$

By induction, this equation holds for $0 \leqslant h \leqslant s$. But the expression is just the eigenvalue of $M_h^T M_h$ on V^*.

The subset S^* of S consisting of matrices with zero eigenvalue on V^* is a subspace of codimension 1, and so must be $\langle M_0^T M_0, \ldots, M_{s-1}^T M_{s-1} \rangle_{\mathbb{R}}$. We have $M_s^T M_s \in S^*$ but $I \notin S^*$.

Now

$$M_h^T M_h = \sum_{i=0}^{s-1} \binom{x_i}{h} A_i + \binom{k}{h} I, \qquad 0 \leqslant h \leqslant s.$$

By Lemma 1, if k is not equal to any x_i, then these equations can be solved for $A_0, \ldots, A_{s-1}, I$ in terms of $M_0^T M_0, \ldots, M_s^T M_s$; in particular, they imply that $I \in S^*$, which is false. So, without loss of generality, we can assume that $k = x_0$. Then, by Lemma 2, D is a multiple of a $(2s-2)$-design D' with no repeated blocks. By what has already been proved (with $s-1$ replacing s), D' has at most $\binom{v}{s-1}$ blocks; by the theorem of Wilson and Ray-Chaudhuri, D' has at least $\binom{v}{s-1}$ blocks. It follows that D' is tight.

COROLLARY. (i) *A 2s-design with $k \leqslant v-s$ has the property that the number of points incident with two blocks takes s values if and only if it has $\binom{v}{s}$ blocks.*

(ii) *In a non-trivial $(2s+1)$-design, the number of points incident with two blocks takes at least $s+1$ distinct values.*

Proof. (i) The sufficiency is the second part of the result of Wilson and Ray-Chaudhuri; the necessity comes from combining the first part of their result with Theorem 1.

(ii) If D is a $(2s+1)$-design in which the number of points incident with two blocks takes just s values, then $b=\binom{v}{s}$ and (considering the derived design D_p) $r=\binom{v-1}{s}$. Since $bk=vr$ ([2] p.57) this implies $k=v-s$; so D is trivial.

Remark. Theorem 1 and the corollary are 'best possible' for small values of s.

For example, the 'pair design' whose blocks are the 2-element sets of points, is quasi-symmetric, with $b=\frac{1}{2}v(v-1)$, and has no repeated blocks. The Steiner system $\mathscr{S}(4, 7, 23)$ ([3] p.610) and its complement also have these properties – they are the only known non-trivial tight 4-designs, and indeed the only known non-trivial tight 2s-designs for any $s \geqslant 2$.

The design of points and planes in affine 4-space over GF(2) is a 3-design (indeed a Steiner system $\mathscr{S}(3, 4, 16)$) in which any two blocks are incident with 0, 1, or 2 points. However, if b and c are disjoint planes, the number of planes disjoint from both b and c depends on whether or not b and c are parallel. So the dual is not a partial design with class number 3.

The Steiner system $\mathscr{S}(5, 8, 24)$ and its residual, and the Steiner system $\mathscr{S}(4, 5, 11)$, are 4-designs in which the number of points incident with two blocks takes just three values. (So their duals are partial designs with class number 3.) The first of these is even a 5-design (compare (ii) of the corollary).[2]

3. Designs almost satisfying $(R.2)$ & $(\bar{R}.3)$

A design will be called *quasi*-3 if the number of blocks incident with three distinct points takes only two values. In this section, symmetric quasi-3 designs and quasi-symmetric 3-designs are considered. First, however, some special classes of designs are defined.

A *projective plane* is a symmetric design with $\lambda=1$. Projective planes have received much investigation; another class considered in the literature is the

[2] The results of this section can be generalised without difficulty to the situation in which vector spaces and subspaces replace sets and subsets.

class of Hadamard designs. A *Hadamard* 2-*design* is a symmetric design with parameters $v=4n-1$, $k=2n-1$, $\lambda=n-1$ for some integer n (greater than 1) called its *order*. A Hadamard 2-design of order n is uniquely extendable ([2] pp. 76, 113) to a 3-design with $v=4n$, $k=2n$, $b_3=n-1$, canonically isomorphic to its complement, which we shall call a *Hadamard* 3-*design* of order n (it is called a double Hadamard design in [3]). There are unique Hadamard 2- and 3-designs of orders 2 and 3. For an integer $d>2$ and a prime power q, $\mathrm{PG}(d, q)$ will denote the symmetric design of points and hyperplanes in the d-dimensional projective geometry over $\mathrm{GF}(q)$; it has $v==(q^{d+1}-1)/(q-1)$, $k=(q^d-1)/(q-1)$, $\lambda=(q^{d-1}-1)/(q-1)$. $\mathrm{PG}(d, 2)$ is a Hadamard 2-design of order 2^{d-1}.

The class of symmetric quasi-3 designs is a large one. It is closed under complementation, and contains all symmetric designs with $\lambda=1$ or $\lambda=2$, all the designs $\mathrm{PG}(d, q)$, and a family of designs with $v=2^{2m}$, $k=2^{2m-1}--2^{m-1}$, $\lambda=2^{2m-2}-2^{m-1}$, $m>1$. An unsolved problem about this class: is it closed under taking duals? (all examples mentioned with $\lambda>2$ are self-dual.)

THEOREM 2. *In a symmetric quasi*-3 *design D with parameters v, k, λ, let the number of blocks incident with three points be either f or g, where* $f<g$. *Clearly* $f\geqslant 0$ *and* $g\leqslant\lambda$; *furthermore, we have:*

(a) D_p *is the dual of a quasi-symmetric design, for any point p.*

(b) *If* $f=0$, *then one of the following is true:*

(i) $g=1=\lambda$, *D is a projective plane;*

(ii) $g=1$, $\lambda=2$;

(iii) $g=2^{d-1}(d>2)$, *D is the complement of* $\mathrm{PG}(d, 2)$;

(iv) $g>1$, $v=g(g^2+5g^2+6g-1)$, $k=g(g^2+3g+1)$, $\lambda=g(g+1)$;

(v) $g=2$, $v=1037$, $k=112$, $\lambda=12$.

(c) *If* $g=\lambda$, *then D is either a projective plane or* $\mathrm{PG}(d, q)$ *for some d, q.*

(d) *If D is a Hadamard* 2-*design of order n, then either* $n=2$ *or* 3, *or D is* $\mathrm{PG}(d, 2)$ *with* $n=2^{d-1}$.

Proof. (a) is immediate from the definitions.

(b) Assume $f=0$. $g=1$ if and only if $\lambda=1$ or $\lambda=2$; so assume $\lambda>2$, $g>1$. Let b be a block; let D_b denote the incidence structure whose points are the points incident with b and whose blocks are the blocks distinct from b. $D'=D_b$ is a 3-design with parameters $v'=k$, $k'=\lambda$, $b_3=g-1$ (for any three points incident with b are incident with $g-1$ more blocks). If p is a point incident with b, then the design $D^*=(D')_p$ has parameters $v^*=k-1$, $k^*=\lambda-1$, $\lambda^*=g-1$, and has just $k-1$ blocks; so D^* is a symmetric design. Also, it is non-trivial.

According to the main theorem of [1], if the derived design of a 3-design D' is a non-trivial symmetric design D^* with parameters v^*, k^*, λ^*, then one of the following is true:

(1) D' is a Hadamard 3-design of order λ^*+1;

(2) $v^*=(g+1)(g^2+2g-1), k^*=g^2+g-1, \lambda^*=g-1,$ for some $g>1$;

(3) $v^*=111, k^*=11, \lambda^*=1$;

(4) $v^*=495, k^*=39, \lambda^*=3$.

In each of these cases, the parameters (v, k, λ, g) can be determined. 2 and 3 yield (iv) and (v) of the theorem.

The Bruck-Ryser-Chowla theorem ([2] pp. 61, 63) shows that 4 does not arise here. (It also restricts the values of g that can occur in 2: for example, if g is even, then $g+2$ is a square.) In case 1, an inclusion-exclusion argument shows that the complement $\bar{D}$ of D is quasi-3 with parameters $\bar{v}=8g-1$, $\bar{k}=4g-1, \bar{\lambda}=2g-1, \bar{f}=g-1, \bar{g}=2g-1=\bar{\lambda}$. It will follow from (c) that $\bar{D}$ is PG$(d, 2)$ for some $d>2$.

(c) Suppose $g=\lambda$. If $\lambda=1$ then D is a projective plane; so assume $\lambda>1$. The line through two points p, q of D is defined to be the set of points incident with every block incident with p and q. So D has the property that any three noncollinear points are incident with f blocks. The Dembowski-Wagner theorem ([2] p. 67) asserts that a design with this property is PG(d, q) for some d, q.

(d) Suppose D is a Hadamard 2-design of order n. Select a point p. D_p is the dual of a quasi-symmetric design. Form a graph whose vertices are the points different from p, with q and r adjacent if the number of blocks incident with p, q, and r takes one particular value. By [3] theorem 3.1 this graph is strongly regular, and the eigenvalues of its adjacency matrix have multiplicities 1, $2n-2$, $2n-1$. Replacing the graph by its complement if necessary, we can assume it has valency less than $2n-1$.

If the graph is not connected, consideration of multiplicities shows that it is the disjoint union of $2n-1$ edges; it and its complement have valencies 1 and $4n-4$. Take two points of D, and count in two ways the number of choices of a third point and a block incident with all three: we obtain

$$(4n-4)f+g=(n-1)(2n-3).$$

So $\lambda=n-1$ divides g, whence $g=\lambda$. By (c), the design is either a projective plane (with $n=2$) or PG$(d, 2)$ (with $n=2^{d-1}$, $d>2$).

Suppose the graph is connected. Inspection of Wielandt's argument in the proof of [6] Theorem 31.2 on primitive permutation groups of degree $2p$ shows that, from our hypotheses, we can conclude that $n=a^2+a+1$ for some positive integer a, and the graph and its complement have valencies $a(2a+1)$ and $(a+1)(2a+1)$. Counting as before,

$$a(2a+1)f+(a+1)(2a+1)g=a(a+1)(2a^2+2a-1).$$

So $2a+1$ divides $a(a+1)(2a^2+2a-1)$. Applying the remainder theorem,

$2a+1$ divides 3, whence $a=1$. The unique Hadamard 2-design of order 3 is quasi-3, since $\lambda=2$.

The class of quasi-symmetric 3-designs is closed under complementation. It contains the plethora of (self-complementary) Hadamard 3-designs (it is conjectured that these exist for every order $n>1$), but appears to contain little else. The only other known designs in this class are the Steiner system $\mathscr{S}(4, 7, 23)$, its complement, and their derived and residual designs (see [3] p. 610). Apart from corollary 2, the result given below is the only restriction I know on designs in this class.

PROPOSITION. *If D is a* 3-*design in which any two blocks are incident with* 0 *or μ points, then $b_3=\mu-1$, and D_p is symmetric for any point p.*

(Now, of course the classification of [1], mentioned earlier, for extensions of symmetric designs applies to this subclass.)

Proof. Any two blocks of D_p are incident with p and hence with $\mu-1$ more points. So D_p is a symmetric design with $\lambda=\mu-1$.

Note. In practice, regularity or near-regularity conditions often arise because a tactical configuration has a large group of automorphisms. For example, if the automorphism group of D is transitive and has rank $t+1$ on the blocks, where the number of points incident with two distinct blocks takes t values, then the dual of D is automatically a partial design with class number t. Again, if D is a nontrivial symmetric design and Aut(D) has two orbits on ordered triples of points, then it has two orbits on ordered triples of blocks also, and D and its dual are quasi-3.

BIBLIOGRAPHY

[1] P. J. Cameron, 'Extending Symmetric Designs', *J. Combinatorial Theory*, to appear.
[2] P. Dembowski, *Finite Geometries*, Springer-Verlag, Berlin-Heidelberg-New York 1968.
[3] J. M. Goethals and J. J. Seidel, 'Strongly Regular Graphs Derived from Combinatorial Designs', *Canad. J. Math.* **22** (1970), 597–614.
[4] D. G. Higman, *Combinatorial Considerations about Permutation Groups*, Mathematical Institute, Oxford 1972.
[5] W M. Kantor, 'On Incidence Matrices of Finite Projective and Affine Spaces', *Math. Z.* **124** (1972), 315–318.
[6] H. Wielandt, *Finite Permutation Groups*, Acad. Pr., New York-London 1964.
[7] R. M. Wilson and D. K. Ray-Chaudhuri, 'Generalisation of Fisher's Inequality to t-Designs', *Amer. Math. Soc. Notices* **18** (1971), 805.

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