M. HENK

A GENERALIZATION OF JUNG'S THEOREM

ABSTRACT. Jung's theorem establishes a relation between circumradius and diameter of a convex body. Half of the diameter can be interpreted as the maximum of circumradii of all 1-dimensional sections or 1-dimensional orthogonal projections of a convex body. This point of view leads to two series of j-dimensional circumradii, defined via sections or projections. In this paper we study some relations between these circumradii and by this we find a natural generalization of Jung's theorem.

INTRODUCTION

Throughout this paper E^d denotes the d-dimensional Euclidian space and the set of all convex bodies $K \subset E^d$ – compact convex sets – is denoted by $\mathcal{K}^d$. The affine (convex) hull of a subset $P \subset E^d$ is denoted by aff(P) (conv(P)) and dim(P) denotes the dimension of the affine hull of P. The interior of P is denoted by int(P) and relint(P) denotes the interior with respect to the affine hull of P. $\|\cdot\|$ denotes the Euclidean norm and the set of all i-dimensional linear subspaces of E^d is denoted by $\mathcal{L}_i^d$. $L^\perp$ denotes for $L \in \mathcal{L}_i^d$ the total orthogonal complement and for $K \in \mathcal{K}^d$, $L \in \mathcal{L}_i^d$ the orthogonal projection of K onto L is denoted by $K \mid L$.

The diameter, circumradius and inradius of a convex body $K \in \mathcal{K}^d$ are denoted by $D(K)$, $R(K)$ and $r(K)$, respectively. For a detailed description of these functions we refer to the book [BF]. With this notation we can define the following i-dimensional circumradii:

DEFINITION. For $K \in \mathcal{K}^d$ and $1 \leqslant i \leqslant d$ let

(i) $$R_\sigma^i(K) := \max_{L \in \mathcal{L}_i^d} \max_{x \in L^\perp} R(K \cap (x + L)),$$

(ii) $$R_\pi^i(K) := \max_{L \in \mathcal{L}_i^d} R(K \mid L).$$

We obviously have $R_\sigma^{i+1}(K) \geqslant R_\sigma^i(K)$, $R_\pi^{i+1}(K) \geqslant R_\pi^i(K)$, $R_\pi^i(K) \geqslant R_\sigma^i(K)$ and $R_\sigma^d(K) = R_\pi^d(K) = R(K)$, $R_\sigma^1(K) = R_\pi^1(K) = D(K)/2$.

Jung's theorem [J] states a relation between the circumradius and the diameter of a convex body. On account of the definition of $R_\sigma^d(K)$, $R_\sigma^1(K)$ we can describe his result as follows:

I would like to thank Prof. Dr J. M. Wills, who called my attention to these generalized circumradii.

Geometriae Dedicata **42**: 235–240, 1992.

JUNG'S THEOREM. *Let $K \in \mathscr{K}^d$. Then*

$$(1.1) \qquad R_\sigma^d(K) \leqslant \sqrt{\frac{2d}{d+1}}\, R_\sigma^1(K),$$

and equality holds if and only if K contains a regular d-simplex with edge length $D(K)$.

In the same way the theorem may be formulated with the circumradii $R_\pi^d(K)$ and $R_\pi^1(K)$. Here we study in general the relations between the i-dimensional and j-dimensional circumradius of both these series and obtain the following results.

1. Results

THEOREM 1. *Let $K \in \mathscr{K}^d$ and $1 \leqslant j \leqslant i \leqslant d$. Then*

$$(1.2) \qquad R_\sigma^i(K) \leqslant \sqrt{\frac{i(j+1)}{j(i+1)}}\, R_\sigma^j(K),$$

and equality holds for $i < j$ if and only if K contains a regular i-simplex with edge length $R_\sigma^j(K)\sqrt{[2(j+1)]/j}$.

THEOREM 2. *Let $K \in \mathscr{K}^d$ and $1 \leqslant j \leqslant i \leqslant d$. Then*

$$(1.3) \qquad R_\pi^i(K) \leqslant \sqrt{\frac{i(j+1)}{j(i+1)}}\, R_\pi^j(K),$$

and equality holds for $i > j$ if and only if an orthogonal projection of K onto an i-dimensional linear subspace contains a regular i-simplex with edge length $R_\pi^j(K)\sqrt{[2(j+1)]/j}$.

Let us remark that both theorems are a generalization of the classical theorem of Jung since for $i = d, j = 1$ the inequalities (1.2) and (1.3) become (1.1).

2. Proofs

To prove these theorems it is necessary to examine in more detail the circumradii of simplices since the circumradius of a convex body K is determined by the circumradius of a certain simplex $\bar{T} \subset K$. This well-known fact is described in the following lemma.

LEMMA 1. *Let $K \subset \mathscr{K}^d$ and let 0 be the center of the circumball of K. Then* *there exists a k-simplex* $\bar{T} \subset K$, $\bar{T} = \operatorname{conv}(\{x^0, \ldots, x^k\})$ *with*

$$0 \in \operatorname{relint}(\bar{T}), \quad R(\bar{T}) = R(K) \quad \text{and} \quad \|x^i\| = R(K), \quad 0 \leqslant i \leqslant k.$$

Proof. Cf. [BF, pp. 9 and 54]. □

With this lemma it is easy to find $(d-1)$-dimensional planes for a simplex which produce the maximal $(d-1)$-circumradius with respect to projections or sections.

LEMMA 2. *Let* $T \in \mathscr{K}^d$ *be a d-simplex,* $\hat{F}$ *a facet of* T *with maximal circumradius and* $\hat{L} \in \mathscr{L}^d_{d-1}$, $\hat{x} \in \hat{L}^\perp$ *with* $\hat{x} + \hat{L} = \operatorname{aff}(\hat{F})$. *Then*

(i) $R^{d-1}_\sigma(T) = R(T \cap (\hat{x} + \hat{L})) = R(\hat{F})$,

(ii) $R^{d-1}_\pi(T) = R(T \mid \hat{L}) = R(\hat{F})$.

Proof. Let $L_{d-1} \in \mathscr{L}^d_{d-1}$ with $R^{d-1}_\pi(T) = R(T \mid L_{d-1})$ and let $T \mid L_{d-1}$ be the convex hull of the points $x^0, \ldots, x^d$, where $x^0, \ldots, x^d$ denote the images of the vertices of T under the projection onto L_{d-1}. Further, let 0 be the center of the circumball of $T \mid L_{d-1}$ and $\bar{T} \subset T \mid L_{d-1}$, $\bar{T} = \operatorname{conv}(\{x^0, \ldots, x^k\})$, $1 \leqslant k \leqslant d-1$, a k-simplex with the properties of Lemma 1.

Now let F be a facet of T containing such $k+1$ vertices which are mapped onto $x^0, \ldots, x^k$ with respect to the orthogonal projection onto L_{d-1}. We have

$$R(\hat{F}) \geqslant R(F) \geqslant R(F \mid L_{d-1}) \geqslant R(\bar{T}) = R^{d-1}_\pi(T);$$

on the other hand $R(\hat{F}) \leqslant R^{d-1}_\sigma(T) \leqslant R^{d-1}_\pi(T)$ and the assertion follows. □

On account of the lemma above we have $R(S)/R^{d-1}_\pi(S) = R(S)/R^{d-1}_\sigma(S) = d/(d^2-1)^{1/2}$ for a regular d-simplex S. This is even an upper bound for every simplex as shown in the next lemma.

LEMMA 3. *Let* $T \in \mathscr{K}^d$ *be a simplex. Then*

(i) $$R(T) \leqslant \frac{d}{\sqrt{d^2-1}} R^{d-1}_\sigma(T),$$

(ii) $$R(T) \leqslant \frac{d}{\sqrt{d^2-1}} R^{d-1}_\pi(T),$$

and equality holds if and only if T *is a regular d-simplex.*

Proof. If T is a regular d-simplex we have equality by Lemma 2. Hence on account of $R^{d-1}_\sigma(T) \leqslant R^{d-1}_\pi(T)$ is suffices to prove the lemma for the $(d-1)$-circumradius $R^{d-1}_\sigma(T)$.

Let 0 be the center of the circumball of T and let $\{x^0, \ldots, x^k\}$ be a suitable subset of the vertices of T, such that $\bar{T} = \operatorname{conv}(\{x^0, \ldots, x^k\})$ has the properties of Lemma 1. If $k < d$ then

$$(2.1) \qquad R(T) = R(\bar{T}) = R_\sigma^{d-1}(T) < \frac{d}{\sqrt{d^2-1}} R_\sigma^{d-1}(T).$$

Hence we may assume that $T = \mathrm{conv}(\{x^0, \ldots, x^d\})$ is a d-simplex with $0 \in \mathrm{int}(T)$ and $\|x^i\| = R(T)$, $0 \leqslant i \leqslant d$.

Let λ be the maximal radius of a d-dimensional ball with center 0, which is contained in T. This ball touches a facet F of T in a point λa, $\|a\| = 1$. Let F be given by $\mathrm{conv}(\{x^1, \ldots, x^d\})$. Since a is a normal vector of $\mathrm{aff}(F)$ we have

$$\|x^i - \lambda a\|^2 = R(T)^2 - \lambda^2, \quad 1 \leqslant i \leqslant d.$$

Hence λa is the center of the circumball of F [BF, p. 54] and it follows that

$$(2.2) \qquad R(T)^2 - R_\sigma^{d-1}(T)^2 \leqslant \lambda^2.$$

For the inradius $r(T)$ of a simplex T we have $r(T) \leqslant R(T)/d$ [F] and so by the choice of λ

$$(2.3) \qquad \lambda^2 \leqslant \frac{R(T)^2}{d^2}.$$

Along with (2.2) this shows inequality (i). If we have equality in relation (i) then, from (2.1), (2.2) and (2.3), it follows that T is a d-simplex with $r(T) = R(T)/d$. This is only possible if T is regular [F]. □

Now we are able to prove the theorems.

PROOF OF THEOREM 1. It obviously suffices to show the inequalities

$$(2.4) \qquad R_\sigma^i(K) \leqslant \frac{i}{\sqrt{i^2-1}} R_\sigma^{i-1}(K), \quad 1 < i \leqslant d.$$

Since the circumradii are invariant with respect to translations we may assume that there is an i-dimensional linear subspace $L_i \in \mathscr{L}_i^d$ with $R_\sigma^i(K) = R(K \cap L_i)$ and 0 is the center of the circumball of $K \cap L_i$. Moreover, let $T \subset (K \cap L_i)$ be a k-simplex with the properties of Lemma 1. Denoting by $R_\sigma^{i-1}(T; L_i)$ the $(i-1)$-circumradius of T with respect to the Euclidean space L_i, we get from Lemma 3

$$(2.5) \qquad R(T) \leqslant \frac{i}{\sqrt{i^2-1}} R_\sigma^{i-1}(T; L_i).$$

By the choice of T we have $R(T) = R_\sigma^i(K)$ and since $R_\sigma^{i-1}(K) \geqslant R_\sigma^{i-1}(T; L_i)$ the inequalities (1.1) are shown.

If an inequality of (1.1) is satisfied with equality for $i > j$ we must have equality in (2.4) and (2.5). By Lemma 3 this means that T is a regular i-simplex which satisfies the relation

$$R(T) = R^i_\sigma(K) = \sqrt{\frac{i(j+1)}{j(i+1)}}\, R^j_\sigma(K). \tag{2.6}$$

Since T is regular we have $R(T) = (i/(2i+2))^{1/2} D(T)$ and by (2.6) we see that T has diameter (edge length) $R^j_\sigma(K)((2j+2)/j)^{1/2}$.

Now let T be a regular i-simplex contained in K with the given edge length. On account of (1.2) we get

$$R(T) = \sqrt{\frac{i}{2i+2}}\, D(T) = \sqrt{\frac{i(j+1)}{j(i+1)}}\, R^j_\sigma(K) \geqslant R^i_\sigma(K). \tag{2.7}$$

Clearly $R(T) \leqslant R^i_\sigma(K)$ and so we can replace '$\leqslant$' by '$=$' in (2.7). □

PROOF OF THEOREM 2. On account of Lemma 3 the proof can be done in the same way as the proof of Theorem 1. □

3. Remarks

(1) If we replace the first maximum condition by a minimum condition in the definition of the circumradii we get two other series of i-circumradii which now start with the half of the width of a convex body. If we further replace the circumradius by the inradius we totally get four series of circumradii and four series of inradii. Some of these functionals are studied in *Computational Geometry* [GK]. For a survey of these generalized circumradii and inradii we refer to [H].

(2) Theorems involving inradius, circumradius, diameter and width have a long tradition in the geometry of convex bodies. In this context we refer to [BL], [BF], [E], [DGK].

References

[BF] Bonnesen, T. and Fenchel, W., *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[BL] Blaschke, W., *Kreis und Kugel*, Veit (2nd edn), W. de Gruyter, Berlin, Leipzig, 1916.

[DGK] Danzer, L. Grünbaum, B. and Klee, V., 'Helly's theorem and its relatives', In *Convexity* (V. Klee, ed.), *Amer. Math. Soc. Proc. Symp. Pure Math.* **13** (1963), 101–180.

[E] Eggleston, H. G., *Convexity*, Cambridge Univ. Press, Cambridge, 1958, 1969.

[F] Fejes Tóth, L. 'Extremum properties of the regular polytopes', *Acta Math. Acad. Sci. Hungar.* **6** (1955), 143–146.

[GK] Gritzmann, P. and Klee, V. 'Inner and outer j-radii of convex bodies in finite-dimensional normed spaces', (to appear in *Discrete Comput. Geom.*)

[H] Henk, M. 'Ungleichungen für sukzessive Minima und verallgemeinerte In- und Umkugelradien konvexer Körper', Dissertation, Universität Siegen, 1991.

[J] Jung, H. W. E. 'Über die kleinste Kugel, die eine räumliche Figur einschließt', *J. reine angew. Math.* **123** (1901), 241–257.

Author's address:

M. Henk,
Mathematisches Institut,
Universität Siegen,
Hölderlinstrasse 3,
D-5900 Siegen,
Germany.

(Received, March 7, 1991)