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PACKING CONVEX BODIES IN THE PLANE WITH DENSITY GREATER THAN 3/4

1. Introduction

A *convex body* is a compact convex set with non-empty interior. A collection of convex bodies in the plane with mutually disjoint interiors, and such that each of these bodies in congruent to a given body K, is called a *packing* of the plane with (copies of) K. If a packing of the plane with K entirely covers the plane, then the packing is called a *tessellation* and we say that K *tessellates* the plane. Let us say that a packing is *uniform* if there exists a tessellation whose every member contains exactly one member of the packing. The *density* of a packing is a real number between 0 and 1 which, intuitively speaking, is supposed to represent the ratio between the sum of the areas of the bodies used for the packing and the area being packed. In the general case, a formal definition of the density of packing of the plane would be somewhat cumbersome, but in the case of uniform packings, it is quite natural to assume it to be the ratio between the area of the body used for the packing and the area of a body which contains it and which tessellates the plane in the proper manner, associated with the packing. The question of the uniqueness of the area of the tessellating body is settled without much trouble. Since in this paper we will deal with uniform packings only, the general definition of density will be left aside. G. D. Chakerian and L. H. Lange [1] proved that every convex body K is contained in a quadrilateral of area at most $\sqrt{2}$ times that of K. Since every quadrilateral tessellates the plane, they concluded that every convex body admits a (uniform) packing of the plane with density at least $\sqrt{2}/2$.

In this paper, as indicated by its title, we will prove the following claim:

THEOREM. *Every convex plane body admits a uniform packing of the plane with density greater than* 3/4.

A result of I. Fáry ([3], or see L. Fejes Tóth's monograph [4], pp. 100–102) should be mentioned here, which states that each convex body can be lattice packed in the plane with density at least 2/3. For lattice packings, which only allow translations of the given body, Fáry's result is the best possible. See also R. Courant's paper [2] for an elegant, elementary proof of Fáry's theorem.

The idea of the proof of our theorem is as follows. We will be concerned with a special kind of hexagon, which we will call a *p-hexagon*. A *p*-hexagon is a hexagon with a pair of parallel and equal length opposite sides. 'Opposite' here means separated by exactly two other sides. Given a convex body K,

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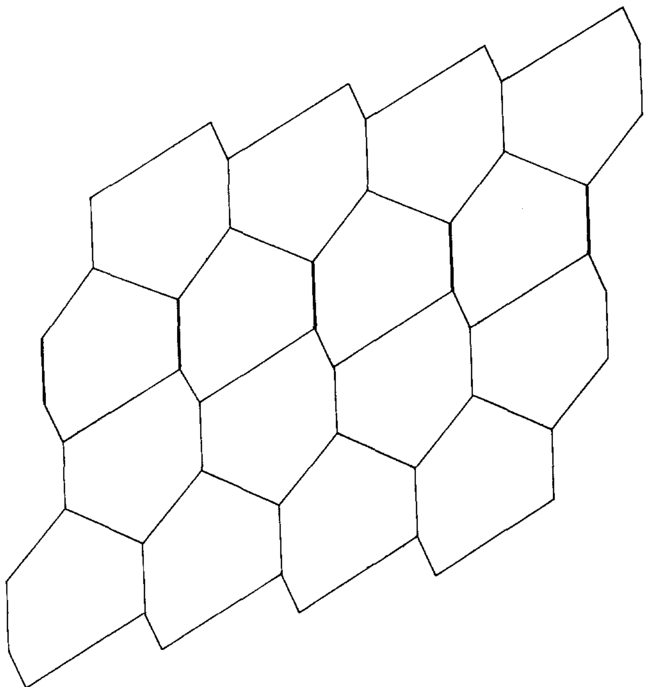

Figure 1.

we will construct a certain p-hexagon H containing K and such that $|H| \leqslant \frac{4}{3}|K|$ ($|S|$ denotes the area of a set S), where the equality only occurs in case K itself is a p-hexagon. Thus, we always can enclose K in a p-hexagon whose area is less than 4/3 times that of K. This is all we need to prove our theorem, since every p-hexagon tessellates the plane. Figure 1 illustrates the tessellation: the hexagons are arranged in rows, each row consisting of the translates of the hexagon, and the adjacent rows being turned upside down with respect to each other.

Remark. Every pentagon with a pair of parallel sides, every quadrilateral and every triangle can be considered a (degenerate) p-hexagon. Each of these figures tessellates the plane in the same manner as a non-degenerate p-hexagon does.

2. The construction of the p-hexagon

Let K be the given convex body and let θ be an arbitrary direction (i.e., a unit vector) in the plane. Denote by m_1 and m_2 the two lines parallel to θ and supporting K, making sure that when θ varies, both m_1 and m_2 depend on θ in a continuous manner. One way to do it is to consider θ to be north

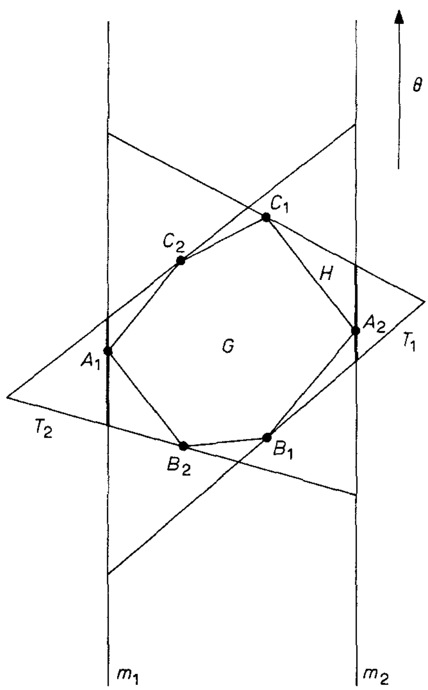

Figure 2.

and let m_1 always be west of m_2. Let T_i (for $i = 1, 2$) be a triangle containing K, with one side on m_i and of minimum area. Let H be the intersection $T_1 \cap T_2$, which is a polygon with at most six sides.

We will now be concerned with the area of H, and, at the same time, fixing θ so that H is a p-hexagon.

Let s_i be the length of the side of T_i which lies on $m_i (i = 1, 2)$, and let h_i be the altitude of T_i perpendicular to θ. Let A_i be a point at which m_i touches K and let B_i and C_i be the midpoints of the sides of T_i which are not parallel to θ (See Figure 2). It is quite easy to show (just as in [5]) that the points B_i and C_i belong to K. Notice now that $\overline{B_iC_i}$ is the unique chord of K which is parallel to m_i and which has the property that the (or some) pair of lines supporting K at the end points of that chord bound a segment on m_i twice the length of that chord. This implies that s_i is uniquely determined by θ, and since the chord $\overline{B_iC_i}$ and the area of T_i depend on θ in a continuous manner, so do s_i and h_i. That in turn yields that the lengths of the sides of H which lie on m_1 and m_2 depend continuously on θ. Therefore there exists a θ such that those two sides of H are of equal length. From now on we will assume θ to be fixed in that specific direction, which makes H a p-hexagon.

Let us notice that the polygon $G = A_1B_2B_1A_2C_1C_2$ (see Figure 2) is contained in K. This yields that $|G| \leqslant |K|$.

Since the area of G equals to the sum of the areas of the quadrilaterals $A_1B_2C_1C_2$ and $B_1A_2C_1C_2$, we get

(1) $$|G| = \tfrac{1}{8}(s_1h_2 + s_2h_1).$$

In the following section we will prove that (i) $|H| \leqslant \frac{4}{3}|G|$, and the equality occurs only if G is a p-hexagon. This is sufficient to prove our theorem, since $|G| \leqslant |K|$ and the equality occurs only if $G = K$.

3. A REDUCTION OF THE GEOMETRIC PROBLEM TO AN ALGEBRAIC ONE AND ITS SOLUTION

Let m be a line perpendicular to θ, denote by P_i (for $i = 1, 2$) the intersection point $m \cap m_i$, denote by Q_i the perpendicular projection on m of the vertex

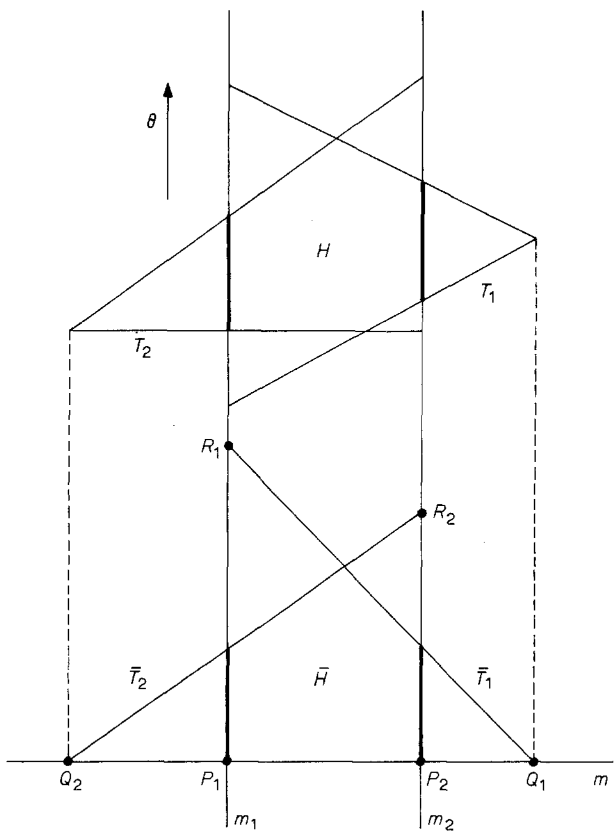

Figure 3.

of T_i which does not lie on m_i and let R_i be a point on m_i whose distance from P_i equals s_i and such that R_1 and R_2 lie on the same side of m (see Figure 3). Let $\bar{T}_i$ be the triangle whose vertices are P_i, Q_i and R_i, and let $\bar{H} = \bar{T}_1 \cap \bar{T}_2$. It is easy to notice that $|H| \leqslant |\bar{H}|$ and that the sides of $\bar{H}$ that lie on m_1 and m_2 are of the same length as those of H. Now, instead of (i), we will prove the following:

(ii) $|\bar{H}| \leqslant \frac{4}{3}|G|$, and the equality occurs only if G is a p-hexagon

from which (i) follows immediately.

Let D denote the intersection point of the hypotenuses of $\bar{T}_1$ and $\bar{T}_2$ and let D_0 be the perpendicular projection of D on m (see Figure 4). Denote the length of the resulting segments on m as follows: $a = Q_2P_1$, $b = P_1D_0$, $c = D_0P_2$, and $d = P_2Q_1$. All of these four numbers are, of course, non-negative, and $a + b > 0$ and $c + d > 0$ or else the triangles $\bar{T}_1$ and $\bar{T}_2$ would not exist. Also, $b + c > 0$ and $DD_0 > 0$, since K has an interior point. Without loss of generality we can assume that $b + c = 2$ and $DD_0 = 2$, since this can be obtained by an application of a suitable affine transformation preserving the right angle between m and θ. Let $x = b - 1$ and get

(2) $b = 1 + x, c = 1 - x, -1 < x < 1.$

The fact that the sides of $\bar{H}$ parallel to θ are of equal length yields that

(3) $$\frac{a}{a+b} = \frac{d}{c+d},$$

from which it follows that $b > 0$ and $c > 0$ and

(4) $$\frac{a}{b} = \frac{d}{c}.$$

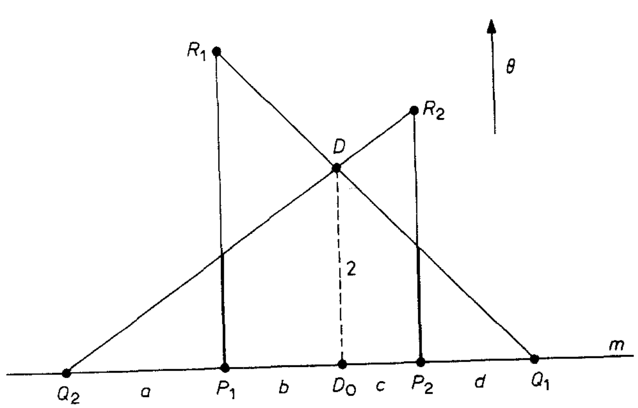

Figure 4.

Then let $y = a/b = d/c (y \geqslant 0)$, and, by (2), get

$$(5) \qquad a = (1+x)y, d = (1-x)y.$$

A further analysis of the configuration of the triangles $\bar{T}_1$ and $\bar{T}_2$ yields the following:

$$(6) \qquad h_1 = b + c + d = 2 + (1-x)y,$$

$$(7) \qquad h_2 = a + b + c = 2 + (1+x)y,$$

$$(8) \qquad s_1 = 2\frac{b+c+d}{c+d} = 2 + 2\frac{1+x}{(1-x)(y+1)}$$

$$(9) \qquad s_2 = 2\frac{a+b+c}{a+b} = 2 + 2\frac{1-x}{(1+x)(y+1)}$$

and

$$(10) \qquad |\bar{H}| = \frac{1}{2}\left(\frac{2a}{a+b} + 2\right)b + \frac{1}{2}\left(\frac{2d}{c+d} + 2\right)c = \frac{4y+2}{y+1}.$$

Also, (1) and (6)–(9) produce

$$(11) \qquad |G| = \frac{(2+y)^2 - (xy)^2}{2(1-x^2)(y+1)}.$$

From (10) and (11) we get directly

$$(12) \qquad 4|G| - 3|H| = 2\left[\frac{(y-1)^2}{y+1} + \frac{4x^2}{1-x^2}\right],$$

where $y \geqslant 0$ and $x^2 < 1$. Hence $4|G| - 3|H| \geqslant 0$ and the equality occurs only if $y = 1$ and $x = 0$. Under these conditions we get $a = b = c = d$, and by (2) and (3), $s_1 = s_2$. In that case $B_1C_1 = B_2C_2$, since $B_iC_i = \frac{1}{2}s_i$ (see Figure 2), and G is a p-hexagon. This proves (ii) and the Theorem from Section 1.

4. Conclusion

By an argument of a topological nature, the Theorem whose proof has just been completed can be somewhat strengthened. Since the collection of affine equivalence classes of all convex plane bodies of area 1 is a compact set, and since the function assigning to each such equivalence class the minimum area of a p-hexagon containing a representative of that class is continuous, there exists a minimum value for that function, taken on a specific element of that compact set. Let us denote that minimum value by Δ. We have proved in this paper that $\Delta < \frac{4}{3}$, thus we can conclude that there exists a number $d > \frac{3}{4}$ (namely $d = \Delta^{-1}$) such that every convex body can be packed in the plane with density at least d. The value of Δ remains unknown.

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