U. DEMPWOLFF AND A. REIFART*

THE CLASSIFICATION OF THE TRANSLATION PLANES OF ORDER 16, I

ABSTRACT. The translation planes of order 16 are completely classified. The exceptional isomorphism $A_8 \simeq GL(4, 2)$ gives a crucial computational approach to this problem

1. INTRODUCTION

All translation planes of order $2^n (n \leqslant 3)$ are known [4]. The purpose of this paper and its successor is to settle the first open case and give the complete classification of the translation planes of order 16. Various partial results have already been obtained; for instance, [6], [7], [8], [10] and [1]. Nearly all of these results use, in some way, detailed information about certain translation planes and classify these planes with these properties. Reference [1] does not quite fit into this row. Here a certain computational approach, which only works in the case of translation planes of order 16, gives a good tool to complete a classification problem even when little information is available. Indeed, the methods of [1] are powerful enough to complete, in this paper, the full classification of the translation planes of order 16.

The planes that will come up (see Section 3) are already known and have fairly large translation complements. The method we employ for the classification (see Section 2), however, discovers translation planes with small automorphism groups better than those with large automorphism groups. In fact, the Desarguesian plane of order 16 is the most difficult to find with our method. This will make the validity of our results highly stable even where an oversight occurs. A new, unknown, translation plane of order 16 would, by the previous classifications, have a 'small' automorphism group. Hence, during our investigation we would have had many cases to handle, where this plane would come up. This remark will become clearer after reading Section 2.

A large part of our work was done on a computer. The printouts comprise well over 1000 pages. However, to make the computations accessible to the reader we shall provide the reader with the necessary computer programs in the second part of the paper.

The terminology we use is standard and may be found in [2] or [11].

* The authors thank the Deutsche Forschungsgemeinschaft for their support of this research.

Geometriae Dedicata **15** (1983) 137–153. 0046–5755/83/0152–0137$02.55.

2. Notation, Preliminary Lemmas and the General Method

We always think of a translation plane $\mathscr{P}$ of order p^n as a $2n$-dimensional vector space W over $GF(p)$ having a collection π (*spread*) $V_\infty, V_0, V_1, \dots, V_{p^n-1}$ of n-dimensional pairwise disjoint subspaces which are called the *components*. The lines of $\mathscr{P}$ different from l_∞ are the cosets $w + U$ for $w \in W$ and U a component. The points off l_∞ are the elements of W. The points on l_∞ are numbered by $i \mathrel{\hat{=}} V_i$ for $i = \infty, 0, 1, \dots, p^n - 1$. As usual, using points $(\infty, 0, 1)$ on l_∞ as points of reference, we introduce coordinates on $\mathscr{P}$ as follows:

Let V be an n-dimensional $GF(p)$-vector space and write $W = V \oplus V$ such that by a suitable choice of the basis we have $V_\infty = \{(0, v) \mid v \in V\}$, $V_0 = \{(v, 0) \mid v \in V\}$ and $V_1 = \{(v, v) \mid v \in V\}$. Then there are elements $1 = t_1, t_2, \dots, t_{p^n-1} \in GL(n, p)$ with $V_i = \{(v, vt_i) \mid v \in V\}$ for $i = 1, 2, \dots, p^n - 1$ and

(†) $t_i t_j^{-1}$ is a fixed-point-free transformation on V for $1 \leqslant i < j \leqslant p^n - 1$.

We call a set $M \subseteq GL(V)$, $M = \{1 = t_1, t_2, \dots, t_{p^n-1}\}$ a *coordinate set if* M satisfies condition (†). If the translation plane $\mathscr{P} = (W, \pi)$ has coordinate set M for some points of reference on l_∞ (which we denote by $(\infty, 0, 1)$), we also write $\mathscr{P} = \mathscr{P}(M)$. Conversely, if M is a coordinate set then M defines on $W = V \oplus V$ in the described fashion a translation plane $\mathscr{P} \cong \mathscr{P}(M)$.

The next lemma shows how a coordinate set M changes if one chooses different points of reference on l_∞.

2.1. *Let M be a coordinate set of $\mathscr{P}$ with respect to* $(\infty, 0, 1)$. *Then a coordinate set with respect to* (x, y, z) *for* $x, y, z \in \{\infty, 0, 1, \dots, p^n - 1\}$ *is:*

(a) $M^{-1} = \{m^{-1} \mid m \in M\}$ *for* $(x, y, z) = (0, \infty, 1)$.

(b) $Mt_i^{-1} = \{mt_i^{-1} \mid m \in M\}$ *for* $(x, y, z) = (\infty, 0, i)$ *and* $1 \leqslant i \leqslant p^n - 1$.

(c) $1 - M = \{1\} \cup \{1 - m \mid m \in M - \{1\}\}$ *for* $(x, y, z) = (\infty, 1, 0)$.

Proof. Set $T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\begin{pmatrix} I & 0 \\ 0 & t_i^{-1} \end{pmatrix}$ or $\begin{pmatrix} I & I \\ 0 & -I \end{pmatrix}$, where T is $2n \times 2n$ matrix decomposed in $n \times n$ blocks and I denotes the identity matrix. Then $V_\infty T$, $V_0 T$, $V_1 T$ are for the cases (a), (b) and (c) of the desired shape. Determining the coordinate set with respect to $V_\infty T$, $V_0 T$, and $V_1 T$ gives the assertions.

REMARK. Lemma 2.1 tells us that for any choice of (x, y, z) as points of reference for $x, y, z \in \{\infty, 0, 1, \ldots, p^n - 1\}$ the coordinate set with respect to this triple is obtained by a successive application of some of the operations

(0) $\qquad M \to M^{-1}, M \to Mm^{-1}$ for some $m \in M$ or $M \to 1 - M$.

Conversely, any successive application of some of the operations (0) to a coordinate set M gives a coordinate set N which defines the same plane with respect to some other points of reference.

2.2. *Let $W = V \oplus V$ and $\mathscr{P} = (W, \pi)$ and $\mathscr{P}' = (W, \pi')$ be translation planes with coordinate sets M and M′ with respect to $(\infty, 0, 1)$ and $(\infty', 0', 1')$, respectively. There is an isomorphism $\psi : \mathscr{P} \to \mathscr{P}'$ with $z^{\psi} = z'$ for $z \in \{\infty, 0, 1\}$ iff there is an $x \in GL(V)$ such that*

(1) $\qquad xMx^{-1} = M'$.

Proof. [3].

REMARK. Let M and N be coordinate sets. Then $\mathscr{P}(M) \simeq \mathscr{P}(N)$ iff N is obtained by M by a successive application of some of the operations (0) and (1).

From now on we restrict our attention to the case $p^n = 16$. We use the fact that $A_8 \simeq GL(4, 2)$.

2.3. *Let V be a 4-dimensional vector space over $GF(2)$ and $\varphi : A_8 \to GL(V)$ an isomorphism. If x^{φ} for $x \in A_8$ is a fixed-point-free automorphism in $GL(V)$, then x has cycle structure (abc), $(abcde)$, $(abc)(de)(fg)$ or $(abc)(defgh)$.*

Proof. Clearly, a fixed-point-free element in $GL(V)$ is not centralized by a transvection. As transvections lie in the centres of Sylow 2-subgroups of $GL(V)$ they correspond to the product of four disjoint transpositions in A_8. So if $r \in A_8$ has order 3 and r^{φ} is fixed-point-free, then the centralizer of r in A_8 only contains involutions whose cycle structure is $(ab)(cd)$. Hence, r is a 3-cycle. Now 7 does but 5 does not divide $|GL(3, 2)|$. Thus elements of order 5 in A_8 map onto fixed-point-free transformations while elements of order 7 do not act fixed-point-free on V. All assertions of 2.3 follow.

Let F be the collection of all elements of A_8 of type (abc), $(abcde)$, $(abc)(de)(fg)$, $(abc)(defgh)$ together with 1. A subset $1 \in S \subseteq F$ is called *compatible* if $st^{-1} \in F$ for all $s, t \in S$. In accordance with definition (†) we call a compatible subset M of F a *coordinate set* if $|M| = 15$.

Let V be again a 4-dimensional vector space over $GF(2)$. Let φ, ψ be two isomorphisms from A_8 onto $GL(V)$. Then $\varphi\psi^{-1}$ is an automorphism of A_8.

Now $\mathrm{Aut}(A_8) = S_8$. So $\psi = \alpha\varphi$, where $\alpha \in S_8$. Let M be a coordinate set in A_8. If $\alpha \in A_8$, then $\mathscr{P}(M^\varphi) \simeq \mathscr{P}(M^\psi)$ by 2.2. Take $\tau \in S_8 - A_8$ and set $\varphi^* = \tau\varphi$ for a (fixed) isomorphism $\varphi: A_8 \to GL(V)$. We write $\mathscr{P} = \mathscr{P}(M)$ for $\mathscr{P} = \mathscr{P}(M^\varphi)$ and $\mathscr{P}^* = \mathscr{P}^*(M)$ for $\mathscr{P}^* = \mathscr{P}(M^{\varphi^*})$. Note that, in general $\mathscr{P}^*(M) \not\simeq \mathscr{P}(M)$. Let $\mathscr{S}$ be the collection of compatible subsets in F. We call $S, S' \in \mathscr{S}$ *equivalent* and write $S \sim S'$ iff S' is obtained by S by a successive application of some of the operations (0) and

(1)′ $\quad S \to xSx^{-1}$ for $x \in S_8$.

Our discussion yields the following lemma:

2.4. *Let $M \in \mathscr{S}$ be a coordinate set.*

(a) $\mathscr{P}(M) \simeq \mathscr{P}(xMx^{-1})$ *for* $x \in A_8$.

(b) $\mathscr{P}^*(M) \simeq \mathscr{P}(xMx^{-1})$ *for* $x \in S_8 - A_8$.

(c) *If* $M' \in \mathscr{S}$, $M \sim M'$, *then* $\mathscr{P}(M) \simeq \mathscr{P}(M')$ *or* $\mathscr{P}(M) \simeq \mathscr{P}^*(M')$.

Let $x = (abc)(defgh) \in F$. We say x is of $(+)$*-type* if x is conjugate in A_8 to (123)(45678). Otherwise x is of $(-)$*-type*. Let $y = (abc)(de)(fg) \in F$. We say y stands in the $(+)$*-representation* if $\mathrm{sgn}\begin{pmatrix} 12345678 \\ \mathrm{abcdefg}* \end{pmatrix} = 1$. Otherwise y stands in the $(-)$*-representation.* Note that $y = (123)(45)(67)$ is a $(+)$-representation while $y = (123)(45)(76)$ is a $(-)$-representation of y. Only for the following lemma shall we use an explicit isomorphism $\varphi: A_8 \to GL(4, 2)$.

2.5. *Let $\varphi: A_8 \to GL(4, 2)$ be an isomorphism chosen as in* [5; II, 2.5]. *Identify $x \in A_8$ with x^φ. Then we have:*

(1) $(abc) + 1 = (acb)$;

(2) $(abcde) + 1 = (abcde)^3(fgh)^\alpha$ *and α has to be chosen as* 1 *or* 2 *such that* $(abcde) + 1$ *is of* $(-)$*-type*;

(3) $(abc)(de)(fg) + 1 = (acb)(df)(eg)$ if $(abc)(de)(fg)$ *is* $(+)$*-representation and* $(abc)(de)(fg) + 1 = (acb)(dg)(ef)$ *if* $(abc)(de)(fg)$ *is a* $(-)$*-representation*;

(4) $(abcde)(fgh) + 1 = (abcde)^4(fgh)$ *if* $(abcde)(fgh)$ *is of* $(+)$*-type and* $(abcde)(fgh) + 1 = (abcde)^2$ *if* $(abcde)(fgh)$ *if of* $(-)$*-type.*

Proof. By the choice of φ one easily verifies that $(123)(45678) + 1 = (123)(48765)$ and $(123)(45)(67) + 1 = (132)(46)(57)$. Now two elements $(ab\ldots)(cd\ldots)\ldots$ and $(a'b'\ldots)(c'd'\ldots)\ldots$ of the same cycle structure are conjugate in S_8 by the element

$$\begin{pmatrix} ab & \ldots & cd & \ldots \\ a'b' & \ldots & c'd' & \ldots \end{pmatrix}.$$

The assertion now follows if we take into consideration that under A_8 all elements of order 3, 5 or 6 in F are conjugate and that F contains precisely two A_8-conjugacy classes of elements of order 15.

Let $\mathscr{M}$ denote the subset of the coordinate sets of $\mathscr{S}$. In order to determine all isomorphism classes of translation planes of order 16 we determine a set $\mathscr{R}$ of representatives of the classes on $\mathscr{M}$ for the equivalence relation $\sim$ we just introduced on $\mathscr{S}$. Then $\{\mathscr{P}(M), \mathscr{P}^*(M) | M \in \mathscr{R}\}$ contains at least one member of each isomorphism class of translation planes of order 16. In order to decide precisely how many isomorphism classes occur we only have to determine weather $\mathscr{P}(M) \simeq \mathscr{P}^*(M)$ holds or not. The program of determining $\mathscr{R}$ is now carried out in the following manner:

Step I. We consider the subset $\mathscr{S}_0 \subseteq \mathscr{S}$ of compatible subsets with $|S| = 4$ or 5 for $S \in \mathscr{S}_0$. Denote by $|T|_k$ the number of elements of order k in $T \in \mathscr{S}$. We find representatives of each equivalence class $\mathscr{C}$ on $\mathscr{S}_0$ in the following order:

(i) $\max\{|S|_3 | S \in \mathscr{C}\} \geqslant 4$;
(ii) $\max\{|S|_3 | S \in \mathscr{C}\} = 3$;
(iii) $\max\{|S|_3 | S \in \mathscr{C}\} = 2$;
(iv) $\max\{|S|_3 | S \in \mathscr{C}\} = 1$;
(v) $\max\{|S|_3 | S \in \mathscr{C}\} = 0$ but $\max\{|S|_5 | S \in \mathscr{C}\} \geqslant 1$;
(vi) $|S|_k = 0$ for $k = 3, 5, 15$ and each $S \in \mathscr{C}$.

Use 2.5 to observe that for each $S \in \mathscr{S}_0$ one of the properties (i)–(vi) is true if we let $\mathscr{C} = [S]$ be the equivalence class containing S.

Step II. Let $\mathscr{C}$ be an equivalence class on $\mathscr{S}_0$. Pick a representative $S \in \mathscr{C}$ and say that we are in case (x) of Step I. Then we determine all $M \in \mathscr{M}$ with $S \subseteq M$. Whenever M contains a subset $T \in \mathscr{S}_0$ such that T belongs to case (y) of I and (y) is handled before (x), we can delete M from further consideration. Note that if M is a coordinate set and $N \subseteq M$ lies in $\mathscr{S}_0$, then the class $\mathscr{C}$ containing N must satisfy one of the conditions (i)–(vi) of I. Thus we are sure to pick at least one representative out of each equivalence class on $\mathscr{M}$.

Step III. Let $\{M_1, M_2, \ldots\}$ be the coordinate sets obtained in Step II. For a known plane $\mathscr{P}(K)$ we compute various different coordinate sets – say K, K', K'', ... A 'structural' description of the sets K, K', K'', ... (see Section 5) will be sufficient to decide quickly whether there is a $K''^{\cdots\prime}$, an index i, and an $x \in A_8$ with $x M_i x^{-1} = K''^{\cdots\prime}$.

3. The translation planes of order 16

We describe the known translation planes of order 16 in the language of coordinate sets. We also determine essential parts of the automorphism group which makes it possible to recognize that these planes are pairwise non-isomorphic. The objective of this section is to show that the list of planes we provide contains a member of each isomorphism class of translation planes of order 16.

1. *The Desarguesian plane of order* 16

Set $M_1 = \{((123)(45678))^k \mid 0 \leqslant k \leqslant 14\}$. Set $M' = M_1 \cup \{0\}$. M', considered as a subset of the set of 4×4 matrices over $GF(2)$, is according to 2.5 isomorphic as a subalgebra to $GF(16)$. It is clear that $\mathscr{P}(M_1)$ is Desarguesian.

2. *The semifield plane with kern* $GF(4)$

Set $M_2 = \{1$, (678), (687), (456), (465), (45678), (45687), (46785), (46875), (123)(47586), (123)(47658), (123)(47)(58), (132)(48)(57), (132)(48576), (132)(48657)$\}$. Now $C_{A_8}(M_2) = \langle(123)\rangle$ which shows that $\mathscr{P}(M_2)$ has kern $GF(4)$. Let H be the translation complement fixing $(0,0) \in W$. As $N_{A_8}(M_2) = \langle(123), (45)(78), (12)(78)\rangle$ we have that the stabilizer $H_{\infty,0,1}$ of the points $\infty, 0, 1$ on l_∞ contains a subgroup isomorphic to $S_3 \times Z_2$.

The equations $M_2 = M_2(456) = (678)M_2$ show that $H_{\infty,0} \simeq (Z_3 \times Z_3 \times Z_3)\cdot(Z_2 \times Z_2)$. Finally, $M_2 + 1 = M_2$. This induces a shear σ with fixed-point ∞ on l_∞. One easily verifies that $\langle \sigma, H_{\infty,0}\rangle$ contains a normal elementary abelian subgroup E of order 16 whose non-trivial elements are shears fixing ∞. Thus $\mathscr{P}(M_2)$ must be the unique semifield plane with kern $GF(4)$ (see[9]).

3. *The semifield plane with* kern $GF(2)$

Set $M_3 = \{1$, (346), (364), (587)(16243), (587)(13426), (287)(16453), (287)(13546), (187)(24536), (187)(26354), (587)(14)(23), (578)(12)(34), (287)(14)(56), (278)(15)(46), (187)(23)(56), (178)(25)(36)$\}$. We observe that (125)(364) normalizes M_3 and that $(346)M_3(346) = M_3$. This induces an elementary abelian group D of order 9 in $\mathrm{Aut}(\mathscr{P}(M_3))$ which fixes the points ∞, 0 on l_∞. With the help of 2.5 one verifies that $(M_3 + 1)^{-1} = M_3 + 1$ and $(M_3(187)(25)(36) + 1)^{-1} = M_3(187)(25)(36)$ so that we have two shears, σ and τ, interchanging on l_∞ the points ∞ and 1 and ∞ and 15, respectively. Further, σ and τ both fix 0 on l_∞. One now checks that the conjugates of σ and τ under D generate an elementary abelian group of order 16 whose non-trivial elements are shears fixing precisely 0 on l_∞. As $C_{A_8}(M_3) = 1$ the plane $\mathscr{P}(M_3)$ has kern $GF(2)$. By [9], $\mathscr{P}(M_3)$ is the unique semifield plane with kern $GF(2)$.

All other known translation planes of order 16 we described in Section 2 of [1]. They all possess a 4-group E of automorphisms such that the non-trivial elements in E are Baer involutions which centralize the same Baer subplane. From [1] we repeat:

Set $E = \langle (45)(67), (46)(57) \rangle$. For $t \in F$ set $t^E = \{ete \mid e \in E\}$ and $M_0 = \{1, t_2 = (123), t_3 = (132)\}$. Then we have:

4. *The Hall plane of order* 16

Set $M_4 = M_0 \cup \{(45678)^E, (45786)^E, (46587)^E\}$. Now $M_4 - M_0$ is a conjugacy class of elements of order 5 of A_5 acting on $R = \{4, 5, 6, 7, 8\}$. Thus $L \simeq A_5$ acting on R normalizes M_4 and induces a subgroup of $\mathrm{Aut}(\mathscr{P}(M_4))$ which fixes the components $V_\infty, V_0, \ldots, V_3$. Kern $\mathscr{P}(M_4) \simeq GF(4)$ as $C_{A_8}(M_4) = \langle (123) \rangle$. As $M_4 = M_4^{-1}$ there is a shear interchanging V_0 and V_∞. Without verifying we mention that for either $i = 2$ or $i = 3$ (the choice of i depends on the isomorphism $\varphi: A_8 \to GL(4,2)$) the map $\sigma: W \ni (v, u) \to (u + vt_i, v)$ defines an automorphism of order 5 fixing the components $V_4, \ldots, V_{15}$. Furthermore, $\langle \sigma \rangle L = \langle \sigma \rangle \times L$.

5 and 6. *The Lorimer–Rahilly plane and the Johnson–Walker plane*

Set $M_5 = M_0 \cup \{(345)^E, (246)^E, (147)^E\}$ and $M_6 = M_0 \cup \{345)^E, (247)^E, (146)^E\}$. The support of the 3-cycles of M_5 or M_6, respectively, forms a projective plane of order 2. Hence the subgroup $X_i (i = 5, 6)$ of A_8, $X_i \simeq GL(3, 2)$ normalizing this plane also normalizes M_i. Assume we have chosen the isomorphism $\varphi: A_8 \to GL(4, 2)$ such that X_5^φ is the stabilizer of a point. Then X_6^φ is the stabilizer of a hyperplane. Moreover, $\mathscr{P}(M_5) \simeq \mathscr{P}^*(M_6)$ and $\mathscr{P}(M_6) \simeq \mathscr{P}^*(M_5)$. However, $\mathscr{P}(M_5) \not\simeq \mathscr{P}(M_6)$, as in both cases a full translation complement is isomorphic to $S_3 \times GL(3, 2)$ (see [10]). With the notation chosen as above, $\mathscr{P}(M_5)$ is the Lorimer–Rahilly plane while $\mathscr{P}(M_6)$ is the Johnson–Walker plane (i.e. these are the only translation planes of order 16 where $\mathscr{P}(M) \not\simeq \mathscr{P}^*(M)$).

7. *The derived semifield plane of order* 16

Set $M_7 = M_0 \cup \{(148)^E, (248)^E, (348)^E\}$. Here $H = \langle E, (123), (456), (23)(56) \rangle \simeq (A_4 \times Z_3) \cdot Z_2$ obviously normalizes M_7. Since $M_7 + 1 = M_7^{-1}$ there is a collineation ρ of order 3 such that ρ fixes the components $V_2, \ldots, V_{15}$ and interchanges V_∞, V_0, V_1. $H\langle \rho \rangle = H \times \langle \rho \rangle$ is a full translation complement.

8. *The Dempwolff plane*

Set $M_8 = M_0 \cup \{(348)^E, (12483)^E, (14832)^E\}$. Since $(48)(56)M_8(34)(56) = M_8$ there is a collineation ρ of order 2 such that $L = \langle \rho, E \rangle \simeq A_5$.

$M_8 = (123)M_8$ means that there is a homology τ of order 3 fixing V_∞ pointwise. Finally $(12)(56)M_8(12)(56) = M_8$ induces an involution σ and one readily checks that $L\langle\tau, \sigma\rangle \simeq \Gamma L(2, 4)$. This, in fact, is a full translation complement.

4. The determination of the coordinate sets

In this section we show how the solution of the crucial problem (i.e. to find at least one coordinate set for every translation plane of order 16) is organized.

By direct counting arguments we see that the list F introduced in Section 2 contains exactly 5825 elements. Thus, in this section we are confronted with the problem of finding $M_1, M_2, \ldots, \in \mathscr{M}$ such that $[M_1] \cup [M_2] \cup \cdots = \mathscr{M}$, where $[M_i]$ denotes the equivalence class on $\mathscr{M}$ containing M_i.

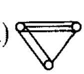
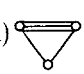
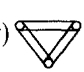
By the computer program provided in Part II of this paper it turns out that the following problem can be solved by a machine in a few seconds. Suppose $S \in \mathscr{S}$ such that $|\{f \in F | S \cup \{f\}$ is compatible$\}| \leqslant 100$, then determine all $M \in \mathscr{M}$ with $S \subseteq M$.

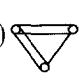
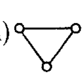
For the sake of completeness we remark that we normally need $|S| = 5$ or $|S| = 4$, provided that there are some side conditions which we will have in a later stage of the proof. As described at the end of Section 2, our first objective will be to find up to equivalence all $M \in \mathscr{M}$ such that M contains at least four elements of order 3. So we need a list of all compatible sets S_0 of size 4 consisting of elements of order 3 only.

Thus we consider $a, b \in F$ of order 3 such that $ab^{-1} \in F$. Then up to conjugation we have one of the following cases:

(i) $a = (123)$, $b = (132)$ with diagram a ○≡○ b (a 3-fold connection)
(ii) $a = (123)$, $b = (124)$ with diagram a ○═○ b (a 2-fold connection).
(iii) $a = (123)$, $b = (145)$ with diagram a ○—○ b (a single connection).

Next consider the possible diagrams (triangles) for three compatible elements of order 3 in F. Clearly, in any such triangle there can be at most one 3-fold connection. Thus we get the following diagrams:

(i)  (ii)  (iii)  (iv) 

(v)  (vi)  (vii) 

It now easily follows that there are no solutions for cases (i) and (ii). Similary, up to conjugation we obtain unique solutions for

(iii) (a, b, c) = ((123), (132), (145))

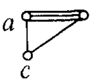
(iv) (a, b, c) = ((123), (124), (125))
(v) (a, b, c) = ((123), (124), (235))

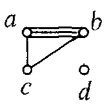
and two solutions for

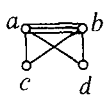
(vi) (a, b, c) ∈ {((123), (124), (156)), ((123), (124), (345))}

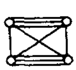
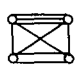
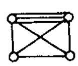
and three solutions for

(vii) (a, b, c) ∈ {((123), (145), (167)), ((123), (145), (246)), ((123), (145), (256))}.

To demonstrate the procedure of establishing diagrams with four points we now handle case (iii) in full detail to obtain a complete list of solutions. (Recall that there were no solutions for cases (i) and (ii).) We start with the diagram a  b and add a fourth point d; in terms of the diagrams we have

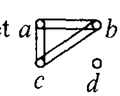

As the triangle $\{a, b, c\}$ contains a 3-fold connection we obtain

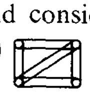
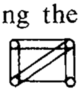
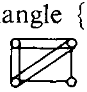

Thus, we finally get the following three cases:

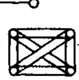
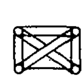
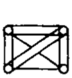
(a)  (b)  (c) 

In case (a) the above arguments show $(a, b, c) = ((123), (132), (145))$ and $d = c^{-1}$, thus yielding the unique solution $\{(123), (132), (145), (154)\}$. In case (b) the triangle diagram (b, c, d) has two solutions, as shown in (vi) above. Furthermore, we have $a = b^{-1}$ and so we have precisely the two solutions $\{(123), (124), (156), (165)\}$ and $\{(123), (124), (345), (354)\}$. For lexicographical reasons these solutions are changed via conjugation into $\{(123), (132), (145), (146)\}$ and $\{(123), (132), (145), (245)\}$, respectively.

Similarly, in case (c) we obtain three solutions: $\{(123), (132), (145), (167)\}$, $\{(123), (132), (145), (246)\}$ and $\{(123), (132), (145), (256)$.

Next we add a point to the triangle of case (iv), and thus we get 

and considering the triangle $\{b, c, d\}$ we have the following three cases:
(α)  (β)  (γ) 

Case (α) yields the diagrams  or  In case (β) we obtain  or  where the first solution is obviously the same as the second one of case (α).

The rest of the considerations are left to the reader.

Table I gives a complete list of compatible sets $S \in \mathscr{S}$ (where 1 is deleted) consisting of four elements of order 3. The third column gives the number of elements $f \in F$ such that $S \cup \{f\}$ is compatible.

As pointed out before, a computer finds in a few seconds all $M \in \mathscr{M}$ with $S \subseteq M$ in each of these 32 cases. It turns out that, except for the Desarguesian plane, all known translation planes of order 16 have a coordinate set with at least five elements of order 3. Thus they all show up during the above computation.

This gives rise to the following inductive procedure indicated in step I of Section 2.

Assume all translation planes of order 16 with a coordinate set with at least n elements of order 3 have been determined. Then consider up to conjugacy sets $S \in \mathscr{S}_0$ such that $|S| = n + 1$ and S contains precisely $n - 1$ elements of order 3. According to Section 2 the set $S - \{1\}$ describes n additional points on l_∞ with respect to $(\infty, 0, 1)$. Changing coordinates means that we now consider compatible sets $\bar{S}$ with $\bar{S} \sim S$. As pointed out in Section 2, any plane $\mathscr{P}(M)$ or $\mathscr{P}^*(M)$ with $S \subseteq M$ can be described by a coordinate set $\bar{S}$ with $\bar{S} \subseteq \bar{M}$, and vice versa. The effect of this 'changing of coordinates' is shown in the following example:

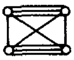
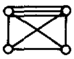
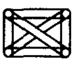
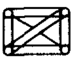
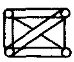
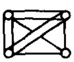
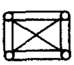
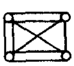
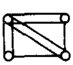
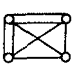
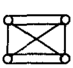
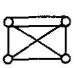
Choose $S = \{1, (123), (124), (13425)(678)\}$. Using 2.1 and 2.5 we get $S + 1 = \{1, (132), (142), (14532)\}$ which, by inversion, is equivalent to $\{1, (123), (124), (12354)\}$. Further, $(S + 1)(132)^{-1} = \{1, (123), (143), (145)\}$.

This example shows two major tools by which to reduce the possibilities for any compatible set S with $n + 1$ elements where $n - 1$ of them have order 3. The following shows the effect in our example:

There are 492 elements in F of order not equal to 3 which are compatible with $\{(123), (124)\}$. Up to conjugacy only 27 of these are different, and by the procedure just explained only two possibilities survive, namely (23)(45)(678) and (123)(45)(78). For both cases $S' = \{1, (123), (124), (23)(45)(678)\}$ and $S'' = \{1, (123), (124), (132)(45)(78)\}$ there are 123 compatible elements of order 5, 6 or 15 in F. Although this number is greater than 100 we started the appropriate program, and after $1\frac{1}{2}$ minutes of computing time we obtained four coordinate sets containing S', and none containing S''. A further example where all $M \in \mathscr{M}$ are determined with $S \subseteq M$ for a $S \in \mathscr{S}_0$, is given in full detail in the Appendix.

As this example illustrates, the philosophy always is: If $|S|$ is small, then using equivalence operations there are only a few possibilities for essentially

TABLE I

Diagram	Solutions S	No. of compatible elements
	(123), (132), (145), (154)	60
	(123), (132), (145), (146)	48
	(123), (132), (145), (245)	76
	(123), (132), (145), (167)	36
	(123), (132), (145), (246)	38
	(123), (132), (145), (256)	48
	(123), (124), (125), (126)	172 (but $N_{S_8}(S)$ is large!)
	No solution	
	(123), (124), (125), (246)	84
	(123), (124), (125), (167)	82
	(123), (124), (125), (345)	64
	(123), (124), (235), (245)	60
	(123), (124), (235), (246)	42
	(123), (124), (235), (154)	58
	(123), (124), (235), (163)	84
	(123), (124), (235), (156)	42
	(123), (124), (235), (165)	36
	(123), (124), (235), (267)	34
	(123), (124), (156), (157)	44
	(123), (124), (156), (176)	60
	(123), (124), (156), (256)	48
	(123), (124), (345), (346)	48
	(123), (124), (156), (178)	24
	(123), (124), (156), (257)	40
	(123), (124), (156), (267)	60
	(123), (124), (156), (275)	32
	(123), (124), (156), (354)	42
	(123), (124), (156), (364)	32
	(123), (145), (167), (246)	22
	(123), (145), (167), (247)	28
	(123), (145), (246), (256)	48
	(123), (145), (256), (346)	40
	(123), (145), (256), (364)	52

extending S by a further compatible element. If $|S|$ is large enough, then the number of elements which are compatible with S is small – in general, not greater than 100 – and so all possible coordinate sets can be computed.

All this is still valid even where there are no elements of order 3, i.e. only elements of order 5, 6 or 15 show up. By Lemmas 2.1 and 2.5 we may assume that any such coordinate set either contains the element (12345) or consists of elements of order 6 only. Using Lemma 2.1 the elimination of the second case can easily be done by hand and is left to the reader.

Since the problem of finding coordinate sets M with no coordinate set M' equivalent to M containing elements of order 3 turns out to be a crucial and delicate point, we give some further details.

There are precisely 1536 elements in F which are compatible with (12345) and have order not equal to 3. Up to conjugacy only 72 of these are different and, using equivalence, we finally obtain that there are only 13 essentially different extensions of (12345), namely (46857), (34567), (36578), (36475), (23)(465)(78), (23564), (23546), (24657), (24)(35)(678), (247) (36)(58), (26534), (265)(34)(78) or (13524). In any of these 13 case we applied the same procedure again:

(i) Determine all permutations in F which are compatible now with two elements.

(ii) Use conjugacy arguments and 2.1 to get all essentially different compatible sets of three elements.

(iii) Select the permutations which are compatible with these three elements (in all cases there are almost 100 of them!) and determine all possible coordinate sets.

5. Identification

When all steps indicated in the previous section are done we have almost 200 different coordinate sets. Thus the problem is to find out which translation planes of order 16 are described by these sets. The following procedure is found to be effective:

Take a distinct coordinate set for any of the known translation planes of order 16, e.g. see Section 3 and apply 2.1. After less than 7 seconds of computation the machine presents a list of many coordinate sets for each of the known planes. Clearly, this is done only for six of the eight known translation planes of order 16: the Desarguesian plane has, up to conjugacy, a unique coordinate set and conjugating a coordinate set of the Lorimer–Rahilly plane with an element of $S_8 - A_8$ will give us a coordinate set of the Johnson–Walker plane.

Next we install three lists for any of these six known planes. In the first we have listed the number of elements of order 3, 5, 6 or 15 that occur in any of these coordinate sets; in the second list we collect short descriptions of some peculiarities of any of the coordinate sets; while in the third list we have coordinate sets themselves. We now proceed with the final identification.

Given any coordinate set, look at the number of elements of order 3, 5, 6 or 15 and compare this distribution with the first list. If one finds such a distribution, then compare the coordinate set with the short description given in the second list. If this coincides with the given coordinate set then look at the appropriate set in the third list and find a permutation in S_8 conjugating the two sets. Note that we have not tried to make the three lists complete! In all cases it turns out that by doing the procedure with another equivalent coordinate set we finally find a conjugate set in the third list.

We illustrate the described identification procedure by an example:

Let us assume that we have to identify the following plane: (246), (264), (578)(13)(46), (375)(18)(46), (358)(17)(24), (387)(15)(46), (358)(17)(46), (578)(13)(24), (578)(13)(26), (358)(17)(26), (375)(18)(26), (387)(15)(24), (375)(18)(24), (387)(15)(26).

We first look at the distribution of elements and see 0–12–0–2, i.e. 0 elements of order 15, 12 elements of order 6, 0 elements of order 5, and 2 elements of order 3. Next we compare 0–12–0–2 with the first list and see that at least the Lorimer–Rahilly plane, the Dempwolff plane, and the derived semifield plane have coordinate sets with this pattern. Thus we next compare the given coordinate set with the corresponding descriptions in the second list.

In the case of the Lorimer–Rahilly plane we find that the elements of order 3 are inverse (which we have!). The 12 elements of order 6 are pairwise inverse (which we do not have!). Here we stop with the Lorimer–Rahilly plane and consider the Dempwolff plane. The elements of order 3 are inverse (which we have!). There are four different 3-cycles in the elements of order 6, and each occurs three times (which we have!). There are 11 distinct transpositions in the elements of order 6 (which we do not have!). Thus, finally we consider the derived semi-field plane to find the same description, but for the last point we have instead: elements with a common 3-cycle also have a common 2-cycle (which we have!). Therefore, we take the precise coordinate set for this plane from the third list, namely (123), (132), (578)(46)(13), (578)(46)(12), (578)(46)(23), (487)(56)(12), (487)(56)(13), (487)(56)(23), (458)(67)(12) (458)(67)(13), (458)(67)(23) and finally try to identify via conjugation. In fact (16342) fits our requirement.

In the pattern of this example we get the proof of the following result:

THEOREM. *Let $\mathscr{P}$ be a translation plane of order* 16, *then $\mathscr{P}$ is isomorphic to one of the planes described in Section* 3.

6. APPENDIX

Here we present a characteristic example in which a compatible subset $S \in \mathscr{S}$ was completed to the coordinate sets $M \in \mathscr{M}$ with $S \subseteq M$. This example was the only case, however, where the Desarguesian plane could and did show up!

There are 181 elements in F compatible with $S = \{1, (123), (132), (45678)\}$. Acting with $N_{S_8}(S)$ we see that only 17 of these lead to essentially different extensions. Obviously the number of compatible elements for S together with any one of these elements comes down to a quantity that can be managed by the machine. These 17 cases were computed together, and after less than $1\frac{1}{2}$ minutes of computing time the following list of planes was obtained. (Note that, in the following table, we always delete the subset S from each coordinate set. We also use the following abbreviations: dsf = derived semi-field plane of order 16, Hall = Hall plane of order 16, Des = Desarguesian plane of order 16.)

Coordinates	Type of plane
(45687), (45768), (46785), (46578), (46857), (47586), (48567), (123)(47586), (123)(48765), (132)(47586), (132)(48765)	Hall
(45687), (45768), (46857), (46587), (48576), (123)(47865), (123)(47586), (123)(48675), (475)(12836), (475)(16238), (475)(18263)	dsf
(45687), (45768), (46857), (46587), (48576), (132)(47865), (132)(47586), (132)(48675), (475)(13826), (475)(16328), (475)(18362)	dsf
(45687), (45867), (46875), (46587), (46758), (47568), (48576), (123)(47865), (123)(48576), (132)(47865), (132)(48576)	Hall
(45687), (45867), (46578), (46758), (47586), (123)(47685), (123)(48765), (123)(48576), (485)(12736), (485)(16237), (485)(17263)	dsf
(45687), (45867), (46578), (46758), (47586), (132)(47685), (132)(48765), (132)(48576), (485)(13726), (485)(16327), (485)(17362)	dsf
(45687), (46785), (46758), (47568), (47586), (123)(48765), (123)(48576), (123)(48657), (576)(12438), (576)(14283), (576)(18234)	dsf
(45687), (46785), (46758), (47568), (47586), (132)(48765), (132)(48576), (132)(48657), (576)(13428), (576)(14382), (576)(18324)	dsf

Coordinates	Type of plane
(45687), (46875), (46857), (48567), (48576), (123)(47865), (123)(47586), (123)(47658), (586)(12437), (586)(14273), (586)(17234)	dsf
(45687), (46875), (46857), (48567), (48576), (132)(47865), (132)(47586), (132)(47658), (586)(13427), (586)(14372), (586)(17324)	dsf
(45786), (45768), (46587), (47685), (47856), (47568), (48576), (123)(46587), (123)(48675), (132)(46587), (132)(48675)	Hall
(45786), (45768), (47856), (47586), (48567), (123)(46875), (123)(46587), (123)(48765), (465)(12837), (465)(17238), (465)(18273)	dsf
(45786), (45768), (47856), (47586), (48567), (132)(46875), (132)(46587), (132)(48765), (465)(13827), (465)(17328), (465)(18372)	dsf
(45786), (45867), (46875), (46587), (46758), (47685), (47856), (47568), (48765), (48576), (48657)	Hall
(45786), (45867), (46875), (47856), (48567), (48576), (48657), (123)(46875), (123)(47658), (132)(46875), (132)(47658)	Hall
(45786), (45867), (46578), (46587), (46758), (47685), (48657), (123)(47685), (123)(48756), (132)(47685), (132)(48756)	Hall
(45786), (45867), (46578), (47586), (48567), (123)(46875), (123)(47685), (123)(48765), (687)(12534), (687)(14235), (687)(15243)	dsf
(45786), (45867), (46578), (47586), (48567), (132)(46875), (132)(47685), (132)(48765), (687)(13524), (687)(14325), (687)(15342)	dsf
(45786), (45867), (46758), (47856), (47568), (47586), (48765), (123)(46857), (123)(48765), (132)(46857), (132)(48765)	Hall
(45786), (45867), (678)(24)(35), (132)(46875), (132)(46857), (132)(47685), (132)(47658), (132)(48765), (132)(48756), (678)(14)(25), (678)(15)(34)	dsf
(45786), (45867), (678)(25)(34), (123)(46875), (123)(46857), (123)(47685), (123)(47658), (123)(48765), (123)(48756), (678)(14)(35), (678)(15)(24)	dsf
(45786), (47856), (456)(27)(38), (132)(46857), (132)(46875), (132)(46587), (132)(47658), (132)(48765), (132)(48675), (456)(17)(28), (456)(18)(37)	dsf
(45786), (47856), (456)(28)(37), (123)(46875), (123)(46857), (123)(46587), (123)(47658), (123)(48765), (123)(48675), (456)(17)(38), (456)(18)(27)	dsf
(46857), (46875), (46587), (47685), (48765), (48576), (48657), (123)(45678), (123)(47586), (132)(45678), (132)(47586)	Hall
(46875), (48576), (586)(23)(47), (586)(12)(47), (123)(45678), (123)(46875), (123)(48576), (132)(45678), (132)(46875), (132)(48576), (586)(13)(47)	dsf

Coordinates	Type of plane
(46875), (48576), (123)(45678), (123)(46875), (123)(48576), (132)(47865), (132)(47586), (132)(47658), (13472), (13247), (14732)	dsf
(46875), (48576), (123)(47865), (123)(47586), (123)(47658), (12347), (12473), (132)(45678), (132)(46875), (132)(48576), (14723)	dsf
(46875), (48657), (476)(23)(58), (476)(12)(58), (123)(45678), (123)(46875), (123)(48657), (132)(45678), (132)(46875), (132)(48657), (476)(13)(58)	dsf
(46875), (48657), (123)(45678), (123)(46875), (123)(48657), (132)(45876), (132)(47586), (132)(47658), (13582), (13258), (15832)	dsf
(46875), (48657), (123)(45876), (123)(47586), (123)(47658), (12358), (12583), (132)(45678), (132)(46875), (132)(48657), (15823)	dsf
(46857), (47586), (48765), (123)(45678), (123)(46857), (123)(47586), (123)(48765), (132)(45678), (132)(46857), (132)(47586), (132)(48765)	Des
(46857), (123)(45678), (123)(46857), (132)(45876), (132)(46857) (132)(47865), (132)(47586), (132)(47658), (132)(48765), (132)(48675), (132)(48756)	Hall
(46857), (123)(45876), (123)(46857), (123)(47865), (123)(47586), (123)(47658), (123)(48765), (123)(48675), (123)(48756), (132)(45678), (132)(46857)	Hall
(47586), (123)(45678), (123)(46875), (123)(46857), (123)(46587), (123)(47687), (123)(48765), (123)(48576), (123)(48657), (132)(45678), (132)(47586)	Hall
(47586), (123)(45678), (123)(47586), (132)(45678), (132)(46875), (132)(46857), (132)(46587), (132)(47685), (132)(48765), (132)(48576), (132)(48657)	Hall
(45786), (47856), (456)(27)(38), (132)(46875), (132)(46857), (132)(46587), (132)(47658), (132)(48765), (132)(48675), (456)(17)(28), (456)(18)(37)	dsf

REFERENCES

1. Dempwolff, U. and Reifart, A., 'Translation Planes of Order 16 Admitting a Baer 4-Group', *J. Comb. Theory (A)* **32** (1982), 119–124.
2. Dembowski, P., *Finite Geometries*, Springer-Verlag, Berlin, Heidelberg, New York, 1968.
3. Hall, M., 'Projective Planes', *Trans. Amer. Math. Soc.* **54** (1943), 229–277.
4. Hall, M., Swift, J. D. and Walker, J. R., 'Uniqueness of Projective Planes of Order Eight', *Tables Aides Comput.* **10** (1956), 186–194.
5. Huppert, B., *Endliche Gruppen I*, Springer, Berlin, Heidelberg, New York, 1967.
6. Johnson, N. L., 'A Note on the Derived Semifield Plane of Order 16', *Aequat. Math.* **18** (1978), 103–111.
7. Johnson, N. L., and Ostrom, T. G., 'Tangentially Transitive Planes of Order 16', *J. Geometry* **10** (1977), 146–163.

8. Johnson, N. L. and Ostrom, T. G., 'The Translation Planes of Order 16 that Admit PSL (2, 7)', *J. Comb. Theory (A)* **26** (1979), 127–134.
9. Knuth, D., 'Finite Semifields and Projective Planes', *J. Algebra* **2** (1965), 182–217.
10. Lorimer, P., 'A Projective Plane of Order 16', *J. Comb. Theory (A)* **16** (1974), 334–347.
11. Lüneburg, H., *Translation Planes*, Springer Berlin, Heidelberg, New York, 1981.

Authors' address:

U. Dempwolff and A. Reifert,
Mathematisches Institut,
Universität Kaiserslautern,
Erwin-Schrödinger-Straße,
6750 Kaiserslautern,
F.R. Germany.

(Received December 3, 1982)