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SMOOTH APPROXIMATION OF POLYHEDRAL SURFACES REGARDING CURVATURES*

ABSTRACT. In this paper we prove that every closed polyhedral surface in Euclidean three-space can be approximated (uniformly with respect to the Hausdorff metric) by smooth surfaces of the same topological type such that not only the (Gaussian) curvature but also the absolute curvature and the absolute mean curvature converge in the measure sense. This gives a direct connection between the concepts of total absolute curvature for both smooth and polyhedral surfaces which have been worked out by several authors, particularly N. H. Kuiper and T. F. Banchoff.

1. CURVATURE OF POLYHEDRAL SURFACES

We consider a *polyhedral surface* M in Euclidean three-space which is defined to be a compact subset $M \subseteq E^3$ which is both

(1) a polyhedron in the usual sense (cf [11] for instance),
(2) homeomorphic to a connected two-dimensional manifold with or without boundary.

This means that there exists a finite triangulation of M such that each edge is contained in a certain (affine) line and each face is contained in a certain (affine) plane. In general, we assume that the faces of M are 3-gons but sometimes we will allow convex n-gons which, of course, can be subtriangulated into 3-gons in a suitable manner. Note that all considerations in this article also hold for immersed surfaces, where immersion means local embedding.

For a vertex $p \in M$ we define the *star* st(p) to be the union of all (closed) 3-gons containing p, and the *link* lk(p) to be the boundary of st(p) in the relative topology of M. Topologically, st(p) is a closed disk, lk(p) is a circle in case $p \in M \setminus \partial M$ and an interval in case $p \in \partial M$. Of course, st(p) consists of a certain number – say m_p – of 3-gons which have certain (positive) angles at p, say $\alpha_i(p)$, $i = 1, \ldots, m_p$. Then the *curvature* $K(p)$ of M at p is defined by

$$(1) \qquad K(p) := \begin{cases} 2\pi - \sum_i \alpha_i(p) & \text{if } p \in M \setminus \partial M \\ \pi - \sum_i \alpha_i(p) & \text{if } p \in \partial M. \end{cases}$$

Roughly speaking, the curvature measures the deviation of M from a flat (i.e., Euclidean) surface. Of course, K is intrinsically defined in the sense

* The present paper is a detailed version of the short announcement [3].

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that it depends only on the inner metric of M (PL-Riemannian metric, for details see [14]).

Now let v_0, e_0, f be the number of vertices, edges and 3-gons in $M \setminus \partial M$, v_1 and e_1 the number of vertices and edges in ∂M, then by the obvious equations $2e_0 + e_1 = 3f$ and $v_1 = e_1$ we can derive directly the *Gauss–Bonnet equation* for the *total curvature*:

$$\begin{aligned}(2)\qquad \sum_{p \in M} K(p) &= 2\pi \cdot v_0 + \pi \cdot v_1 - \sum_p \sum_i \alpha_i(p) \\ &= \pi(2v_0 + v_1 - f) \\ &= \pi(2v_0 + v_1 - 2e_0 - e_1 + 2f) \\ &= 2\pi \chi(M).\end{aligned}$$

The following gives some motivation for the fact that the curvature K defined above can be considered as a suitable analogue of the Gaussian curvature of a smooth surface:

Let $U(p)$ be a small neighborhood of a vertex $p \in M \setminus \partial M$ which topologically is a disk. Obviously there is a smooth approximation of $\mathrm{st}(p) \setminus U(p)$ by suitable cylinders with zero curvature; inside of $U(p)$ let us choose an arbitrary smooth approximation (with respect to the Hausdorff metric). In this manner $\mathrm{st}(p)$ can be approximated by a smooth surface $F(p)$ where $\partial F(p)$ is a geodesic m_p-gon with sum of the interior angles $\pi m_p - \sum_i \alpha_i(p)$. Hence, the classical Gauss–Bonnet formula for $F(p)$ says that the total curvature of $F(p)$ is just

$$\int_{F(p)} K \, do = \pi m_p - \sum_i \alpha_i(p) - (m_p - 2)\pi = K(p).$$

So we see that $K(p)$ is nothing but the total Gaussian curvature (concentrated in one point) of an approximating smooth surface. Clearly this approximation can be done globally, which leads to the following.

PROPOSITION 1. *Let M be a compact polyhedral surface without boundary. Then there exists a sequence $(M_n)_{n \in \mathbb{N}}$ of smooth surfaces in E^3 (each M_n homeomorphic to M) such that*

(i) $M_n = M$ *outside of the* $1/n$*-neighborhood of the* 1*-skeleton of* M,

(ii) $M_n \underset{n \to \infty}{\longrightarrow} M$ *with respect to the Hausdorff metric,*

and such that in addition the following curvature convergence property (CCP) *is satisfied for the Gaussian curvature* K:

CCP(K) *for every open set $U \subseteq E^3$ where ∂U contains no vertex of M the following sequence converges*

$$\int_{U \cap M_n} K_n \, \mathrm{d}o_n \underset{n \to \infty}{\longrightarrow} \sum_{p \in U \cap M} K(p)$$

where K_n and $\mathrm{d}o_n$ *denote the Gaussian curvature and the area element of M_n respectively.*

Note that CCP(K) can be interpreted as a weak convergence with respect to the curvature measure where we mean by *curvature measure* in the polyhedral case the discrete Dirac measure in the vertices weighted with the curvature and in the smooth case the area measure weighted with the Gaussian curvature regarded as a function.

Of course, CCP makes sense not only for the Gaussian curvature K but also for other curvatures, e.g., the absolute Gaussian curvature or the mean curvature. Nevertheless, it is much more difficult to get a sequence satisfying CCP for the absolute Gaussian curvature because there is no theorem of the Gauss–Bonnet type in this case.

The main theorem of the present paper proves the analogue of Proposition 1 for the absolute Gaussian curvature, the mean curvature and the absolute mean curvature.

For a smooth oriented surface without boundary, the *mean curvature measure* is defined to be the area measure weighted with the mean curvature regarded as a function. For an edge e of a polyhedral-oriented surface without a boundary, let $H(e)$ denote the signed angle between the outer normals of the two adjacent faces ($-\pi < H(e) < \pi$) which has to be taken as negative if there are outer normals which meet (we call the edge 'concave' in this case). Then the *mean curvature measure* $\int H$ for an oriented polyhedral surface can be defined by

$$\int_U H := \sum_e H(e) \cdot \mathrm{length}(e \cap U)$$

for a Borel set U in M, where the sum ranges over all edges e in M. If the orientation is changed, then $\int H$ changes its sign. The *absolute mean curvature measure* $\int |H|$, defined by

$$\int_U |H| := \sum_e |H(e)| \cdot \mathrm{length}(e \cap U)$$

does not depend on the orientation and, therefore, it can also be defined for nonorientable immersed surfaces.

2. Absolute curvature of polyhedral surfaces

By analogy with the case of Gaussian curvature considered above, in this section we want to define a suitable polyhedral analogue of the absolute Gaussian curvature $|K|$. There exists[1], for instance, an everywhere flat (i.e., $K \equiv 0$) polyhedral torus in E^3, so the absolute polyhedral curvature $|K(p)|$ would not satisfy the fundamental inequality that the total absolute curvature of a closed surface M is greater or equal to $2\pi(4 - \chi(M))$.

The most familiar proof of that inequality in the smooth case uses the measure of unit normals perpendicular to supporting planes. This motivates the following definitions for a vertex p of a polyhedral surface without a boundary:

Let $A(p) \subseteq S^2$ be the set of all exterior unit vectors perpendicular to local supporting planes of M in p and let $\gamma(p) := \text{area}(A(p))$ be the exterior angle at p. Of course, $0 \leqslant \gamma(p) \leqslant 2\pi$.

Now we decompose the curvature $K(p)$ into its 'positive' and its 'negative' part. Let us define

$$K_+(p) := \gamma(p) \qquad K_-(p) := K_+(p) - K(p),$$

then clearly $K(p) = K_+(p) - K_-(p)$ and we define the *absolute curvature* $K_*(p)$ at p by

$$K_*(p) := K_+(p) + K_-(p). \tag{3}$$

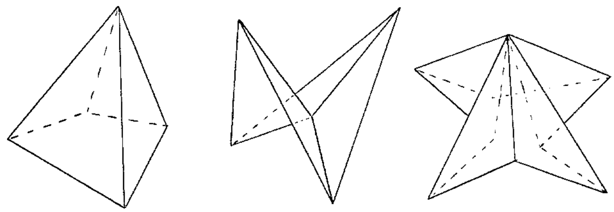
Another interpretation of $K_+(p)$ and $K_-(p)$ is the following:

1st case: Let p be a vertex having some local supporting plane of M in p. Then p lies on the boundary of the convex hull of $\text{st}(p)$. Let $\text{sth}(p)$ denote the star of p in the boundary of the convex hull of st(p). $A(p)$ for M is the same as for $\text{sth}(p)$, hence $\gamma(p)$ for M equals $\gamma(p)$ for $\text{sth}(p)$ and this is equal to the curvature $K(p)$ of $\text{sth}(p)$ in p because in general $\gamma(q) = K(q)$ for vertices q of a convex polytope. Hence $K_+(p)$ in M is equal to $K(p)$ in $\text{sth}(p)$. Hence $K_-(p) = K_+(p) - K(p)$ (in M) equals the sum of the interior angles of $\text{st}(p)$ at p minus the corresponding sum of $\text{sth}(p)$. We can now see easily that $K_-(p) = 0$ if and only if M is locally convex in p.

2nd case: There is no local supporting plane of M in p. Then $K_+(p) = \gamma(p) = 0$, hence $K_-(p) = -K(p) > 0$.

In any case we have $K_-(p) \geqslant 0$, $K_+(p) \geqslant 0$, $K_*(p) \geqslant 0$. The equality $K_*(p) = 0$ holds if and only if M is locally convex at p with $K(p) = 0$ which means that p

[1] This was shown by the first named author in a talk at the Oberwolfach Conference on convex bodies, 1977.



A B C

Fig. 1.

is not a proper vertex, i.e., p lies in the relative interior of a natural face or edge of M.

Summarizing, we have the following *four cases of vertices* (see Figure 1):

(A) $K = K_+ = K_* > 0$ *and* $K_- = 0$

(*purely positive curvatures: locally convex case*)

(B) $-K = K_- = K_* > 0$ *and* $K_+ = 0$

(*purely negative curvature: saddle-like case, no open set of local supporting planes*)

(C) $K_- > 0$ *and* $K_+ > 0$

(*mixed case with an open set of local supporting planes*)

(D) $K_* = 0.$

(*not a proper vertex*)

Let us define the *total absolute curvature* $\mathrm{TA}(M)$ of M by

$$\mathrm{TA}(M) := \sum_{p \in M} K_*(p) = \sum_{p \in M} (K_+(p) + K_-(p)). \tag{4}$$

By analogy with the smooth case we have:

PROPOSITION 2. *Let M be a polyhedral surface without a boundary. Then the following inequality is valid:*

$$\mathrm{TA}(M) \geqslant 2\pi(4 - \chi(M)).$$

Proof. Clearly M cannot be contained in some plane. Hence, for each unit vector $z \in S^2$ there is at least one global supporting plane of M with z as the outer normal. This implies

$$\sum_{p \in M} K_+(p) \geqslant 4\pi.$$

On the other hand, by the Gauss–Bonnet Equation (2) we have

$$\sum_p K_+(p) - \sum_p K_-(p) = 2\pi\chi(M)$$

which then implies

$$\sum_p K_*(p) \geqslant 8\pi - 2\pi\chi(M).$$

Remark. Equality holds if and only if for all vertices p with $K_+(p) > 0$, the local convex hull coincides locally with the global convex hull. Usually M is called *tight* in this case (cf. [6], for some recent combinatorial results see also [4]).

For later use let us state here the following

LEMMA 1. *$K(p)$ and $K_*(p)$ depend continuously on the vertices of M, more precisely: let all vertices in M be fixed except one vertex p_0 which will be moved, then the curvatures K and K_* at the points of $M\backslash\mathrm{st}(p_0)$ remain unchanged while at the points of $\mathrm{st}(p_0)$ they depend continuously on p_0.*

In particular, one can change M a little bit so that no two edges are collinear and no three edges are coplanar, such that the change of K and K_* is arbitrarily small.

The proof of Lemma 1 follows directly from the definitions of K and K_* because, obviously, the angles $\alpha_i(p)$ and $\gamma(p)$ depend continuously on the vertices.

To see that, in fact, K_* is a suitable analogue for the absolute curvature of a smooth surface, the same argument as in Proposition 1 will not work. Of course, an arbitrary local smoothing will, in general, produce a very high total absolute curvature. So in order to prove a similar approximation theorem for the total absolute curvature, we have to use the more specific arguments which are given in Section 4. In Section 3 we will compare our definition with some others which can be found in the literature (see [1], [2], [6], [7], [12]).

3. Morse relations for curvature and critical points

There are several approaches to the notion of curvature by the use of the average of the number of critical points – weighted with some index – of the so-called height functions. This has been done in a purely combinatorial way by T. F. Banchoff for polyhedra (cf. [1], [2]), by R. Schneider for the convex ring (cf. [12]), and in a more topological way by N. H. Kuiper for a larger class of subspaces in E^n. For our case of surfaces, we will describe these approaches shortly and show that all are equivalent.

For $z \in S^2$ let $z^* : M \to \mathbb{R}$ denote the *height function* $z^*(p) := \langle z, p \rangle$ where

$\langle\ ,\ \rangle$ denotes the Euclidean inner product. By compactness of M, almost all z^* are *regular* in the sense that $z^*(p) \neq z^*(q)$ for vertices $p \neq q$. For some regular z^* and some vertex $p \in M$, let $c := z^*(p)$ be the p-level, then $(z^*)^{-1}(c) \cap \mathrm{lk}(p)$ consists of a finite number of points, say $N(p, z)$.

In case $p \in M \backslash \partial M$, $N(p, z)$ is even, and T. F. Banchoff defined in [2] an *index* by $i(p, z) := 1 - \frac{1}{2} N(p, z)$.

In case $p \in \partial M$, let $L(p, z)$ be the number of points in $\mathrm{lk}(p) \cap \partial M$ whose z^*-level is less than c. So $N(p, z) + L(p, z)$ will be even and, similarly, we can define the index $i(p, z) := 1 - \frac{1}{2}(N(p, z) + L(p, z))$. Set $L(p, z) = 0$ for $p \in M \backslash \partial M$.

Now let us consider local supporting planes, and we see that $p \in M \backslash \partial M$ is an extremum for some z^* if and only if $N(p, z) = 0$. This leads to the equation

$$K_+(p) = \tfrac{1}{4} \int_{z \in S^2} (i(p, z) + |i(p, z)|)\, \mathrm{d}o. \tag{5}$$

On the other hand T. F. Banchoff has shown (see [2], Thm. 3)

$$K(p) = \tfrac{1}{2} \int_{z \in S^2} i(p, z)\, \mathrm{d}o, \tag{6}$$

so it follows

$$K_-(p) = \tfrac{1}{4} \int_{z \in S^2} (|i(p, z)| - i(p, z))\, \mathrm{d}o \tag{7}$$

and, of course

$$K_*(p) = \tfrac{1}{2} \int_{z \in S^2} |i(p, z)|\, \mathrm{d}o. \tag{8}$$

(6) can be considered as an analogue of the *Theorema egregium* in the smooth case, if one uses the right-hand side as an (extrinsic) definition of curvature (cf. [2]).

It can be seen easily that $\frac{1}{2}(N(p, z) + L(p, z))$ is just the number of components of the space $\{x \in \mathrm{st}(p) \mid z^*(x) < z^*(p)\}$. In that way our definition turns out to be a special case of the corresponding definitions of curvature used by R. Schneider (see [12]) and N. H. Kuiper (see [7]). Adapting Kuiper's notation from [7], we have the equations

$$\sum K_+ = 4\pi \cdot \tau_0 \qquad \sum K_- = 2\pi \cdot \tau_1$$
$$\sum K = 2\pi \cdot \tau_{\mathrm{alt}} \qquad \sum K_* = 2\pi \cdot \tau.$$

For more details see [5], for instance. The *topological critical point theory* developed by Morse in [9] and N. H. Kuiper in [7] states that for each $z \in S^2$

$$\sum_{p \in M} i(p, z) = \chi(M),$$

$$\sum_{p \in M} |i(p, z)| \geqslant \begin{cases} 4 - \chi(M) & \text{if} \quad \partial M = \varnothing \\ 2 - \chi(M) & \text{if} \quad \partial M \neq \varnothing . \end{cases}$$

This leads to alternative proofs of the Gauss–Bonnet formula and Proposition 2. The latter can be easily extended to the case $\partial M \neq \varnothing$ by

$$\mathrm{TA}(M) \geqslant 2\pi(2 - \chi(M)).$$

Note that $i(-, -)$ and $|i(-, -)|$ can be regarded as curvature measures with two arguments: a set of points in M and a set of directions in S^2. This point of view has been extensively studied by R. Schneider in [12] for the convex ring in E^n which turns out to be a suitable class of subspaces for considerations of that kind. From a similar point of view, R. Langevin has studied in [9] the case of complex algebraic hypersurfaces with isolated singularities.

4. The main theorem

Our main theorem states that Proposition 1 remains valid if we replace the Gaussian curvature of the smooth surfaces M_n by the absolute curvature or mean curvature and the curvature of the given polyhedral surface M by the absolute curvature K_* or the mean curvature.

THEOREM. *Let M be a compact polyhedral surface without boundary. Then there exists a sequence $(M_n)_{n \in \mathbb{N}}$ of smooth surfaces in E^3 (each M_n homeomorphic to M) such that*

(i) $M_n = M$ *outside of the* $1/n$*-neighborhood of the* 1*-skeleton of* M,

(ii) $M_n \xrightarrow[n \to \infty]{} M$ *with respect to the Hausdorff metric*

and such that in addition the following curvature convergence properties (CCP) are satisfied:

For every open set $U \subseteq E^3$ such that in ∂U there is no vertex of M and, at most, countably many points lying on edges of M we have for $n \to \infty$:

$$\mathrm{CCP}(K) \quad \int_{U \cap M_n} K_n \, do_n \longrightarrow \sum_{p \in U \cap M} K(p),$$

$$\text{CCP}(K_*) \int_{U \cap M_n} |K_n| \, \mathrm{d}o_n \longrightarrow \sum_{p \in U \cap M} K_*(p),$$

$$\text{CCP}(H) \int_{U \cap M_n} H_n \, \mathrm{d}o_n \longrightarrow \int_{U \cap M} H,$$

$$\text{CCP}(|H|) \int_{U \cap M_n} |H_n| \mathrm{d}o_n \longrightarrow \int_{U \cap M} |H|,$$

$$\text{CCP}(A) \int_{U \cap M_n} \mathrm{d}o_n \longrightarrow \mathit{area}(U \cap M),$$

where K_n, H_n and do_n *denote the Gaussian, the mean curvature and the area element of M_n respectively. The same remains valid if M is an orientable immersed surface (the M_n being only immersed) and it remains valid (except the* CCP(H) *which is not defined) if M is a nonorientable surface. In this case $U \cap M$ and $U \cap M_n$ can be replaced by 'local' intersections of U with M and M_n regarding the self intersections of M.*

Note that CPP may be regarded as a weak convergence with respect to the corresponding curvature measure.

The proof will be given in *three steps*:

In a *first step* (see Section 4.1), for a given polyhedral surface M we construct another one which coincides with M outside an arbitrarily small neighborhood of the vertices of M and whose vertices are only of the following three standard types:

(1) $K_-(p) = 0$ (*locally convex*)
(2) *Only* 4 *edges meet at p and M has at p no local supporting plane* (*hence* $K_+(p) = 0$) *and all interior angles at p are smaller than* π.
(3) *Only* 4 *edges meet at p and one interior angle at p is equal to* π (*hence* $K_+(p) = 0$) *and the other interior angles at p are smaller than* π.

In a *second step* (see Section 4.2), for a given polyhedral surface $\tilde{M}$ with vertices only of the three standard types, we construct a C^1 *surface* which coincides with $\tilde{M}$ outside of an arbitrarily small neighborhood of the 1-skeleton of $\tilde{M}$ and which is built up by finitely many pieces of planes, cylinders, cones, spheres and elliptic tori fitting together along pieces of straight lines and ellipses. More specifically, in the second step some neighborhood of each edge of $\tilde{M}$ (outside of an arbitrarily small neighborhood of the vertices) is replaced by an orthogonal cylinder over a piece of a circle which is perpendicular to this edge and which fits tangentially to the adjacent faces.

In a *third step* (see Section 4.3), for a given C^1 surface $\tilde{\tilde{M}}$ of that kind we construct a *smooth surface* which coincides with $\tilde{\tilde{M}}$ outside an arbitrarily small neighborhood of the union of curves where $\tilde{\tilde{M}}$ is not smooth. More specifically, in the third step each cylinder arising from the second step is replaced by an orthogonal cylinder over a piece of an embedded smooth plane curve joining two straight lines.

Each of the three steps will be done with a view to the assertions of the theorem, i.e., each of the three approximations will satisfy the curvature convergence properties in itself. So, in some sense, the assertion of the theorem will be decomposed into Propositions 3, 4, 5 in Sections 4.1, 4.2, and 4.3.

To simplify our proof the following technical remark is useful:

Remark. In each of the three steps described above we can make a certain kind of homothetic deformation: Let M be a polyhedral surface and $\tilde{M}$ be an arbitrary approximating surface which coincides with M outside some ε-neighborhood of the 1-skeleton of $M(\varepsilon > 0$ being sufficiently small) and which replaces each edge of M (outside the ε-neighborhood of the vertices of M) by an orthogonal cylinder over a certain piece of an embedded plane curve perpendicular to this edge, where this cylinder tangentially fits to the adjacent faces.

Note that this includes the first step described above, where we have to choose cylinders over an angle (in this case the edges outsides of the ε-neighborhood of the vertices of M remain unchanged). It also includes the second step and a combination of the second and third step.

Now for an arbitrary number $0 < c \leqslant 1$ define the *homothetic deformation* $\tilde{M}_c$ of $\tilde{M}$ as follows: Let $\tilde{M}_c$ coincide with $\tilde{M}$ outside the ε-neighborhood of the 1-skeleton of M. For each vertex $p \in M$ $\tilde{M}_c \cap U_{c\cdot\varepsilon}(p)$ is defined to be the homothetic contraction of $\tilde{M} \cap U_\varepsilon(p)$ to the centre p with factor c.

Outside the $U_{c\cdot\varepsilon}(p)$, each edge of M is replaced by the same orthogonal cylinder as in $\tilde{M}$, but with a homothetic contraction of factor c to this edge. This, of course, will be an orthogonal cylinder again. Obviously this contraction fits together to a surface $\tilde{M}_c$ of the same types as $\tilde{M}$ (polyhedral, C^1, smooth) which coincides with M outside of the $c\cdot\varepsilon$-neighborhood of the 1-skeleton of M.

Passing to the limit $c \to 0$ we can state the following:

LEMMA 2. *Let $M, \tilde{M}$ and $\tilde{M}_c$ be as described above. Assume that $\tilde{M}$ has finite area and finite $\int_{\tilde{M}} |H|$. Then for every open set $U \subseteq E^3$ such that ∂U contains no vertex of M and, at most, countably many points lying on edges of M, we have the following convergences:*

(1) $\lim_{c\to 0} \mathrm{d}(M, \tilde{M}_c) = 0$ *where* d(,) *denotes the Hausdorff metric,*
(2) $\lim_{c\to 0} area(U \cap \tilde{M}_c) = area(U \cap M)$,

(3) $\lim_{c\to 0}\int_{U\cap\tilde{M}_c} K = \int_{U\cap M} K$
(4) $\lim_{c\to 0}\int_{U\cap\tilde{M}_c} H = \int_{U\cap M} H.$
(5) $\lim_{c\to 0}\int_{U\cap\tilde{M}_c}|H = \int_{U\cap M}|H|$ *provided that each cylinder in* $\tilde{M}$ *replacing some edge of* M *is a cylinder over a piece of an embedded convex curve.*

Proof. (1) is clear by construction.

For each vertex $p\in M$ we have

$$\text{area}\,(U_{c\varepsilon}(p)\cap\tilde{M}_c) = c^2\,\text{area}\,(U_\varepsilon(p)\cap\tilde{M})$$

$$\int_{U_{c\varepsilon}(p)\cap\tilde{M}_c}|H| = c\cdot\int_{U_\varepsilon(p)\cap\tilde{M}}|H|.$$

For each edge e of M let $V_\varepsilon(e)$ denote the ε-neighborhood of e minus the ε-neighborhood of its two endpoints. Then we have

$$\lim_{c\to 0}\text{area}\,(V_{c\varepsilon}(e)\cap\tilde{M}_c) = \lim_{c\to 0}(c\cdot\text{area}\,(V_\varepsilon(e)\cap\tilde{M})) = 0,$$

$$\lim_{c\to 0}\int_{V_{c\varepsilon}(e)\cap\tilde{M}_c} H \quad = \lim_{c\to 0}\int_{V_\varepsilon(e)\cap\tilde{M}} H = H(e)\cdot\text{length}(e),$$

$$\lim_{c\to 0}\int_{V_{c\varepsilon}(e)\cap\tilde{M}_c}|H| \quad = \lim_{c\to 0}\int_{V_\varepsilon(e)\cap\tilde{M}}|H|$$

$= |H(e)|\cdot\text{length}(e)$, where the last equality holds provided that the cylinder in $\tilde{M}$ replacing e is convex.

Now (2), (4), and (5) follow directly from the equations above, and (3) holds by the Gauss–Bonnet equation

$$\int_{U_{c\varepsilon}(p)\cap\tilde{M}_c} K = \int_{U_\varepsilon(p)\cap\tilde{M}} K = K(p)$$

(cf. Prop. 1).

The problem of finding a sequence M_n satisfying the CCP(K), CCP(H), CCP($|H|$), CCP(A) in our theorem is reduced by Lemma 2 to the construction of only one suitable $\tilde{M}$. On the other hand, the total absolute curvature of $\tilde{M}_c$ considered above is equal to the total absolute curvature of $\tilde{M}$ independently of c. So in order to get a sequence which also satisfies the CCP(K_*) we have to change $\tilde{M}$ in another way which will be described below.

We can use the argument of the homothetic deformation described above in the following way. All that has to be done is to give a suitable approximation of the ε-neighborhood of the vertices which fits together with the cylinders replacing the edges outside. For this reason we will describe this approximation for the *tangent cone* Cp of a vertex p which is defined to be the infinite cone over $\mathrm{lk}(p)$ with centre p (cf. [13]). Clearly, for every number $c \in (0, \infty)$ there is a homothetic deformation with factor c from Cp to Cp.

4.1. *Reduction of Polyhedral Vertices to those of Standard Type*

PROPOSITION 3 (*Reduction process*). *Let M be a compact polyhedral surface without a boundary. Then, for every sufficiently small $\varepsilon > 0$ there exists a polyhedral surface $\tilde{M} = \tilde{M}(\varepsilon)$ (homeomorphic to M) with the following properties*:

(1) *all vertices of $\tilde{M}$ are of the three standard types (see above),*
(2) *$M = \tilde{M}$ outside of the ε-neighborhood U_ε of the vertices of M,*
(3) $\mathrm{d}(M, \tilde{M}) = O(\varepsilon)$ *where* $\mathrm{d}(\ ,\ )$ *denotes the Hausdorff metric,*
(4) *for each vertex $p \in M$ we have*

(i) $K(p) = \sum_{q \in U_\varepsilon(p)} \tilde{K}(q)$,
(ii) $0 \leqslant \sum_{q \in U_\varepsilon(p)} \tilde{K}_*(q) - K_*(p) = O(\varepsilon)$,
(iii) $\int_{U_\varepsilon(p) \cap \tilde{M}} |H| = O(\varepsilon)$,
(iv) $area(U_\varepsilon(p) \cap \tilde{M}) = O(\varepsilon)$

where $K, \tilde{K}, K_*, \tilde{K}_*, H, \tilde{H}$ denote the curvature, absolute curvature and mean curvature of M and $\tilde{M}$ respectively. $\tilde{M}$ can be chosen so that the number of its vertices is bounded independently of ε. In particular, passing to the limit $\varepsilon \to 0$ we get the following: For every open set $U \subseteq E^3$ where ∂U contains no vertex of M we have

$$\lim_{\varepsilon \to 0} \sum_{q \in U \cap \tilde{M}} \tilde{K}_*(q) = \sum_{q \in \bar{U} \cap M} K_*(q).$$

Proof. For the proof of the reduction process we can assume that all faces are strictly convex, in particular that all interior angles of the faces are smaller than π, and that no three edges are coplanar (cf. Lemma 1).

Now let us consider a fixed vertex p.

We consider two faces f_1 with edges e_1, e_2 and f_2 with edges e_2, e_3 of our polyhedral surface M such that e_1, e_2, e_3 meet in p. e_2 is called *reducible* in p if e_1, e_2, e_3 are the only edges meeting in p which are contained in the convex cone with the vertex p generated by e_1, e_2 and e_3.

Let us now assume that e_2 is reducible. Then we can choose three points p_1 on e_1, p_2 on e_2, p_3 on e_3 in some given neighborhood of p such that the angles α_1, α_2 at p_2 and the angle β_1 at p_1 are smaller than a given positive number ε (as indicated in Figure 2) such that the intersection of the closed tetrahedron $p\,p_1\,p_2\,p_3$ with M consists only of the union of the triangles $p\,p_1\,p_2$ and $p\,p_2\,p_3$.

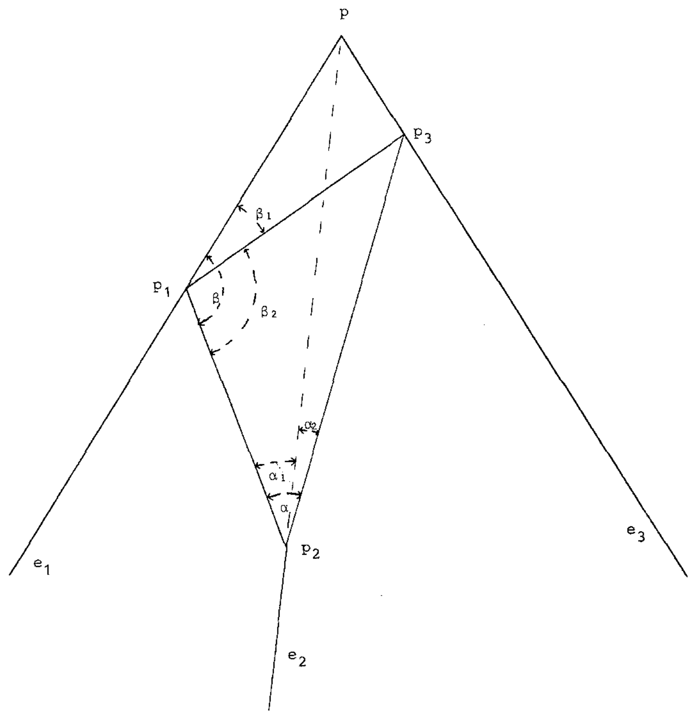


Fig. 2.

If we replace the triangles pp_1p_2 and pp_2p_3 by the triangles pp_1p_3 and $p_1p_2p_3$ we get a new polyhedral surface $\tilde{M}$ homeomorphic to M. Let us denote the curvatures of vertices in $\tilde{M}$ by $\tilde{K}$ etc. The new vertex p_2 is locally convex $(\tilde{K}_-(p_2)=0)$ with curvature $\tilde{K}(p_2)=\tilde{K}_+(p_2)=\tilde{K}_*(p_2)=\alpha_1+\alpha_2-\alpha<2\varepsilon$. p_1 and p_3 are of standard type 3. Thus $\tilde{K}_+(p_1)=\tilde{K}_+(p_3)=0$ and $\tilde{K}_*(p_1)=\tilde{K}_-(p_1)=-\tilde{K}(p_1)=\beta_1+\beta_2-\beta\leqslant 2\beta_1<2\varepsilon$ because $\beta_2\leqslant\beta+\beta_1$.
By the Gauss–Bonnet Equation (2) the total curvature remains invariant under this process, hence $K(p)=\tilde{K}(p)+\tilde{K}(p_1)+\tilde{K}(p_2)+\tilde{K}(p_3)$.

An edge e, which meets the vertex p, is called *non-essential* (in p) if e lies in the convex cone with vertex p generated by all edges meeting in p except e. Note that $K_+(p)=\tilde{K}_+(p)$ if e_2 is non-essential. This implies:

$$\begin{aligned} K_*(p) &= 2K_+(p)-K(p)\\ &= 2\tilde{K}_+(p)-\tilde{K}(p)-\tilde{K}(p_1)-\tilde{K}(p_3)-\tilde{K}(p_2)\\ &= \tilde{K}_*(p)+\tilde{K}_*(p_1)+\tilde{K}_*(p_3)-\tilde{K}_*(p_2); \end{aligned}$$

thus

$$|K_*(p)-\tilde{K}_*(p)-\tilde{K}_*(p_3)|=|\tilde{K}_*(p_1)-\tilde{K}_*(p_2)|<2\varepsilon$$

and

$$0 < \tilde{K}_*(p) + \tilde{K}_*(p_1) + \tilde{K}_*(p_2) + \tilde{K}_*(p_3) - K_*(p) = 2\tilde{K}_*(p_2) < 4\varepsilon.$$

LEMMA 3. *Let p be a vertex of the polyhedral surface M such that no three edges meeting in p are coplanar and all interior angles of the faces at p are smaller than π. Then either there is an edge which is reducible and non-essential in p as well or p is of one of the two standard types* 1 *or* 2.

Proof. In the following, by a *spherical polygon* we mean a simple closed curve on the unit sphere S^2 which consists of finitely many pieces of great circles. The intersection of the tangent cone Cp with the unit sphere with centre p is a spherical polygon which we will call P. The edges of P are arcs of great circles with lengths smaller than π. Given two non-antipodal points p_1, p_2 on the sphere, we denote by arc $p_1 p_2$ the open arc with a length less than π of the great circle connecting p_1 and p_2.

Note that an edge is reducible if and only if the arc connecting the two neighbors of the corresponding vertex in P does not meet P. Now we need the following

LEMMA 4. *Let P' be a spherical polygon with at least three vertices such that all edges of P' have a length smaller than π and no three vertices lie on the same great circle. At least one of the two simple connected regions with P' as the boundary, say S, contains no great circle. Then we can triangulate S into spherical triangles with interior angles strictly smaller than π without adding any new vertex.*

Proof. We use induction over the number n of vertices of P'. For $n = 3$, Lemma 4 is trivial. Assume that it has been proved for polygons with less than n vertices ($n \geqslant 4$). Given S with P' having n vertices, we choose any vertex v with an interior angle less than π. Such a vertex exists because no great circle is contained in S. Denote the other endpoints of the edges meeting in v by v_1 and v_2.

1st case. arc $v_1 v_2$ lies in S. Hence we can split off the spherical triangle $v_1 v_2 v$ and the remaining polygon has $n-1$ vertices.

2nd case. arc $v_1 v_2$ is not contained in S. Then there is at least one further vertex of P' inside the closed spherical triangle $v_1 v_2 v$. At least one of these, say w, has the property that no vertex of P' lies in the interior of the intersection of the spherical triangle $v_1 v_2 v$ with the hemisphere which contains v and has v_1 and w on its boundary. Hence, arc vw is contained in S and defines a decomposition of S into two parts, each of which has less than n vertices.

We now continue the proof of Lemma 3. A subset of the sphere is called *spherically convex* if the cone generated by that subset over the centre of the sphere is convex. We consider two cases:

1st case. There is a great circle which does not meet P. Then the spherical convex hull S of P is not the sphere. If P is the boundary of S the vertex p of M is locally convex, thus of standard type 1. Otherwise $S\setminus P$ has at least one connected component for which P is not its boundary. Choose one of them. It has then exactly one edge which is not an edge of P. If it has n vertices it has $n-1$ edges in common with P. We triangulate the chosen component as in Lemma 4 into $n-2$ spherical triangles. At least one of these has two edges in common with P.

The edge in M corresponding to the vertex at which these two edges in P meet is reducible and non-essential by construction.

2nd case. Every great circle meets P. Then the spherical convex hull of P is the sphere and $K_+(p)=0$ (by assumption no three vertices lie on a great circle) and M has no local supporting plane at p. We can triangulate each of the two open subsets of the sphere with P as the boundary, as in Lemma 4. In each of the two triangulations there are at least two triangles, each of which have two edges in common with P. A vertex where such a pair of edges meet corresponds to a reducible edge in M. The two triangulations cannot have such a vertex in common because the interior angles of the triangles are less than π. Thus, there are at least four reducible edges meeting in p.

If there are only four edges meeting at p, then p is of standard type 2. So let us assume that there are more than four edges meeting at p. Thus P has at least five vertices.

The two triangulations given above yield a triangulation of S^2.

Choose some vertex v_1 of P. The antipodal point v_1' is contained in the interior of some convex spherical triangle $v_2\, v_3\, v_4$ of the given triangulation of S^2. Thus, the spherical convex hull of $v_1\, v_2\, v_3\, v_4$ is equal to S^2. Choose a fifth vertex v_5 of P. Then there are three vertices $w_1\, w_2\, w_3$ of P such that the spherical convex hull of $w_1\, w_2\, w_3\, v_5$ is equal to S^2. There are, at most, three vertices in $\{v_1, v_2, v_3, v_4\} \cap \{w_1, w_2, w_3\}$. All other vertices correspond to non-essential edges. Because there are at least four reducible edges meeting at p, there is at least one edge which is reducible and non-essential in p. This completes the proof of Lemma 3.

For each vertex p which is not one of the three standard types we can apply the reduction process with $K_+(p)=\tilde{K}_+(p)$, because there is a reducible non-essential edge at p, as shown in Lemma 3. Now we iterate the reduction process until all vertices of the polyhedral surface are of standard types. We can do the reduction process for a vertex p so that the new vertices lie in a given ε-neighborhood U_ε of p and so that the curvatures satisfy $K(p)=\sum_{q\in U_\varepsilon}\tilde{K}(q)$ and $0 \leqslant \sum_{q\in U_\varepsilon}\tilde{K}_*(q) - K_*(p) \leqslant 4\varepsilon\cdot(m-3)$, where m is the number of edges meeting in p.

This proves assertion (4) (ii) of Proposition 3. The other assertions follow by additional homothetic deformation with some factor c_ε (depending on ε) using Lemma 2.

4.2. C^1 Approximation of Polyhedral Vertices of Standard Type

PROPOSITION 4. *Let M be a compact polyhedral surface without a boundary which has only vertices of the three standard types. Then, for every sufficiently small $\varepsilon > 0$ there exists a C^1 surface $\tilde{M} = \tilde{M}(\varepsilon)$ homeomorphic to M with the following properties*:

(1) *$\tilde{M}$ is built up by a finite number of pieces of planes, cylinders, cones, spheres and elliptic tori fitting together along pieces of straight lines and ellipses,*

(2) *$M = \tilde{M}$ outside the ε-neighborhood of the 1-skeleton of M. Inside the ε-neighborhood of each edge (minus the ε-neighborhood of its endpoints) $\tilde{M}$ consists of an orthogonal cylinder over an arc of a circle such that the cylinder fits tangentially to the adjacent faces,*

(3) $\mathrm{d}(M, \tilde{M}) = O(\varepsilon)$ *where* $\mathrm{d}(\ ,\ )$ *denotes the Hausdorff metric,*

(4) *for each vertex $p \in M$ we have*

(i) $\int_{\tilde{M} \cap U_\varepsilon(p)} \tilde{K}\, \widetilde{\mathrm{d}o} = K(p)$,

(ii) $0 \leqslant \int_{\tilde{M} \cap U_\varepsilon(p)} |\tilde{K}|\, \widetilde{\mathrm{d}o} - K_*(p) = O(\varepsilon)$

(iii) $\int_{\tilde{M} \cap U_\varepsilon(p)} |\tilde{H}|\, \widetilde{\mathrm{d}o} = O(\varepsilon)$

(iv) $\int_{\tilde{M} \cap U_\varepsilon(p)} \widetilde{\mathrm{d}o} = O(\varepsilon)$

where $K, \tilde{K}, H, \tilde{H}$ denote the curvature and mean curvature of M and $\tilde{M}$ respectively, K_* the absolute curvature of M and $\widetilde{\mathrm{d}o}$ the area element of $\tilde{M}$.

In particular, passing to the limit $\varepsilon \to 0$ we get the following: For every open set $U \subseteq E^3$ such that ∂U contains no vertex of M we have

$$\lim_{\varepsilon \to 0} \int_{U \cap \tilde{M}} |\tilde{K}|\, \widetilde{\mathrm{d}o} = \sum_{p \in U \cap M} K_*(p).$$

Proof. We will use the notion of the *parallel surface* M_r of M at a distance r, which is defined to be the set M_r of the points $p \in E^3$ with the distance $\mathrm{d}(p, M) = r$. Locally there are two disjoint sheets of M_r and for oriented M we globally can speak of the outer parallel surface M_r^+ ((+)-parallel surface) and the inner parallel surface M_r^- ((−)-parallel surface).

Let us choose some neighborhood U of the vertices and let r be a positive number which is sufficiently small (for the approximation it will tend to zero). Outside U we get a C^1 approximation by replacing some neighborhood of each edge by an orthogonal cylinder over an arc of a circle of radius r which is perpendicular to this edge and which fits tangentially to the adjacent faces. This is the same as taking any of the two triple parallel surfaces $((M_r^+)_{2r}^-)_r^+$ or $((M_r^-)_{2r}^+)_r^-$.

Now our task is to extend this approximation to the neighborhood $U(p)$ of each vertex p, but in general these triple parallel surfaces are not differentiable near the vertices. In order to describe a suitable approximation we first approximate the tangent cones Cp:

First case. If Cp is convex we take the double parallel surface $(Cp_r^-)_r^+$ taking the natural orientation of Cp. Obviously the parallel cone Cp_r^- is convex and polyhedral. The $(+)$-parallel surface of it is also convex and it consists of finitely many pieces of planes, spheres and cylinders with radius r which fit differentiably together along pieces of straight lines and circles. Furthermore, outside of some neighborhood of p $(Cp_r^-)_r^+$ coincides with $((Cp_r^-)_{2r}^+)_r^-$. Thus, the approximation of M near the vertex p and the approximation outside fit together.

Second case. Let Cp be a cone which has only four edges and which has no supporting plane at p and all of whose interior face angles are, at most, π (this corresponds to standard type 2 or standard type 3 without a supporting plane). It is easy to see that in this case the four edges are necessarily creased in an alternating manner. Now let us consider the parallel surface Cp_r^+ for an arbitrary orientation of Cp. This parallel surface consists of two differentiable pieces, each of which consists of two pieces of planes differentiably connected by a piece of a cylinder. The curve where the surface is not differentiable is the intersection of those two pieces. Now the intersection of such two cylinders (with same radii) is an ellipse and the intersection of a cylinder with a plane (not parallel to the cylinder) is also an ellipse. Hence, that curve consists of at most five straight lines and ellipses. There is no supporting plane of Cp at p. Hence, the two differentiable pieces of the parallel surface have no common tangent at any point of their intersection. Hence, the curve is differentiable.

Let λ be some number (not depending on r) with $0 < \lambda < 1$ such that the smallest radius of curvature of the curve is greater than λr. The double parallel set $(Cp_r^+)_{\lambda r}^-$ is differentiable everywhere and it consists of pieces of cylinders, planes and pieces of elliptic tori arising from the intersection ellipses (an *elliptic torus* we define to be the parallel surface of an ellipse in E^3).

Note that the plane spanned by such an intersection ellipse separates the two half-cylindrical pieces (or half-cylinder and half-plane, respectively) of which it is the intersection. Moreover, let X denote the inner normal vector of the ellipse in that plane and let Y be the inner tangent vector on the half cylinder (or half-plane respectively) which in addition is perpendicular to the ellipse. Then we have $\langle x, y\rangle < 0$. This means that the corresponding pieces of elliptic tori are 'inner' pieces of the tori, i.e., they have nonpositive curvature.

Third case. Let Cp be a cone which has only four edges such that one interior face angle is equal to π and the other angles are smaller than π, and which has some supporting plane at p (this corresponds to standard type 3 with supporting plane). The two edges of Cp which are collinear are creased in the same direction because there is a supporting plane. One of the other edges is creased in the same direction and the last one is creased in the opposite direction. Choose the orientation of Cp such that three edges are 'convex' and one edge is 'concave'. This results in the $(+)$-parallel surface Cp_r^+ being differentiable near the convex edges. It consists of pieces of planes and cylinders and it is differentiable, except along the curve coming from the concave edge. As in the second case, this curve consists of a piece of a straight line and finitely many pieces of ellipses. It ends in a point of differentiability (where the two cylinders have a common tangent plane). Let λ be some number (not depending on r) with $0 < \lambda < 1$ such that the smallest radius of curvature of the curve is greater than λr. The double parallel surface $(Cp_r^+)_{\lambda r}^-$ is differentiable everywhere and it consists of pieces of cylinders, planes and 'inner' pieces of elliptic tori all of which have nonpositive curvatures, just as in the second case. So we can take in the second and third cases $(Cp_r^+)_{\lambda r}^-$ as the approximation near p (which coincides with $(M_r^+)_{\lambda r}^-$ near p).

Note that in each of the three cases, the parallel surface near the vertex p which we have chosen has exactly the same positive and negative curvature (hence, the same absolute curvature) as the vertex p before. But, unfortunately, in case 2 and 3 there remains the problem of fitting together the different parallel surfaces inside and outside U. The problem is that, necessarily, the two parallel surfaces of the faces will be different parallel faces which do not coincide; similarly from the edges we get cylinders with different radii.

To overcome this problem let us consider a vertex p of the surface M (before taking parallel surfaces) and the neighborhood $U(p)$ considered above which we may also regard as lying in the tangent cone Cp. Let us choose a number $\delta > 0$ such that

$$Cp \cap \{q / \|p - q\| \leqslant 2\delta\} \subset U(p).$$

For each edge e meeting p, let e_δ denote the point on e satisfying $\|e_\delta - p\| = \delta$. Now join all these points e_δ by a closed curve γ lying inside $U(p)$ in the following way: each edge e yields two vectors $X(e)$, $Y(e)$, each lying in one of the two faces meeting in e, such that $X(e)$ and $Y(e)$ are perpendicular to E. $X(e)$ and $Y(e)$ span a plane orthogonal to e provided that the faces are not coplanar, which we can assume without loss of generality. Now let us construct the curve γ as follows: near the points e_δ γ follows the straight lines spanned by $X(e)$, $Y(e)$ respectively, and both pieces of straight lines lying in the same face are joined differentiably by a single arc of a circle. See Figure 3 which shows this construction inside one face for two different cases.

Because we are working in the cone Cp we get a second curve $\tilde{\gamma}$ by homothetic reduction of the size by a fixed ratio, say $\frac{1}{2}$. Let V and $\tilde{V}$ denote the interiors of γ and $\tilde{\gamma}$ respectively.

Now let us apply the different processes of taking the parallel surfaces described above. Inside $\tilde{V}$ we take the double parallel surface (after choosing

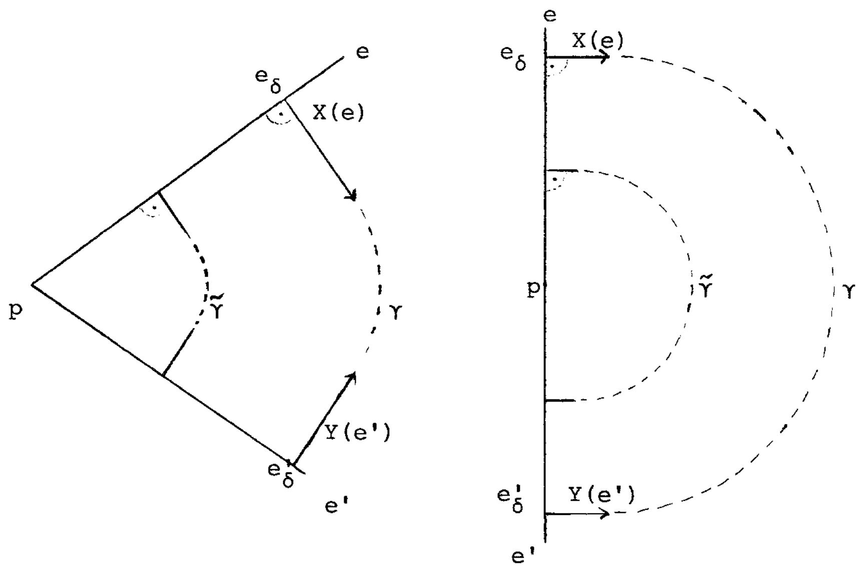


Fig. 3.

some local orientation) and outside V the triple parallel surface such that the middle parts of the faces outside V remain unchanged. Note that we have chosen r sufficiently small and that our choice of δ, γ and $\tilde{\gamma}$ has been made independently of r.

Now we have to fit together both different parallel surfaces. Note that its boundaries are parallel curves c and $\tilde{c}$ of γ and $\tilde{\gamma}$ respectively which are of class C^1 and consist of finitely many pieces of straight lines and circles. Of course, c and $\tilde{c}$ are not homothetic to each other but, in any case, there is a natural bijection between those points of c and $\tilde{c}$ where c and $\tilde{c}$ are not of class C^2. Let us join the corresponding points by straight lines. Furthermore, let us join the corresponding pieces of straight lines in c and $\tilde{c}$ (which are parallel) by plane 4-gons, and the corresponding pieces of circles (which lie in parallel planes) by pieces of oblique cones.

The union W of those pieces of planes and cones is homeomorphic to $V\backslash\tilde{V}$ and obviously it forms a C^1 surface. Now take the union $\mathscr{M}$ of W with the both parallel surfaces inside of V and outside of V. This will be a continuous surface homeomorphic to the old one but, of course, it will not be differentiable along c and $\tilde{c}$. But the angle between the right and left limit tangent planes will be of order $O(r)$ in this case, so it is more easy to give a differentiable approximation than in the case of the old surface. Let us choose a sufficiently small number $\rho > 0$ which is smaller than $\min(\lambda r, (1-\lambda)r)$ and take the triple parallel surface $((\mathscr{M}_\rho^+)_{2\rho}^-)_\rho^+$ of the surface $\mathscr{M}$ constructed above. Independently of the orientation in the region where $\mathscr{M}$ is already of class C^1, nothing will be changed. In a neighborhood of the curves c and $\tilde{c}$, the surface $\mathscr{M}$ will be changed in a similar way as in the construction of the double parallel surface described above. After the process it consists of pieces of planes, cylinders, cones and elliptic tori. Note that this is true only if we choose ρ as sufficiently small. Now if we look at the additional total absolute Gaussian curvature arising from this process, it is easy to see that this is of the order $O(r)$ independently of ρ. This proves assertion (4) (ii) of Proposition 4. The other assertions follow by additional homothetic deformation with some factor c_r (depending on r) using Lemma 2.

4.3. *Smooth Approximation*

The construction explained in Section 4.2 gives C^1 surfaces which are built up by finitely many pieces of planes, cylinders, elliptic cones, elliptic tori or spheres which are joined together along pieces of straight lines and ellipses. There remains the question of whether such surfaces can be approximated smoothly, i.e., of class C^∞. That the answer is yes will follow from Proposition 5 which, in fact, deals with a somewhat more general situation:

PROPOSITION 5. *Assume that $F: M \to E^3$ is a C^1 immersion of a closed surface M such that there is a decomposition $M = \bigcup_{i=1}^n M_i$ into pieces M_i which are compact surfaces with a boundary. Assume that $F|_{M_i}$ is of class C^k $(2 \leqslant k \leqslant \infty)$ and that the boundary of M_i consists of finitely many C^k curves, $i = 1, \dots, n$. Then for every sufficiently small $\varepsilon > 0$ there exists a C^k immersion $\tilde{F} = \tilde{F}(\varepsilon): M \to E^3$ (which is an embedding provided that F is also an embedding) such that*

(1) *$F = \tilde{F}$ outside the ε-neighborhood of the union of curves where F is not of class C^k,*

(2) $\|F - \tilde{F}\|_{\sup} = O(\varepsilon)$,

(3) $\int_M \left| |K| \, \mathrm{d}o - |\tilde{K}| \, \mathrm{d}\tilde{o} \right| = O(\varepsilon)$,

(4) $\int_M \left| |H| \, \mathrm{d}o - |\tilde{H}| \, \mathrm{d}\tilde{o} \right| = O(\varepsilon)$,

(5) $\int_M |\mathrm{d}o - \mathrm{d}\tilde{o}| = O(\varepsilon)$,

where $K, \tilde{K}, H, \tilde{H}$, $\mathrm{d}o, \mathrm{d}\tilde{o}$ denote the Gaussian and mean curvature and the area element of F and $\tilde{F}$ respectively. In particular for every open set $U \subseteq M$ we have

$$\lim_{\varepsilon \to 0} \int_U |\tilde{K}| \, \mathrm{d}\tilde{o} = \int_U |K| \, \mathrm{d}o,$$

$$\lim_{\varepsilon \to 0} \int_U |\tilde{H}| \, d\tilde{o} = \int_U |H| \, do,$$

$$\lim_{\varepsilon \to 0} \int_U d\tilde{o} = \int_U do.$$

COROLLARY 5′. *For every C^1 immersion $F: M \to E^3$ satisfying the assumptions of proposition 5 the following relations (which are well known for C^2 immersions) remain valid:*

$$\int_M K \, do = 2\pi \cdot \chi(M) \quad (\textit{Gauss-Bonnet formula}),$$

$$\int_M |K| \, do \geqslant 2\pi(4 - \chi(M)).$$

As usual, we define a compact surface with a boundary to be of class C^k if it can be extended to an open surface of class C^k. Note that Proposition 5 becomes false if we replace the word 'compact' by 'open' and assume $M = \bigcup_{i=1}^n \mathring{M}_i$ because the total absolute curvature of such a C^1 surface may be infinite.

Proof. By assumption, the set of points where M is not of class C^k is contained in the boundaries of the M_i, whose union is just the set

$$A := \bigcup_i \bigcap_{j \neq i} (M_i \cap M_j).$$

By assumption, this set consists of finitely many C^k curves. Without loss of generality, we can assume that each $\gamma_{ij} = M_i \cap M_j$ is such a curve, otherwise choose a subdivision of the M_i's. What we have to do is to change a neighborhood of A such that the surface will be of class C^k. This will be done in two steps: at first for the relative interior of the γ_{ij} and then for the endpoints. Of course, there are only finitely many different endpoints $p_1, \ldots, p_m$ of the γ_{ij}, $i, j = 1, \ldots, n$.

Now for a fixed sufficiently small number $\rho > 0$ let $U_\rho(p_i)$ denote the open ρ-neighborhood of p_i, $i = 1, \ldots, m$ and set $U_\rho := \bigcup_{i=1}^m U_\rho(p_i)$ where we can assume that the $U_\rho(p_i)$ are disjoint. (It does not matter whether we take the distance ρ with respect to the inner metric or with respect to the ambient space, because the difference is of higher order.)

At first we will change the surface outside $U_{\rho/4}$ and afterwards inside U_ρ. Now, for fixed $i \neq j$ the intersection curve γ_{ij} is a common boundary

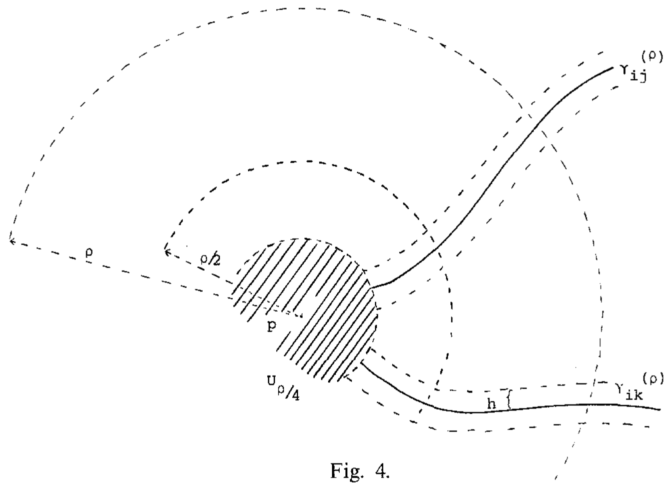


Fig. 4.

piece of M_i and M_j (in case $M_i \cap M_j \neq \emptyset$). Define $\gamma_{ij}^{(\rho)}$ to be the restriction of γ_{ij} to $M \backslash U_{\rho/4}$. Because different γ_{ij} do not meet each other in their relative interiors the $\gamma_{ij}^{(\rho)}$ are disjoint and, hence, there exists a number h depending on ρ such that $\rho/4 > h > 0$ and all h-neighborhoods of all $\gamma_{ij}^{(\rho)}$ are separated, meaning that their closures are disjoint (see Figure 4).

So we can consider each $\gamma_{ij}^{(\rho)}$ separately; we will omit the index ρ if there is no danger of confusion. Because M_i is compact and of class C^k it can be extended to an open surface of class C^k (which we also denote by M_i) containing γ_{ij}. Hence, there exist coordinates (x, y) in the h-neighborhood of γ_{ij} which are defined for $|x| \leqslant h$ such that γ_{ij} appears as $\{x = 0\}$ and $\partial/\partial x$, $\partial/\partial y$ form an orthonormal frame along γ_{ij}.

Because M_i and M_j are joined along γ_{ij} of class C^1, their tangent planes coincide along γ_{ij} and we have the following parametrization of $M_i \cup M_j$ in the h-neighborhood of γ_{ij}:

$$(M_i \cup M_j)(x, y) = M_i(x, y) + f(x, y) \cdot \mathcal{N}$$

where $\mathcal{N}$ denotes the unit normal of the extended M_i, and f is a C^1 function which is indentically zero in $\{x \leqslant 0\}$ and which is of class C^k with bounded second derivatives in $\{x > 0\}$:

$$f = O(x^2), \qquad f_x, f_y = O(x), \qquad f_{xx}, f_{xy}, f_{yy} = O(1).$$

Now let $\psi: \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\psi(x) = 0 \quad \text{for } x \leqslant \tfrac{1}{2},$$

$$\psi(x) = 1 \quad \text{for } x \geqslant 1,$$

$$\psi' \geqslant 0 \quad \text{everywhere}$$

and define

$$M_{ij}^{(h)}(x, y) := M_i(x, y) + \psi\left(\frac{x}{h}\right)\cdot f(x, y)\cdot \mathcal{N}.$$

Of course, nothing has been changed for $x \notin (h/2, h)$, and $M_{ij}^{(h)}$ is of class C^k everywhere and still remains an immersion. It is easily checked that the first and second partial derivatives of the function $\psi(x/h)\cdot f(x, y)$ are bounded (independently of h), hence the Gaussian and mean curvature $K^{(h)}$ and $H^{(h)}$ of $M_{ij}^{(h)}$ are bounded and, hence,

$$\int_{0 \leqslant x \leqslant h} \left| |K|\,\mathrm{d}o - |K^{(h)}|\,\mathrm{d}o^{(h)} \right| = \int |K|\,\mathrm{d}o + \int |K^{(h)}|\,\mathrm{d}o^{(h)}$$
$$= O(h) \leqslant O(\rho);$$

similarly for H, $H^{(h)}$.

This procedure of changing an h-neighborhood of $\gamma_{ij}^{(\rho)}$ can be made globally (i.e., for each γ_{ij}) where we use the assumption that all these h-neighborhoods are disjoint. This leads to a C^k-immersion

$$G^{(\rho)}: M \backslash U_{\rho/4} \to E^3.$$

Now it remains to act in a similar way inside U_ρ. The $U_\rho(p_i)$ are disjoint, hence we can work independently for each point. Let p be one of those points. After applying a translation and a rotation of E^3, we can assume that the tangent plane of the given surface F at p is just the $(x, y, 0)$-plane as a linear subspace of E^3. Using polar coordinates (r, ϕ) in that plane, we can describe $G^{(\rho)}$ in the following way:

$$(*) \qquad G^{(\rho)}(r, \phi) = (r\cos\phi, r\sin\phi, f(r, \phi)), \quad r \geqslant \frac{\rho}{4}$$

where f is a C^k function satisfying

$$f, f_\phi, f_{\phi\phi} = O(r^2), \qquad f_r, f_{r\phi} = O(r), \qquad f_{rr} = O(1).$$

Now recall the function ψ used above and define

$$(**) \qquad F^{(\rho)}(r, \phi) := \begin{cases} (r\cos\phi, r\sin\phi, 0) & \text{for } 0 \leqslant r \leqslant \dfrac{\rho}{4}, \\ \left(r\cos\phi, r\sin\phi, \psi\left(\dfrac{r}{\rho}\right)\cdot f(r, \phi)\right) & \text{for } \dfrac{\rho}{4} \leqslant r. \end{cases}$$

By definition, $F^{(\rho)}$ is planar inside of $\{r \leqslant \rho/2\}$ and hence it is of class C^k

everywhere. Clearly $F^{(\rho)}$ is an immersion and it coincides with $G^{(\rho)}$ in the region $\{r \geqslant \rho\}$.

It remains to to estimate the additional absolute Gaussian and mean curvatures in the region $\{\rho/2 \leqslant r \leqslant \rho\}$.
From (*) it follows by straightforward calculation

$$K = \frac{r^2 f_{rr}(rf_r + f_{\phi\phi}) - (rf_{r\phi} - f_\phi)^2}{(r^2 + r^2 f_r^2 + f_\phi^2)^2}$$

and

$$H = \frac{rf_{rr}(r^2 + f_\phi^2) - 2f_r f_\phi (rf_{r\phi} - f_\phi) + (r^2 f_r + rf_{\phi\phi})(1 + f_r^2)}{2(r^2 + r^2 f_r^2 + f_\phi^2)^{3/2}}.$$

Doing the same for (**) we only have to replace f by $\tilde{f} := \psi(r/\rho)\cdot f$. It is easily checked that

$$\tilde{f}, \tilde{f}_\phi, \tilde{f}_{\phi\phi} = O(r^2), \qquad \tilde{f}_r, \tilde{f}_{r\phi} = O(r), \qquad \tilde{f}_{rr} = O(1).$$

This implies by straightforward calculation that the curvatures $\tilde{K}$ and $\tilde{H}$ of $F^{(\rho)}$ both are bounded (independently of ρ), hence

$$\int_{r \leqslant \rho} \left| |K| \, \mathrm{d}o - |\tilde{K}| \, \tilde{\mathrm{d}o} \right| \leqslant \int |K| \, \mathrm{d}o + \int |\tilde{K}| \, \tilde{\mathrm{d}o} = O(\rho^2);$$

similarly for $H, \tilde{H}$.

Of course, this process just described can be made simultaneously for all points $p_1, \ldots, p_m$ leading to an immersion $\tilde{F} : M \longrightarrow E^3$ (depending on ρ) which is of class C^k everywhere and which is an embedding, provided that F was an embedding.

By construction, the assertions (1) and (2) of Proposition 5 are satisfied for $\varepsilon = \rho$; similarly we have

$$\int_M \left| |K| \, \mathrm{d}o - |\tilde{K}| \, \tilde{\mathrm{d}o} \right| = \int_{U_\rho \cap \{x = h\}} \left| |K| \, \mathrm{d}o - |\tilde{K}| \, \tilde{\mathrm{d}o} \right| = O(\rho),$$

$$\int_M \left| |H| \, \mathrm{d}o - |\tilde{H}| \, \tilde{\mathrm{d}o} \right| = O(\rho) \quad \text{and} \quad \int_M |\mathrm{d}o - \tilde{\mathrm{d}o}| = O(\rho).$$

Proof of the main theorem: Now to complete the proof of our main theorem we combine the three approximations given in Propositions 3, 4 and 5. Using the well-known argument of taking a diagonal sequence we finally get a sequence $M_n = \tilde{M}(\varepsilon_n)$ of smooth surfaces M_n satisfying

$$\lim_{n \to \infty} \int_{U \cap M_n} |K_n| \, \mathrm{d}o_n = \lim_{\varepsilon \to 0} \int_{U \cap \tilde{M}} |\tilde{K}| \, \tilde{\mathrm{d}o} = \sum_{p \in U} K_*(p)$$

for every open set U such that ∂U contains no vertex of M. This is the assertion of the CCP(K_*). The other CCP's are satisfied similarly, in fact they already hold for weaker approximations (cf. Lemma 2).

Because all the essential steps in the proofs have been done locally (i.e., in some ε-neighborhood of the vertices), all assertions also remain valid for immersed surfaces. Similarly, in our constructions we never used global orientations but only local ones. So all assertions remain valid for non-orientable surfaces, expect for the CCP(H) which is not defined because H depends on the orientation.

5. Other related questions

In the context of the problems discussed in Section 4 there remain some natural questions:

(1) Does the main theorem also hold for compact surfaces with a boundary?
(2) Does the similar result hold in the opposite direction, i.e., approximating a given smooth surface by polyhedral ones?
(3) Can the main theorem be sharpened so that the total absolute curvature of the approximating smooth surface exactly equals the total absolute curvature of the given polyhedral one? This seems to be very interesting for tightness problems.

We first want to answer question 1 negatively (in some sense):

PROPOSITION 6. *There exists a compact polyhedral surface M with a boundary such that there is no sequence $(M_n)_{n\in\mathbb{N}}$ of smooth surfaces homeomorphic to M which simultaneously fulfills the assertion of the main theorem for M, ∂M and $M\backslash\partial M$.*

This follows from the equation

$$TA(M) = TA(M\backslash\partial M) + \tfrac{1}{2}TA(\partial M)$$

in the smooth case on the one hand and the fact that this equation does not hold for polyhedral surfaces, in general. See [5] for a detailed discussion of that topic including the necessary conditions for that equality in the polyhedral case.

Concerning question 2, we would like to conjecture that such a theorem is true. In fact this direction of approximating smooth surfaces by polyhedral ones should be the more easy part and it might also include the case of compact surfaces with a boundary. (see Added in Proof)

Question 3 (Banchoff's question) seems to be open. It could be answered negatively if there existed a tight polyhedral surface of some topological type for which it is known that there is no smooth tight realization. But so far as we know, the only candidate for such a counterexample without a boundary could be the surface with $\chi = -1$ (unknown case). Among the surfaces with a boundary, the Moebius band is such a counterexample (see [8]), as is the torus with a disc removed (see [5], [15]).

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ADDED IN PROOF

Question 2 has been positively answered in a recent paper by J. Cheeger, W. Müller and R. Schrader: 'Lattice Gravity or Riemannian Structure on Piecewise Linear Spaces', I.H.E.S. Preprint 1981. There are also given higher dimensional analogues for various curvatures including the scalar curvature and the Gauss–Bonnet integrand.

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