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GENERALIZED METRIC GEOMETRIES OF ARBITRARY DIMENSION*

1. INTRODUCTION

A generalized metric geometry is a structure that satisfies Hilbert's Incidence Axioms and some simple orthogonality conditions, and admits reflections in all points and lines. This paper extends to arbitrary dimensionality the notions of two and three dimensional generalized metric geometries studied by Bachmann [1, §2] and Scherf [6, §1], respectively. Among the questions considered are the following. How should a reflection in an arbitrary flat be defined? For which flats do there exist reflections? When are these reflections unique? When do two reflections commute? What is the composition of two commuting reflections?

The axioms for generalized metric geometry fall into three groups. The first group consists of Hilbert's Incidence Axioms [3, §2], modified to admit arbitrary dimensionality. Lenz's Orthogonality Axioms [5, §1], modified to admit elliptic models, constitute the second group. The third group, the Symmetry Axioms, postulates the existence of reflections in all points and lines.

Models of the Incidence and Orthogonality Axioms are called *orthogonal geometries*. These were studied in great detail in two earlier papers [7; 8]. For definitions and results concerning orthogonal geometries in general, the reader is referred to these works.

A common method of proof in the present paper is the reduction of a question about flats of arbitrary dimension to one about coplanar families of points and lines. Since every plane in a generalized metric geometry is a generalized metric plane in the sense of Bachmann, his results can then be applied. This method yields directly, for instance, the uniqueness of the reflection in a point or line. It is used also in proving the most important result of the paper: there exists a reflection in a flat x if and only if x is orthocomplemented in the sense of [7]; in this case the reflection is unique.

A later paper will study *metric geometries* of arbitrary dimension: generalized metric geometries in which the familiar Three-Reflections Princi-

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ples hold. That theory is equivalent, in the finite dimensional case, to Kinder's [4]; his representation theorem will be extended to the infinite dimensional case.

2. Orthogonal collineations

Let $(r, G, E, \perp)$ be an orthogonal geometry. Thus, r is a set called the *entire space*, and its singleton subsets are called *points*. Further, G and E are families of subsets of r called *lines* and *planes*, respectively; and $\perp$ is a binary relation on G called *orthogonality*. Finally, $(r, G, E, \perp)$ satisfies the Incidence and Orthogonality Axioms of [7, §2 and §3].

An *orthogonal collineation* of $(r, G, E, \perp)$ is a permutation of the entire space r satisfying the following three conditions: (i) the image of a line is a line; (ii) the image of a plane is a plane; and (iii) the images of two orthogonal lines are themselves orthogonal. It is easy to show that the orthogonal collineations form a subgroup of the symmetric group of r. A monomorphism $\pi \to \pi^*$ from this subgroup to the automorphism group of the lattice of all flats can be defined by setting $\pi^*(x) = \pi[x]$ for each orthogonal collineation π and each flat x. Henceforth, π and π^* will not be distinguished. The Proposition below lists some relations preserved by an arbitrary orthogonal collineation π. Its proof is straightforward; consult [7] and [8] for the notions involved.

PROPOSITION. *If x is a flat, then x and $\pi(x)$ have the same dimension and codimension. If o is a point in x, then $\pi([o, x]) = [\pi(o), \pi(x)]$; thus $\pi(\text{reach } x) = \text{reach } \pi(x)$, and x is orthocomplemented if and only if $\pi(x)$ is. If y is any flat, then $x \perp y$ if and only if $\pi(x) \perp \pi(y)$. When the geometry is elliptic, $\pi(x^{\perp}) = \pi(x)^{\perp}$.*

3. Symmetry axioms

A *reflection in a point o* is a self inverse orthogonal collineation σ_o that leaves every line through o fixed, but not elementwise fixed. A *reflection in a line g* is a self inverse orthogonal collineation σ_g that leaves g elementwise fixed, and every plane through g fixed, but not elementwise fixed. It follows that each flat x properly including o or g is fixed, but not elementwise fixed, by σ_o or σ_g, respectively.

A *generalized metric geometry* is an orthogonal geometry that satisfies the following *Symmetry Axioms* S1 and S2.

AXIOM S1. *In each point there is a reflection.*

AXIOM S2. *In each line there is a reflection.*

Henceforth, $(r, G, E, \perp)$ is assumed to be a generalized metric geometry.

4. Subgeometries

For each flat x of dimension $m \geqslant 2$, let G_x and E_x denote the families of all lines and planes, respectively, in x. Moreover, let $\perp_x$ denote the restriction to G_x of the relation $\perp$. Clearly, $(x, G_x, E_x, \perp_x)$ is then an m dimensional generalized metric geometry; its point and line reflections are also obtained by restriction. If the geometry $(r, G, E, \perp)$ is elliptic, then so is $(x, G_x, E_x, \perp_x)$. The following Proposition follows from a result of Ahrens [2, App.].

PROPOSITION. *Each line passes through three distinct points.*

Therefore, if e is a plane, then $(e, G_e, \perp_e)$ is a generalized metric plane in the sense of Bachmann [1, §2].

5. Point reflections

Proofs of theorems about point reflections generally differ from and are much simpler than those of the corresponding theorems about reflections in arbitrary flats. Therefore, these results are gathered here; the general theory is developed in later sections.

PROPOSITION 5.1. *In each point o there is a unique reflection σ_o.*

Proof. Suppose σ and τ are reflections in o. Let p be an arbitrary point and e be a plane through o and p. The restrictions to e of σ and τ are reflections in o with respect to the generalized metric plane $(e, G_e, \perp_e)$, hence $\sigma(p) = \tau(p)$ by [1, §2].

COROLLARY 5.2. *Let π be an orthogonal collineation and o be a point. Then $\pi\sigma_o\pi^{-1} = \sigma_{\pi(o)}$.*

Proof. The left hand side of the equation is easily seen to be a reflection in $\pi(o)$.

PROPOSITION 5.3. *A point p is fixed under the reflection in a point o if and only if $o = p$ or (when the geometry is elliptic) p lies in $o^\perp$.*

Proof. Suppose p is fixed. Let e be any plane through o and p. By [1, §2], $o = p$ or p lies in a line polar to o with respect to the generalized metric plane $(e, G_e, \perp_e)$. In the latter case, p lies in $o^\perp$.

Conversely, suppose p lies in $o^\perp$. Let g be a line through o; then $g \perp o^\perp$, hence $g=\sigma_o(g)\perp\sigma_o(o^\perp)$. Thus $\sigma_o(o^\perp)$ is a hyperplane polar to o, hence $\sigma_o(o^\perp)=o^\perp$. Since $o^\perp$ and op are fixed, so is their intersection p.

COROLLARY 5.4. *A flat y is fixed under the reflection in a point o if and only if y is incident with o or (when the geometry is elliptic) with $o^\perp$; y is elementwise fixed if and only if $y=o$ or (when the geometry is elliptic) y lies in $o^\perp$.*

Proof. By Proposition 5.3, if $y=o$ or y lies in $o^\perp$, then y is elementwise fixed, and conversely. Suppose y is fixed, but does pass through o. Then each point p in y is fixed because op and y are fixed.

COROLLARY 5.5. *For any points o and p, $\sigma_o=\sigma_p$ if and only if $o=p$.*

Proof. By Proposition 5.3, if $o\neq p$, then σ_o and σ_p have different sets of fixed points.

COROLLARY 5.6. *The reflections in points o and p commute if and only if $o=p$ or (when the geometry is elliptic) p lies in $o^\perp$.*

Proof. Suppose $\sigma_o\sigma_p=\sigma_p\sigma_o$, so that $\sigma_p=\sigma_o\sigma_p\sigma_o$. By Corollary 5.2, σ_p is the reflection in $\sigma_o(p)$. By Corollary 5.5, $p=\sigma_o(p)$. By Proposition 5.3, $o=p$ or p lies in $o^\perp$. The argument is reversible.

6. Reflections in arbitrary flats

A *reflection in a flat* x is a self inverse orthogonal collineation σ that satisfies the following two conditions: (i) if y is a flat incident with x, then $\sigma(y)=y$; and (ii) if y properly includes x, then $u(p)\neq p$ for some point p in y. (This definition is clearly consistent with those already given for reflections in points and lines.) The empty flat and the entire space have exactly one reflection: the identity. If x is a flat of dimension 2 or more and σ is a reflection in a flat w in x, then the restriction to x of σ is clearly a reflection in w with respect to the generalized metric geometry $(x, G_x, E_x, \perp_x)$.

PROPOSITION 6.1. *A reflection in a flat x leaves every perpendicular to x fixed, but not elementwise fixed.*

Proof. Let g be a perpendicular to x, so that x is not empty, nor a point, nor the entire space; let $o=g\cap x$; and let σ be a reflection in x. Then $\sigma(g)$ lies in $\sigma(x\vee g)=x\vee g$, hence $\sigma(g)$ lies in $k\vee g$ for some line k through o in x, because the geometry over o is projective. Evidently, g and $\sigma(g)$ are both perpendiculars to k through o in the plane $k\vee g$, hence $g=\sigma(g)$ by [7, Axiom O3].

Suppose g were elementwise fixed. There exists a point p in $x\vee g$ such that $\sigma(p)\neq p$, so that p lies neither in x nor in g. (See Figure 1a.) There is a line k through o in x such that op lies in $k \vee g$; and p lies in perpendiculars g' and k' to g and k, respectively. Let $q = g \cap g'$ and $s = k \cap k'$. Evidently, g' and $\sigma(g')$ are both perpendiculars to g through q in the plane $k \vee g$, hence $g' = \sigma(g')$. By [7, Proposition 3.4], $k' \perp x$, hence $\sigma(k') = k'$. Thus $g' = k'$, since both pass through p and $\sigma(p)$. (See Figure 1b.) By the Proposition in

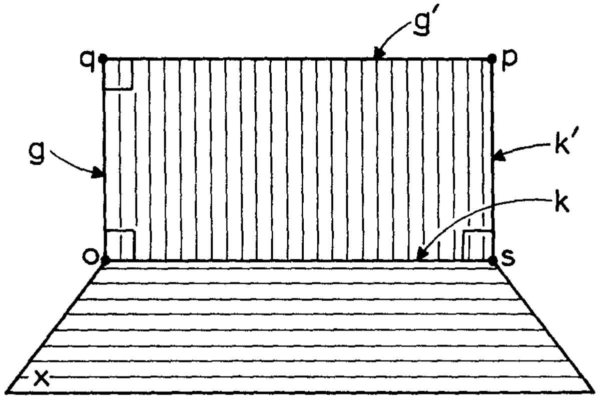

Fig. 1a.

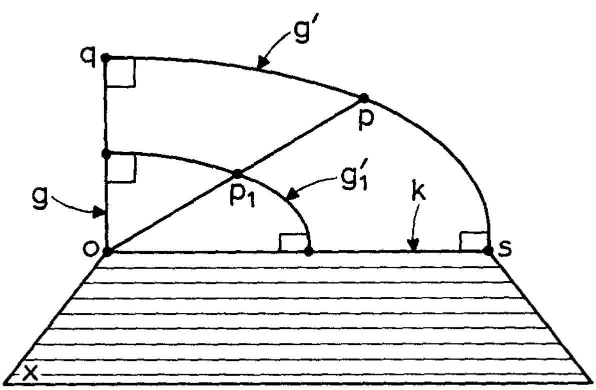

Fig. 1b.

§4, there exists a point p_1 in op distinct from o and p. By the above argument, p_1 lies in a perpendicular g'_1 to g and k. Thus p and p_1 lie in $o^\perp$, so that o lies in $o^\perp$, contradiction. Therefore, g must not be elementwise fixed.

COROLLARY 6.2. *Let o be a point in a flat x and p be a point in a perpendicular to x through o. If σ is a reflection in x, then $\sigma(p) = \sigma_o(p)$.*

Proof. Clearly, x is not a point, and it may be assumed that p does not lie in x. Let q be a point in $x - o$ and $e = opq$. By Proposition 6.1, σ leaves e

fixed, but not elementwise fixed. Hence the restriction to e of σ is the reflection in oq with respect to the generalized metric plane $(e, G_e, \perp_e)$. A similar statement holds of σ_o. The result follows from [1, §2].

COROLLARY 6.3. *In an orthocomplemented flat there is at most one reflection.*

Proof. Let σ and τ be reflections in an orthocomplemented flat x, and let p be any point. Then p lies in a perpendicular g to x; let $o=g\cap x$. By Corollary 6.2. $\sigma(p)=\sigma_o(p)=\tau(p)$.

The most important result of this paper is Proposition 6.5: there exists a reflection in a flat x if and only if x is orthocomplemented. The proof given below requires a lemma showing the existence of reflections in two and three dimensional flats. With little extra effort, the lemma can be stated in a much more general form, giving a formula for the reflection in an arbitrary finite dimensional flat. This is the content of the next Proposition. (Proposition 7.10 is an analogous result for orthocomplemented finite codimensional flats.)

PROPOSITION 6.4. *Let $x=g_1\vee\cdots\vee g_n$, where g_1 to g_n constitute a finite family of n mutually orthogonal lines through a point o. Define*

$$\sigma_x = \begin{cases} \sigma_o\sigma_{g_1}\cdots\sigma_{g_n} & \text{if } n \text{ is even,} \\ \sigma_{g_1}\cdots\sigma_{g_n} & \text{if } n \text{ is odd.} \end{cases}$$

Then σ_x is the reflection in x.

Proof. The result is true for $n=1$. Assume that it holds for some value $n=m\geqslant 1$. The result will then be demonstrated for the case $n=m+1$. Arguments like that of Corollary 6.2 show that σ_o and σ_{g_1} to σ_{g_n} all commute. Thus σ_x is a self inverse orthogonal collineation: clearly, it leaves fixed each flat through x.

Case 1: m is odd. First, it will be shown that each point p in $x=g_1\vee\cdots\vee+$ $+g_{m+1}$ is fixed. Since the geometry over o is projective, p lies in $e=g\vee g_{m+1}$ for some line g through o in $w=g_1\vee\cdots\vee g_m$. (See Figure 2.) By [7, Axiom O2 or O3], p lies in a perpendicular k to g in e; let $q=g\cap k$. By Corollary 6.2,

$$\sigma_{g_1}\cdots\sigma_{g_m}(p)=\sigma_w(p)=\sigma_q(p).$$

It follows that $\sigma_x(p)=\sigma_{g_{m+1}}\sigma_o\sigma_q(p)$. By restriction to the generalized metric plane $(e, G_e, \perp_e)$, it is seen that $\sigma_{g_{m+1}}\sigma_o\sigma_q(p)=\sigma_g\sigma_q(p)=p$.

Now suppose y is a flat properly including x. By [7, Proposition 4.11], there exists a perpendicular g_{m+2} to x through o in y, and a point p in g_{m+2}

such that $\sigma_o(p) \neq p$. By Corollary 6.2,

$$\sigma_x(p) = \sigma_o \sigma_{g_1} \cdots \sigma_{g_{m+1}}(p) = \sigma_o^{m+2}(p) = \sigma_o(p) \neq p.$$

Thus y is not elementwise fixed.

Case 2: m is even. The argument here is the same as that for Case 1; only the two displayed formulas differ. They take the following forms:

$$\sigma_o \sigma_{g_1} \cdots \sigma_{g_m}(p) = \sigma_w(p) = \sigma_q(p)$$
$$\sigma_x(p) = \sigma_{g_1} \cdots \sigma_{g_{m+1}}(p) = \sigma_o^{m+1}(p) = \sigma_o(p) \neq p.$$

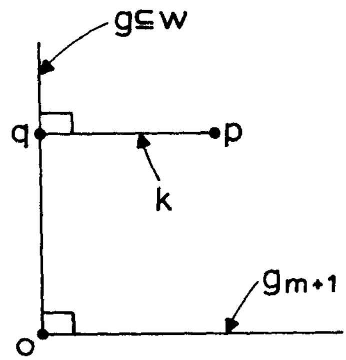

Fig. 2.

PROPOSITION 6.5. *There exists a reflection in a flat x if and only if x is orthocomplemented.*

Proof. It may be assumed that x is not empty, nor a point, nor the entire space. Suppose σ_x is a reflection in x. It will be shown that an arbitrary point p not in x lies in the reach of x. There exists a point q in $x \vee p$ such that $\sigma_x(q) \neq q$. It will be shown that $q\sigma_x(q) \perp x$, so that q lies in the reach of x. Then the Exchange Law implies that $x \vee p = x \vee q$, hence p lies in the reach of x.

It remains to show that $q\sigma_x(q) \perp x$. Let s be any point in $x - o$. (See Figure 3.) Since $\sigma(q)$ lies in $\sigma_x(x \vee qs) = x \vee qs$, there is a line k through s in x such that $\sigma_x(q)$ lies in $e = k \vee qs$. Now,

$$\sigma_x(e) = \sigma_x(k) \vee \sigma_x(q) \vee \sigma_x(s) = k \vee \sigma_x(q) \vee s \subseteq e,$$

hence $\sigma_x(e) = e$. The restriction to e of σ_x is thus the reflection in k with respect to the generalized metric plane $(e, G_e, \perp_e)$, hence $q\sigma_x(q) \perp k$. Let $o = q\sigma_x(q) \cap k$. To prove that $q\sigma_x(q) \perp x$, it suffices to show that $q\sigma_x(q) \perp l$ for each line l through o in x. Let $f = l \vee q$; then

$$\begin{aligned}\sigma_x(f) &= \sigma_x(l \vee oq) = \sigma_x(l) \vee \sigma_x(o) \vee \sigma_x(q) \\ &= l \vee o \vee \sigma_x(q) = f.\end{aligned}$$

The restriction to f of σ_x is thus the reflection in l with respect to the generalized metric plane $(f, G_f, \perp_f)$. Therefore, $q\sigma_x(q) \perp l$. The proof that the existence of a reflection in x implies that x is orthocomplemented is complete.

Conversely, suppose x is orthocomplemented. A reflection σ_x in x can be constructed as follows: for each point o in x, define $\sigma_x(o)=o$; for each point

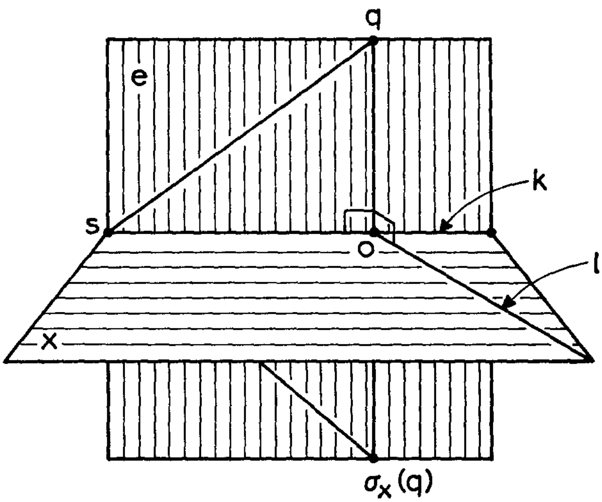

Fig. 3.

p not in x, let g be a perpendicular to x through p, let $o=g\cap x$, and define $\sigma_x(p)=\sigma_o(p)$. First, it must be verified that this construction provides a unique value for $\sigma_x(p)$. Suppose p lies in two distinct perpendiculars g and g' to x. Let $o=g\cap x$ and $o'=g'\cap x$, so that $o\neq o'$. By [8, Proposition 3.5], o and o' lie in $p^\perp$, hence p lies in $o^\perp$ and $o'^\perp$. By Proposition 5.3, $\sigma_o(p)=p=\sigma_{o'}(p)$. Thus, to each point p there is assigned a unique value $\sigma_x(p)$. Clearly, σ_x is a self inverse permutation of the entire space that satisfies the following two conditions: (i) if y is a flat incident with x, then $\sigma_x(y)=y$; and (ii) if y properly includes x, then $\sigma_x(p)\neq p$ for some point p in y.

It remains to show that σ_x is an orthogonal collination. Let p_1 to p_4 be any four points. For $i=1$ to 4, p_i lies in a perpendicular g_i to x; let $o_i=g_i\cap x$. Define $w=o_1\vee\cdots\vee o_4$. By Corollary 6.4, there is a reflection σ_w in w. By Corollary 6.2, $\sigma_x(p_i)=\sigma_{o_i}(p_i)=\sigma_w(p_i)$ for $i=1$ to 4. Then p_1 to p_3 are collinear if and only if their images under σ_w, hence under σ_x, are collinear. Suppose p_1 to p_3 are not collinear; similarly, p_1 to p_4 are coplanar if and only if $\sigma_x(p_1)$ to $\sigma_x(p_4)$ are, and $p_1p_2\perp p_1p_3$ if and only if $\sigma_x(p_1p_2)\perp\sigma_x(p_1p_3)$. Therefore σ_x is an orthogonal collineation.

Henceforth, the unique reflection in an orthocomplemented flat x will be denoted by σ_x.

The following trivial result is a generalization of Corollary 5.2.

PROPOSITION 6.6. *Let π be an orthogonal collineation and x be an orthocomplemented flat. Then $\sigma_{\pi(x)} = \pi\sigma_x\pi^{-1}$.*

7. Fixed flats and commuting reflections

The first results in this section characterize the flats fixed under a reflection. This information is used to determine when the reflections in two flats coincide, and when they commute. These results generalize those in §5.

LEMMA 7.1. *Let x be an orthocomplemented flat, g be a line, $\sigma_x(g)=g$, and let $g \cap x$ be a point. Then $g \perp x$.*

Proof. Let $o = g \cap x$. By [7, Proposition 4.11], there is a perpendicular l to x through o in $x \vee g$; and l lies in $g \vee k$ for some line k in x. Let $e = k \vee l$. By Proposition 6.1. the restriction to e of σ_x is the reflection in k with respect to the generalized metric plane $(e, G_e, \perp_e)$. Thus $g \perp k$ and g is not elementwise fixed. It follows that for any line k' through o in x, the restriction to $e' = k' \vee g$ of σ_x is the reflection in k' with respect to the generalized metric plane $(e', G_{e'}, \perp_{e'})$, hence $g \perp k'$. Therefore, $g \perp x$.

PROPOSITION 7.2. *The fixed points of the reflection in an orthocomplemented flat x are those in x and (when the geometry is elliptic) in $x^{\perp}$.*

Proof. It may be assumed that x is not empty, nor a point (by Proposition 5.3), nor the entire space. If p is a point in $x^{\perp}$, then p lies in two distinct perpendiculars to x by [8, Proposition 3.5], hence in two distinct fixed lines; hence p is fixed. Conversely, suppose p is a fixed point not lying in x. Let o and o' be two distinct points in x, so that op and $o'p$ are distinct fixed lines. By Lemma 7.1, they are perpendiculars to x, hence p lies in $x^{\perp}$.

PROPOSITION 7.3. *Let x and y be orthocomplemented flats. Then $\sigma_x(y)=y$ if and only if x is incident with y, or $x \perp y$, or (when the geometry is elliptic) $x^{\perp}$ is incident with y; moreover, y is elementwise fixed if and only if y lies in x or (when the geometry is elliptic) in $x^{\perp}$.*

Proof. Suppose $x \perp y$, so that there exists a point o in $x \cap y$ and flats x' and y' such that

$$o \subseteq x' \subseteq [o, y] \qquad x = x' \vee (x \cap y)$$
$$o \subseteq y' \subseteq [o, x] \qquad y = y' \vee (x \cap y).$$

Let p be a point in y. Since the geometry over o is projective, p lies in $k \vee l$ for some lines k in $x \cap y$ and l in y'. By Proposition 6.1, $\sigma_x(l)=l$, hence $\sigma_x(p)$ lies in $\sigma_x(k \vee l)=k \vee l$, hence in y. Thus y is fixed.

Suppose y properly includes $x^{\perp}$. Let p be a point in y. There is a perpendicular g to x through p. By [8, Corollary 3.10], $g \perp x^{\perp}$, hence g lies in y. By Proposition 6.1, $\sigma_x(p)$ lies in g, hence in y. Thus y is fixed.

The only remaining nontrivial part of this Proposition is the claim that if y is a fixed flat passing through a point p that is not fixed, and if y does not pass through x, then $x \perp y$. Under these hypotheses, there is a perpendicular g to x through p; then g lies in y because p and $\sigma_x(p)$ are distinct points in $g \cap y$. Thus, the point $o=g \cap x$ lies in $x \cap y$. Define $y'=y \cap [o, x]$. Clearly, $(x \cap y) \vee y'$ lies in y; it will be shown next that $(x \cap y) \vee y'=y$.

Let p' be a point in y. It must be shown that p' lies in $(x \cap y) \vee y'$. If p' is fixed, then p' lies in x or $x^{\perp}$ by Proposition 7.2. In the first case, p' lies in $x \cap y$; in the second, p' lies in $[o, x]$, hence in y', by [8, Corollary 3.10]. Thus it may be assumed that p' is not fixed. There is a perpendicular g' to x through p'. By the argument of the last paragraph, g' lies in y, hence the point $o'=g' \cap x$ lies in $x \cap y$. If $o=o'$, then p' lies in $[o, x]$, hence in y'. Therefore, assume $o \neq o'$. (See Figure 4.) By [7, Axiom O3], there is a perpendicular l to oo' through o in $oo'p'$. By [7, Proposition 3.4], l lies in $[o, x]$, hence in y'. Thus, p' lies in $o' \vee l$, hence in $(x \cap y) \vee y'$.

Now define $x'=x \cap [o, y]$. Clearly, $(x \cap y) \vee x'$ lies in x. The proof that $x \perp y$ will be complete once it is shown that an arbitrary point q in $x-y$ lies in $(x \cap y) \vee x'$. There is a perpendicular k to y through q. If $\sigma_x(k) \neq k$, then q would lie in $y^{\perp}$. Since this cannot be true for all points q in $x-y$, it may be assumed that k is fixed. Let $s=k \cap y$, so that s is fixed. By Proposition 7.2, s lies in x or $x^{\perp}$. In the latter case, $os \perp oq$ by [8, Proposition 3.7], hence $oq \perp y$ by [7, Proposition 3.4] hence q lies in $x'=x \cap [o, y]$, and thus in $(x \cap y) \vee x'$. In the former case, the same argument holds if $o=s$. Suppose $o \neq s$. Then q

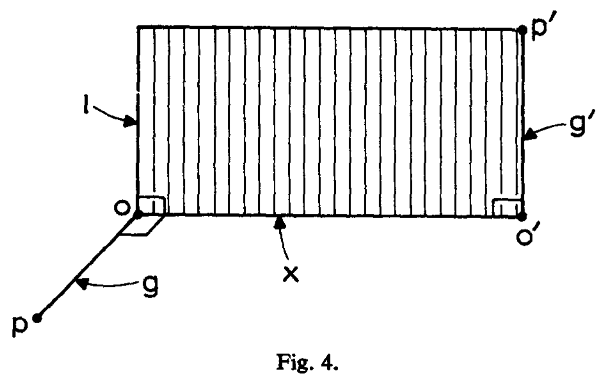

Fig. 4.

lies in $os \vee l$, where l is the perpendicular to os through o in oqs. By [7, Proposition 3.4], $l \perp y$, so that q lies in $(x \cap y) \vee x'$.

COROLLARY 7.4. *Let x and y be orthocomplemented flats. Then $\sigma_x = \sigma_y$ if and only if $x = y$ or (when the geometry is elliptic) $x = y^\perp$.*

Proof. If $\sigma_x = \sigma_y$, then $x = y$ or $x = y^\perp$ by Proposition 7.2. If the geometry is elliptic, then σ_x is a reflection in $x^\perp$ by Proposition 7.3.

COROLLARY 7.5. *Let x and y be orthocomplemented flats. Then $\sigma_x \sigma_y = \sigma_y \sigma_x$ if and only if x is incident with y, $x \perp y$, or (when the geometry is elliptic) y is incident with $x^\perp$ or $\sigma_x(y) = y^\perp$.*

Proof. By Proposition 6.6, $\sigma_x \sigma_y = \sigma_y \sigma_x$ if and only if $\sigma_{\sigma_x}(y) = \sigma_x \sigma_y \sigma_x = \sigma_y$, hence, by Corollary 7.4, if and only if $\sigma_x(y) = y$ or $\sigma_x(y) = y^\perp$. The result then follows from Proposition 7.3.

The next results give succinct descriptions of the compositions of two commuting reflections σ_x and σ_y when $x \perp y$ or x is incident with y or $y^\perp$. Unfortunately, no such description is at hand for the case $\sigma_x(y) = y^\perp$.

LEMMA 7.6. *Let x, y, and z be orthocomplemented flats, $x \perp y$, $x \vee y = z$, and let $o = x \cap y$ be a point. Then $\sigma_x \sigma_y = \sigma_z \sigma_o$.*

Proof. By Corollary 7.5, σ_x, σ_y, and σ_o commute, so that $\sigma = \sigma_x \sigma_y \sigma_o$ is a self inverse orthogonal collineation. It will be shown next that σ leaves each point p in z fixed. Since the geometry over o is projective, p lies in $k \vee l$ for some lines k and l through o in x and y, respectively. From the definition of $x \perp y$, it follows that x lies in $[o, y]$; thus $k \perp y$ and $l \perp x$. Therefore, k is elementwise fixed under σ_x, and fixed, but not elementwise fixed, under σ_y and σ_o; moreover, l is elementwise fixed under σ_y, and fixed, but not elementwise fixed, under σ_x and σ_o. Therefore, the plane $e = k \vee l$ is fixed under σ_x, σ_y, and σ_o; their restrictions to e are the reflections in k, l, and o, respectively, with respect to the generalized metric plane $(e, G_e, \perp_e)$. Thus $\sigma(p) = \sigma_x \sigma_y \sigma_o(p) = p$.

Clearly, an arbitrary flat w that properly includes z is fixed under σ. To conclude that $\sigma = \sigma_z$, it remains to show only that w is not elementwise fixed. By [7, Proposition 4.11], there exists a perpendicular g to z through o in w. Let k be any line through o in x. (See Figure 5.) Then k is elementwise fixed under σ_x, and g is fixed, but not elementwise fixed. Thus the restriction to $f = g \vee h$ of σ_x is the reflection in k with respect to the generalized metric plane $(f, G_f, \perp_f)$. Moreover, each line m through o in f is fixed, but not elementwise fixed, under σ_o and, since $m \perp y$, under σ_y. Thus the restrictions to f of σ_o and σ_y are both equal to reflection in o with respect to the

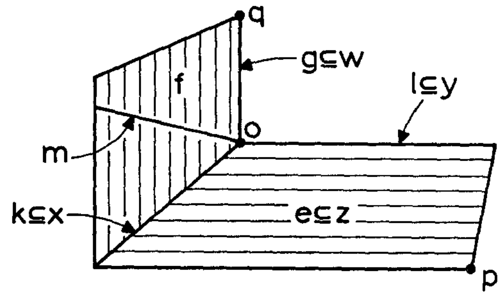

Fig. 5.

generalized metric plane $(f, G_f, \perp_f)$. It follows that the restriction to f of $\sigma = \sigma_x \sigma_y \sigma_o$ is equal to that of σ_x. Thus g, and hence w, is not elementwise fixed under σ.

PROPOSITION 7.7. *If x, y, and $x \vee y$ are orthocomplemented flats and $x \perp y$, then $x \cap y$ is orthocomplemented and $\sigma_x \sigma_y = \sigma_{x \cap y} \sigma_{x \vee y}$.*

Proof. Define $x' = x \cap [o, y]$. By [7, Proposition 5.2], $x = (x \cap y) \vee x'$. Short computations using this equation show that $x \cap y$ and x' are orthocomplemented. From the relations

$$\begin{array}{ll} x = x' \vee (x \cap y) & x' \cap (x \vee y) = o \\ x' \perp x \cap y & x' \cap y = o \\ x \vee y = x' \vee y & x' \perp y \end{array}$$

and Lemma 7.6 follow the equations

$$\sigma_{x'} \sigma_{x \cap y} = \sigma_x \sigma_o \qquad \sigma_{x'} \sigma_y = \sigma_{x \vee y} \sigma_o .$$

From these equations and Corollary 7.5 follows the desired result:

$$\sigma_x \sigma_y = \sigma_o \sigma_{x \cap y} \sigma_{x'} \sigma_y = \sigma_o \sigma_{x \cap y} \sigma_{x \vee y} \sigma_o = \sigma_{x \cap y} \sigma_{x \vee y}$$

COROLLARY 7.8. *Let x and y orthocomplemented flats, and suppose x lies in y. Let o be a point in x, and $w = [o, y]$. Then w is orthocomplemented, $w \perp y$, $w \vee y = r$, $w \cap y = x$, and $\sigma_x \sigma_y = \sigma_w$.*

Proof. The statement is itself a sketch of a proof. The details, which involve several applications of the Modular Law, are left to the reader. (Note that the last equation, with Corollary 7.4, implies that w is independent of the choice of o in x.) It is left to the reader to restate Corollary 7.8 to apply to the case where the geometry is elliptic and x lies in $y^{\perp}$. The next Corollary is a special case of Proposition 7.7.

COROLLARY 7.9. *Let o be a point in an orthocomplemented flat* x. *Then* $\sigma_o\sigma_x=\sigma_{[o,x]}$.

The final result of this paper is the analog of Proposition 6.4 for finite codimensional flats. The proof, an easy application of Proposition 6.4, Corollary 7.9, and [7, Corollary 4.7], is left to the reader.

PROPOSITION 7.10. *Let x be an orthocomplemented flat of finite codimension n. Then there exist* $n+1$ *mutually orthogonal orthocomplemented hyperplanes* h_1 *to* h_n such that

$$x = h_0 \cap \ldots \cap h_n \quad \sigma_x = \sigma_{h_0} \cdots \sigma_{h_n}.$$

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