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MULTIPLE INTERSECTIONS OF DIAGONALS OF REGULAR POLYGONS, AND RELATED TOPICS

ABSTRACT. Our main concern is to investigate geometrically all sets of three concurrent chords of regular polygons or, equivalently, all adventitious quadrangles (that is, all quadrangles such that the angle between every pair of the six sides is an integral multiple of π/n radians). Most of our results are stated without proof. The proofs are elementary, often consisting of straightforward verification; to include them would make the paper much longer and less readable.

1. INTRODUCTION

1.1. This article is the result of investigations by the author, in collaboration with Dr P.A.B. Pleasants and Dr N.M. Stephens of University College Cardiff, into a problem posed by Colin Tripp [13]; Dr G.R.H. Greaves, also of University College Cardiff, has contributed some useful ideas.

The starting-point of the problem is a well-known geometrical puzzle: find the angle θ in Figure 1 using elementary 'pure' geometry. It is shown in [13] that $\theta = 30°$, so the figure has an interesting property: the angle between each pair of the six sides of the quadrangle $BCDE$ is a multiple of $10° = \pi/18$ radians. Tripp therefore says that the quadrangle $BCDE$ is *adventitious*, because it occurs by chance: if we start with the same triangle ABC and choose other multiples of 10° for the angles BCE and CBD, the angle θ will not in general be a rational number of degrees. This idea can be generalized [13]:

DEFINITION. A quadrangle is *n-adventitious*, where n is a positive integer, if the angle between each pair of the six sides of the quadrangle is an integral multiple of π/n radians.[1] A quadrangle is *adventitious* if it is n-adventitious for some integer n.

Tripp's problem is essentially this: enumerate all adventitious quadrangles, and prove their existence using only elementary pure geometry. A partial solution to the problem of enumeration has been given by Meek and Tripp [11], but an independent method of investigation is given in the present paper.

1.2. If in Figure 1 we draw the circumcircle of BCD, we obtain Figure 2. Since all angles of the quadrangle $BCDE$ are multiples of 10°, it is easily seen

[1] Tripp uses the term *adventitious quadrilateral*, but since our figure has four vertices and six sides, *quadrangle* is more appropriate.

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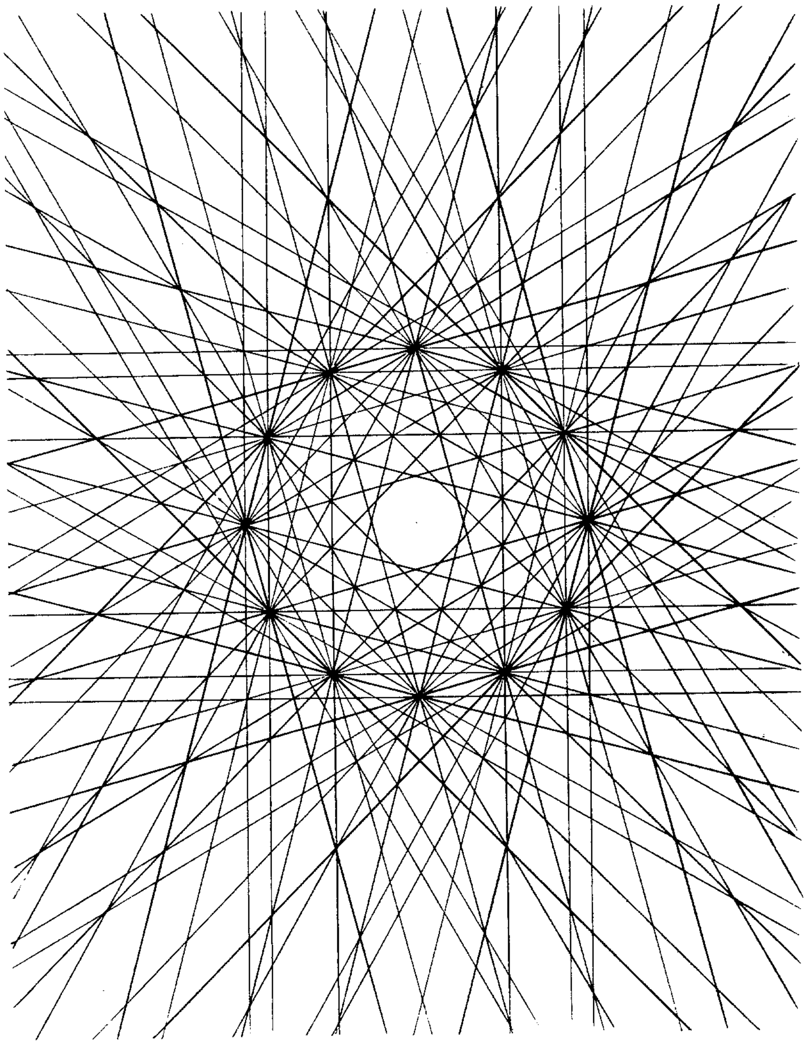


Frontispiece

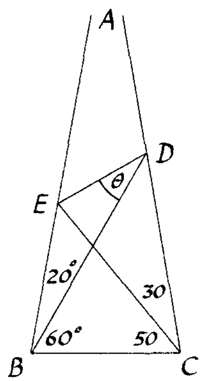


Fig. 1.

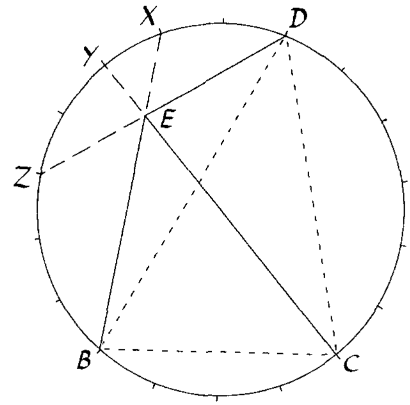


Fig. 2.

that $B\ C\ D\ X\ Y Z$ are six vertices of a regular 18-gon inscribed in the circle. Thus we have three intersecting diagonals BX, CY, DZ of a regular 18-gon.

In general, if $BCDE$ were a non-cyclic n-adventitious quadrangle, BX, CY, DZ would be intersecting chords of a regular n-gon. (This idea is due to Greaves and Lunnon.) Clearly then our original problem is equivalent to the following:

> *Find all triple intersections of diagonals of regular polygons, and prove their existence using elementary pure geometry.*

(The existence of *cyclic* n-adventitious quadrangles $BCDE$ is trivial: we simply take B, C, D, E to be any four vertices of a regular n-gon.)

We approached the problem in three main stages: (i) initial geometrical ideas, (ii) the algebraic determination of *all* triple intersections, (iii) the geometrical proof of the existence of all these triple intersections. We later discovered that (ii) was solved forty years ago by Bol [2] (see Section 4), but we do introduce some new algebraic ideas here.

It is gratifying to have geometrical proofs of existence, but an algebraic approach seems necessary to make sure that we have exhausted all possibilities.

Unlike Bol and Harborth [6, 7] we shall be concerned with triple intersections both inside *and outside* the circle;[2] thus one or more of the intersecting diagonals may be a tangent (see Figures 7 and 8). Three intersecting diagonals will be called a *triplet*. Since the number of multiple intersections rapidly becomes large as n increases (frontispiece) we shall regard triplets that can be obtained from each other by rotation or reflection as being identical.

[2] Although we use the term 'polygon', it is more convenient to think of n points equally spaced round a circle.

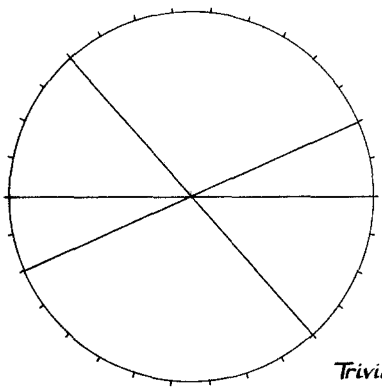


Fig. 3.

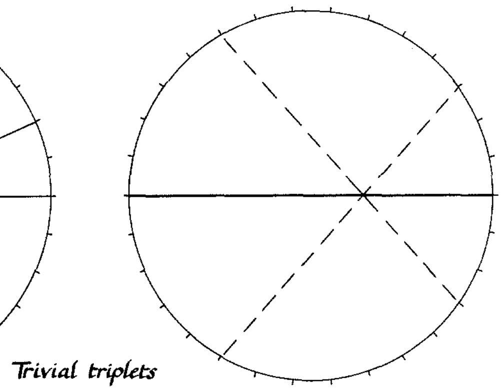


Fig. 4.

2. Initial geometrical ideas

2.1. Triplets of the types illustrated in Figures 3 and 4 are *trivial*; they can be called *central triplets* and *symmetric trivial triplets* respectively.

2.2. If n is even, let A, B, C be vertices of a regular n-gon such that the arcs BC, CA, AB all have even length (Figure 5). Let J be the incentre of triangle ABC. Since AJ bisects $\angle BAC$, P bisects the arc BC and so P is a vertex of the n-gon; similarly, so are Q and R. We can also obtain a triplet by taking J to be an excentre of ABC. In all cases (incentre or excentre) J is the orthocentre of triangle PQR (Figures 6 and 7), so the triplet AP, BQ, CR will be called an *orthic triplet*.

If triangle PQR is obtuse, its orthocentre will lie outside the circle (Figure 7) and for suitable positions of P, Q, R one chord will be a tangent (Figure 8).

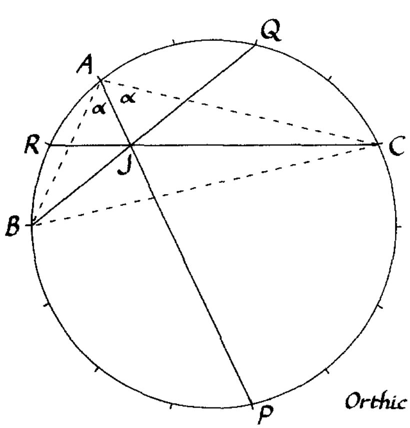


Fig. 5.

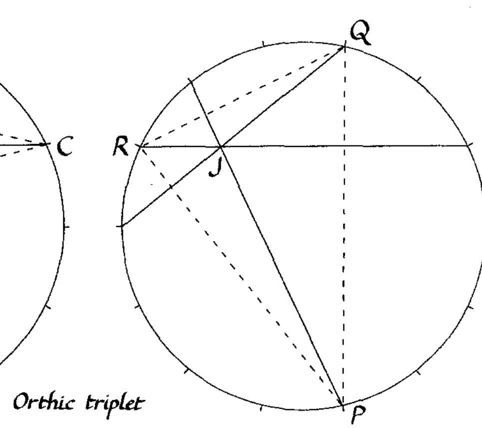


Fig. 6.

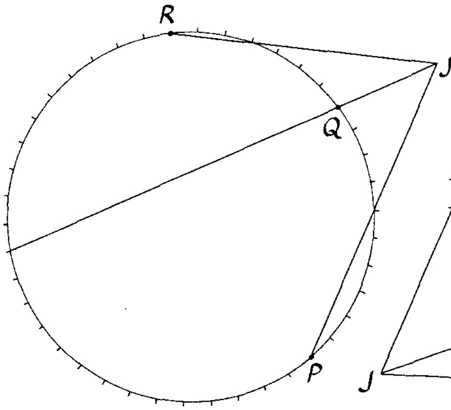


Fig. 7.

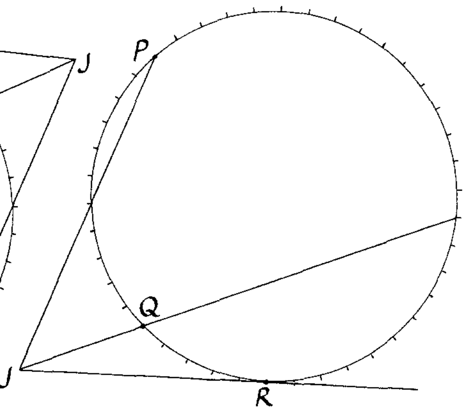
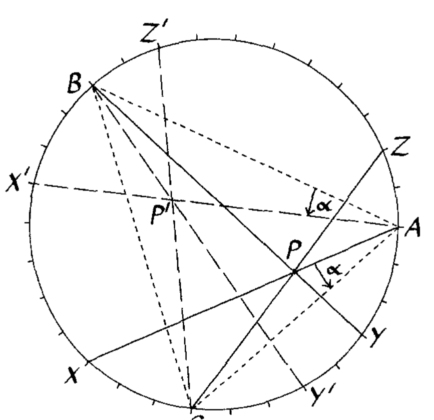


Fig. 8.

There are many non-trivial triplets that are not orthic triplets (the triplet in Figure 2 for example). In 2.3 and 2.4 we describe two methods of generating one triplet from another; these two methods can be combined to produce non-orthic triplets.

2.3. Let AX, BY, CZ be a triplet, with P as the point of intersection (Figure 9). Let AX' be the line such that $\angle BAX' = \angle XAC$ (it is important that these two angles have the same sign) and define BY', CZ' similarly, as shown in the figure. Then X', Y', Z' are vertices of the polygon, since equal angles on a circle are subtended by equal arcs; moreover AX', BY', CZ' are



Isogonal conjugation

Fig. 9.

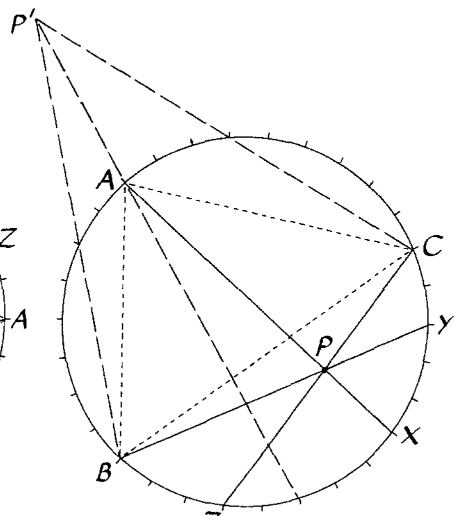


Fig. 10.

concurrent in a point P' called the *isogonal conjugate* of P with respect to triangle ABC. The existence of this isogonal conjugate can be proved using similar triangles, trilinear coordinates [3b, p. 93], trigonometry [3a, p. 49] or the geometry of reflections [1, p. 16].

This process of obtaining one triplet from another is called *isogonal conjugation*, and the second triplet is an *isogonal conjugate* of the first. Since each diagonal has two ends, we can choose the triangle ABC in $2^3 = 8$ ways when a triplet is given. Hence a triplet will in general have eight isogonal conjugates. (In Figure 10, for instance, we start with the same triplet as in Figure 9, but we obtain a different isogonal conjugate by using a different triangle.) Each of these in turn has eight conjugates, so it would seem that by repeating the process of isogonal conjugation we can generate a large number of triplets. However, if we start with a given triplet, the process of successive isogonal conjugation yields a total of only fifteen triplets (or fewer in special cases), all related to each other in a symmetric manner that we shall discuss in detail in Section 3, and forming a *conjugacy class* of triplets. Six of these triplets have intersection points inside the circle (*internal* triplets) and nine outside (*external* triplets).

The isogonal conjugate of the incentre of a triangle, or of an excentre, is itself, and the isogonal conjugate of the orthocentre is the circumcentre; it is easily verified that the conjugacy class of an orthic triplet consists of a central triplet, six symmetric trivial triplets, and four orthic triplets each counted twice. Hence we can speak of an *orthic class* or a *trivial class* of triplets. We can therefore generate no new types of triplet by isogonal conjugation alone from trivial and orthic triplets: we need also to use the process described in 2.4.

2.4. Suppose we have a triplet with two equal diagonals, as in Figure 11. The line of symmetry of these diagonals is also a line of symmetry of the n-gon:

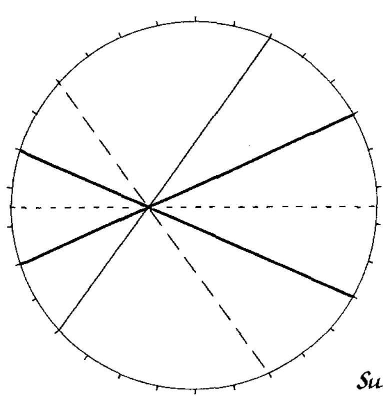


Fig. 11.

Substitution

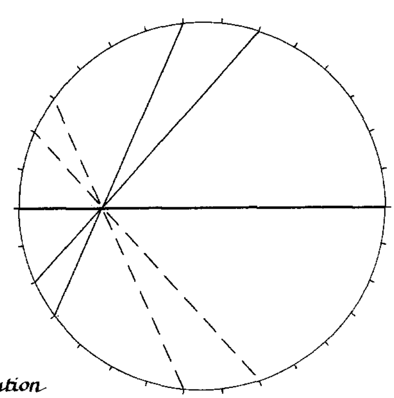


Fig. 12.

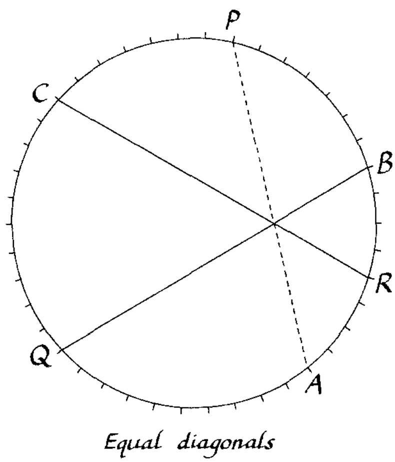


Fig. 13.

If we reflect the third diagonal in this line of symmetry, we obtain four intersecting diagonals, or five if the line of symmetry is itself a (diametral) diagonal. From these four or five diagonals we can select (in one or more ways) three diagonals forming a triplet that will in general belong to a different conjugacy class. This process is called *substitution.*

We may also apply the process of substitution if one diagonal of the initial triplet is a diameter, in which we reflect (Figure 12).

If the line of symmetry in Figure 11 is not a diagonal of the n-gon, it will be a diagonal of a $2n$-gon, so substitution can be used to pass from an n-triplet to a $2n$-triplet, and *vice-versa.*

2.5. Using the notation of 2.2, suppose that n is a multiple of 6, and that the arc QR of the triangle PQR is one-third of the circumference (Figure 13); we then easily prove that the altitudes through Q, R give equal chords QB, RC of the n-gon. Thus we can perform a substitution to obtain a new triplet. The conjugacy class of this new triplet (which will not in general be an orthic class) may contain other triplets on which we can perform a substitution; the process can be continued, and thus we obtain many triplets.

2.6. After failing as yet to find any other methods of generating triplets, we now ask three questions: (i) are there any triplets if n is odd? (ii) are there any triplets other than trivial and orthic triplets if n is not a multiple of 6? (iii) can we generate *all* triplets from orthic triplets by substitution and isogonal conjugation? The answer to each question is 'no', but the answer to (iii) only just fails to be 'yes'. We obtain the answers by determining all triplets algebraically in Section 4.

2.7. The process of obtaining one conjugacy class from another by substitution implies and requires that each of the two classes contains a triplet with either two equal chords or a diameter. A class containing no such triplet is said to be *inaccessible*, as it cannot be obtained in one or more steps from an orthic class using substitutions. (This argument does not guarantee that every class containing a triplet with either two equal chords or a diameter *is* accessible from an orthic class, although this is in fact the case.) In Section 4 we obtain various inaccessible classes, but not many, which is why the answer to (iii) is 'no'.

3. Conjugacy classes

3.1. We remarked at the end of 1.2 that triplets that can be obtained from each other by rotation or reflection of the n-gon are regarded as being identical. Let **A** denote a given triplet (or rather, a set of identical triplets). The fact that only fifteen triplets (including **A**) can be obtained from **A** by successive isogonal conjugations may be verified by tedious but straightforward calculation; the work is most easily done if we label the vertices of the n-gon either by roots of unity (taking the circle as unit circle in the Argand diagram) or by integers modulo n. How are these triplets related to each other?

Suppose **A** consists of the chords XY, ZT, UV. Let **B** denote the isogonal conjugate of **A** obtained by using the triangle XZU, and let **C** denote the isogonal conjugate obtained from the 'opposite' triangle YTV. We call **B** and **C** *opposite conjugates* of **A**; it is easily verified that **A** and **C** are opposite conjugates of **B**, and similarly **A** and **B** are opposite conjugates of **C**. Let us denote this situation by (**ABC**) (or (**CBA**), or (**BAC**), etc.). We may denote the remaining six conjugates of **A** by **D**, **E**, **F**, **G**, **H**, **K**, where (**ADE**), (**AFG**), (**AHK**).

There are six more triplets that can be obtained from **A** by two successive isogonal conjugations; it is easily verified that if we label them **L**, **M**, **N**, **P**, **Q**, **R** in a suitable order, we have

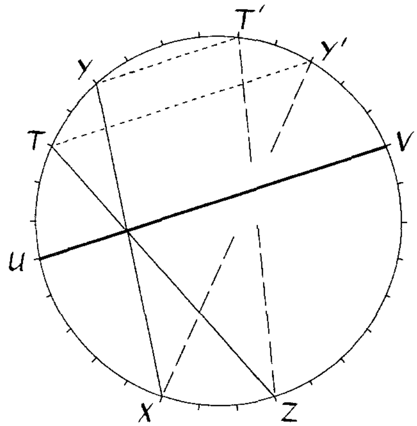
(**BEL**) (**BGM**) (**BKN**) (**CDL**) (**CFM**) (**CHN**) (**DFQ**)

(**DHP**) (**EGQ**) (**EKP**) (**FHR**) (**GKR**) (**LMQ**) (**LNP**)

(**MNR**) (**PQR**).

3.2. If we now refer to **A**, **B**, . . . , **R** as 'points' and (**ABC**) etc. as 'lines', we have a configuration of fifteen points and twenty lines. This is a familiar configuration, and is most easily visualised by taking six general 3-spaces in 4-space, which meet by fours in fifteen points, by threes in twenty lines, and by twos in fifteen planes; the points and lines in each 3-space form a Desargues configuration.

3.3. Let XY, ZT, UV be the triplet **A** of intersecting chords (Figure 14). Draw lines through Y and T parallel to UV meeting the circle again at T' and Y' respectively. Then the hexagon $XYT'ZTY'$ is inscribed in the circle;



Greaves transformation

Fig. 14.

$XY \cap ZT$ lies on UV, and $YT' \cap TY'$ lies on UV (at infinity), and hence $T'Z \cap Y'X$ lies on UV by Pascal's theorem. Hence XY', ZT', UV is another triplet, obtained from the original one by a *Greaves transformation*. If we transform **A** by drawing lines through X and Z parallel to UV, we obtain a reflection of the original transform, so the two transforms are identical. However, we obtain a distinct transform by drawing lines through Y and Z (or X and T) instead of through Y and T.

Also, instead of drawing lines parallel to UV, we can use XY or ZT. Thus from **A** we can obtain six triplets using Greaves transformations. In the notation of 3.1, these are the six triplets **L**, **M**, **N**, **P**, **Q**, **R**. Each pair of the fifteen triplets in 3.1 are either isogonal conjugates or Greaves transforms of each other, but not both. A Greaves transformation is the product of two isogonal conjugations, and an isogonal conjugation is the product of two Greaves transformations.

4. The algebraic determination of triplets

4.1. Let XY, ZT, UV be intersecting diagonals of an n-gon. Take the circle as unit circle in the Argand diagram and one vertex of the n-gon as unit point. Then each vertex is represented by an n-th root of unity. Suppose X, Y, ... are represented by x, y, ... The condition for XY, ZT, UV to be concurrent is

$$(1) \qquad \begin{cases} xyz + xyt + ztu + ztv + uvx + uvy \\ -uvt - uvz - xyv - xyu - zty - ztx = 0, \end{cases}$$

which may be written as

$$\begin{vmatrix} xy & x+y & 1 \\ zt & z+t & 1 \\ uv & u+v & 1 \end{vmatrix} = 0,$$

or

$$(x-t)(u-y)(z-v)+(y-z)(v-x)(t-u)=0,$$

from which we deduce Bol's result

$$XT.UY.ZV = YZ.VX.TU \qquad [2, \text{p. } 15].$$

Thus a triplet produces twelve roots of unity in two sets of six ($xyz, \ldots,$ $-uvt, \ldots$) whose sum is zero. Since the choice of a different vertex as unit point has the effect of multiplying all these roots of unity by the same number, it is their ratios only that are important.

If we label the six ends of the diagonals in a different order (which may be done in 48 ways), or if we reflect the triplet in a line of symmetry of the polygon, we simply permute the twelve roots of unity within the two sets of six (the same permutation in each set), or else we interchange the two sets. What is more interesting is that *the fifteen triplets in a conjugacy class all produce the same two sets of six roots of unity.*

The number of ways of permuting the roots of unity within their two sets and perhaps interchanging the two sets is $6! \times 2$. The number of permutations caused by relabelling the vertices of a triplet and perhaps reflecting it is 48×2. We see that $(6! \times 2)/(48 \times 2) = 15$, the number of triplets in a conjugacy class.

Let us write $xyz = a$, $xyt = b$, $ztu = c$, $ztv = d$, $uvx = e$, $uvy = f$, $xyztuv = s$. Then equation (1) becomes

$$(2) \qquad \begin{cases} a+b+c+d+e+f \\ -\dfrac{s}{a}-\dfrac{s}{b}-\dfrac{s}{c}-\dfrac{s}{d}-\dfrac{s}{e}-\dfrac{s}{f}=0. \end{cases}$$

Thus *a conjugacy class determines, and is determined by, the roots* of unity a, b, c, d, e, f and s.

The problem of finding all triplets is therefore reduced to the problem of finding all roots of unity satisfying (2), where $s^3 = abcdef$. Since we only need the ratios of $x, y, z, \ldots$ we can obtain them by the formulae

$$(3) \qquad x=e, \quad y=f, \quad z=\frac{s}{b}, \quad t=\frac{s}{a}, \quad u=\frac{s^2}{abd}, \quad v=\frac{s^2}{abc}.$$

All solutions of (2), except for two omissions, were found by Dr Pleasants before we discovered that the calculations had already been done by Bol [2]. We give a list of solutions in 4.3.

4.2. If we map each n-th root of unity x onto x^k, where $(k, n) = 1$, this mapping determines an automorphism of the field generated by the n-th roots of unity. Thus (1) implies

$$\begin{aligned} & x^k y^k z^k + x^k y^k t^k + \cdots \\ - & u^k v^k t^k - u^k v^k z^k - \cdots = 0. \end{aligned}$$

Hence x^k, y^k; z^k, t^k; u^k, v^k are ends of three intersecting diagonals. This triplet is said to be *isomorphic* to the original triplet. It is obtained by *multiplying the original triplet by k* (thinking of the vertices of the n-gon as being represented by integers modulo n) (figure 15). The angles between chords of the triplet are also multiplied by k. We see that if we multiply two isogonally conjugate triplets by k, then the new triplets are isogonally conjugate. Thus we can define *isomorphic conjugacy classes.*

The number of integers k less than n and prime to n is $\phi(n)$, but, since multiplication by -1 (i.e. by $n - 1$) merely reflects a triplet, multiplication by k and $-k$ (i.e. $n - k$) have the same effect. Thus a conjugacy class is isomorphic to at most $\frac{1}{2}\phi(n)$ classes, including itself. If there are r values of k $(0 < k < \frac{1}{2}n, (k, n) = 1)$ for which $k\mathscr{C} = \mathscr{C}$, then the conjugacy class $\mathscr{C}$ is isomorphic to $\frac{1}{2}\phi(n)/r$ classes.

It should be noted that 'multiplication by k' is an operation whose validity has been proved only algebraically and not geometrically. However, the idea can be used to shorten geometrical investigations. For instance, if we are considering accessibility and have shown that the class $\mathscr{C}$ can be obtained from the orthic class $\mathscr{O}$ by substitutions, we immediately deduce that $k\mathscr{C}$ can be obtained from the orthic class $k\mathscr{O}$ by substitutions.

When giving a list of conjugacy classes, we need therefore only give one conjugacy class from each set of isomorphic classes. By this means, Bol's list of 65 sporadic classes (see 4.3) is reduced to 16, a considerable saving.

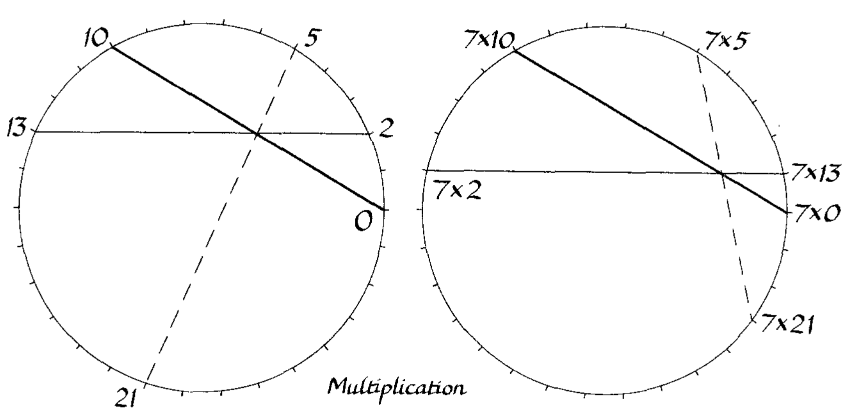


Fig. 15.

TABLE I

Non-orthic conjugacy classes

$[k, 6]$ denotes the least common multiple of k and 6; $\zeta^k = 1$; $\omega^3 = \xi^5 = \eta^7 = 1$, $i^2 = -1$, $j^2 = i$, $\lambda^3 = \omega$.

Type	n	a	b	c	d	e	f	s	x	y	z	t	u	v	Bol type
I	[k, 6]	1	ω	ω^2	ζ^{-1}	$-\omega\zeta^2$	$-\omega^2\zeta^2$	ζ	1	ω	$-\omega\zeta^{-1}$	$-\omega^2\zeta^{-1}$	$-\omega\zeta$	$-\omega^2$	A
II	[k, 6]	ω	ω^2	ζ^2	ζ^3	$-\zeta^3$	$-\zeta^4$	ζ^4	1	ζ	$-\omega\zeta$	$-\omega^2\zeta$	$-\zeta^2$	$-\zeta^3$	B, D
III	[k, 6]	1	ω	ω^2	$-\zeta^2$	ζ^5	$-\zeta^5$	ζ^4	ζ	$-\zeta$	ω^2	1	$-\omega^2\zeta^2$	ζ^4	$C\zeta$
A	30	1	ω	ξ	ξ^2	$\omega\xi$	$-\omega\xi$	-1	0	10	1	11	8	27	3(30)
B	30	ξ	ξ^2	ξ^3	ξ^4	$-\omega$	$-\omega^2$	ω	0	12	10	22	3	13	13(30)
C	30	ξ	ξ^2	ξ^2	$\omega\xi^2$	$\omega\xi^3$	$-\omega$	-1	0	14	11	25	9	22	12(30)
D	30	1	ξ	ξ	ξ^2	$-\omega\xi^2$	$-\omega^2\xi^2$	ω	0	6	2	8	3	11	2(30)
E	42	1	η^2	η^3	η^5	$\omega\eta^3$	$-\omega^2\eta$	$-\omega$	0	12	5	17	8	27	5(42)
F	60	ξ	ξ^2	ξ^3	ξ^4	$-\omega$	$-i\omega^2$	$-i\omega$	0	24	5	29	8	33	8(60)
G	60	1	ξ	ξ^2	$-i\xi$	$-\omega\xi^3$	$-\omega^2\xi^3$	i	0	26	3	29	21	54	12(60)
H	60	ω	$\omega\xi^2$	$\omega\xi^3$	$-\xi$	$i\xi^2$	$-i\xi^2$	-1	0	30	1	37	5	51	9(60)
J	60	ω	ξ^2	ξ^3	$-\omega^2\xi$	$i\xi^2$	$-i\xi^2$	$-\omega$	0	30	11	25	17	21	1(60)
K	60	1	$\omega\xi^3$	ξ^3	$-\omega^2\xi$	$i\xi^4$	$-i\xi^4$	$-\omega$	0	30	25	47	19	39	19(60)
L	60	1	ξ^2	$\omega\xi^2$	$-\omega^2\xi$	i	$-i$	$-\omega$	0	30	5	43	9	49	18(60)
M	84	η^2	η^4	η^6	$-\omega$	$i\omega\eta$	$-i\omega\eta$	-1	0	42	1	47	17	77	4(84)
N	84	η	η^2	η^4	$-\omega$	$i\omega$	$-i\omega$	-1	0	42	7	53	13	61	7(84)
P	90	$\lambda\xi^3$	1	$\omega\lambda\xi^3$	ξ	$-\omega^2\xi^4$	$\omega^2\lambda\xi^3$	$-\xi^3$	0	37	12	38	29	43	1(90)
Q	120	ξ	ξ^2	$-\omega$	i	$\omega\xi j$	$-\omega\xi j$	-1	0	60	1	71	7	103	3(120)
R	210	η	η^2	η^4	$-\xi$	$\omega\xi^2$	$\omega^2\xi^2$	-1	0	60	31	101	45	162	4(210)

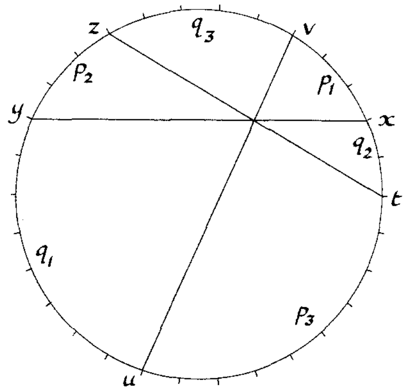


Fig. 16.

4.3. Apart from orthic classes, there are two kinds of conjugacy class:

(a) *General classes.* These classes, of which there are three types, occur whenever n is a multiple of 6.

(b) *Sporadic classes.* These classes occur only for particular values of n.

Table I gives, for each class or type of class, (i) a symbol to denote the class, (ii) the value of n, (iii) the values of a, b, c, d, e, f; s, (iv) the values of x, y; z, t; u, v for one triplet in the class (not necessarily the triplet given by the formulae (3)) given as roots of unity for the general classes and as integers modulo n for the sporadic classes, (v) the corresponding symbol in Bol's list.[3]

We emphasize again that, in the case of the sporadic classes, only one class is listed from each set of isomorphic classes. The reader who consults Bol's list may wish to know which of Bol's classes are isomorphic and to which of our types they belong. This information is given in table II.

4.4. Conjugacy classes of types I, II, III (i.e. general classes) are all accessible. A complete description of all the ways in which orthic classes and general classes are linked by substitutions is too involved to give here. Suffice it to say that we can go from orthic to type I, and from type I to type II, using substitutions without a diameter, and from type II to type III using a substitution with a diameter.

[3] Bol lists a triplet by giving the 'distances' p_1, p_2, p_3, q_1, q_2, q_3 between consecutive vertices (figure 16). Bol's triplets are always interior. There are four misprints in his list; the corrected triplets are:

D.	$(6k)$	k	$2r$	$k-2r$	r	$k-2r$	$3k+r$
21.	(60)	18	3	11	16	7	5
4.	(90)	23	9	5	32	19	2
1.	(210)	121	45	1	18	14	11

TABLE II

$n = 30$	
	A: 3, 7, 8, 10; B: 13; C: 1, 6, 9, 12; D: 2, 4, 5, 11.
$n = 42$	
	All Bol's classes are isomorphic.
$n = 60$	
	F: 8, 10; G: 5, 6, 12, 20; H: 4, 9, 13, 22; J: 1, 11, 14, 21; K: 2, 16, 17, 19; L: 3, 7, 15, 18.
$n = 84$	
	M: 1, 3, 4, 5, 6, 8; N: 2, 7.
$n = 90, 120, 210$	
	All Bol's classes are isomorphic.

The sporadic classes A–F are accessible: using the equal diagonals xy and zt (as given in Table I) and performing a substitution without a diameter, we obtain in each case a general or an orthic triplet. (For example, if we perform a substitution on A we obtain the triplet with vertices 0 10, 13 14, 8 27; this does not occur in Bol's list of sporadic classes, so it is general or orthic.)

The sporadic classes J–N and Q are accessible: replacing the diameter xy by the reflection of the diagonal zt in the diameter, we obtain in each case a general or an orthic triplet in the $\frac{1}{2}n$-gon.

The sporadic classes G and H are accessible. From G we can only reach K by substitution (using the equal chords xy and zt for instance) but K is accessible. From H we can only reach J, K, L, A, B, C and D by substitutions (with or without a diameter) but all these are accessible.

Since no triplets in classes P and R contain either equal diagonals or a diameter, these classes are not accessible. We shall discuss these classes further in Section 6.

The general classes and the accessible sporadic classes are linked by substitutions in many ways; only the simplest ways are described above.

5. Intersections of more than three diagonals

5.1. We have already seen (Figures 11 and 12) how a triplet containing two equal diagonals or a diameter gives a symmetric figure of four or five intersecting diagonals. We call these *symmetric quadruplets and quintuplets*; in a sense they are trivial since they arise naturally once we have obtained the triplets.

The asymmetric quadruplet and quintuplet in Figures 17 and 18 were originally found by the author by accurate drawing, checked by algebra.[4] The

[4] Accurate drawing produces many apparent triplets and quadruplets, but non-genuine ones are easily exposed by multiplication (4.2), which turns apparently concurrent lines into lines that clearly do no concur. Triplets that survive this simple test can then be checked using condition (1) of 4.1.

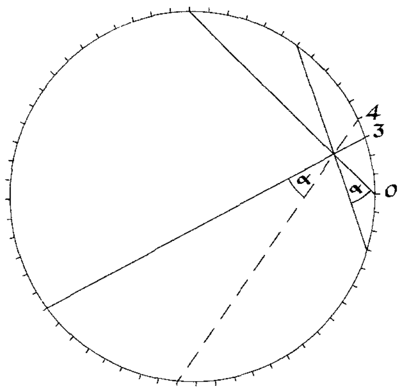


Fig. 17.

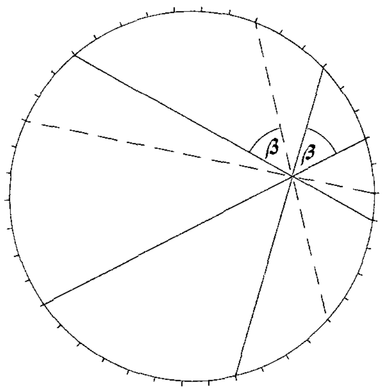


Fig. 18.

asymmetric quintuplet turns out to be unique (5.2). Both these figures contain an orthic triplet, which suggests a method of obtaining further quadruplets.

Suppose AP, BQ, CR is an orthic triplet, the intersection point J being the orthocentre of PQR, and suppose XY is another diagonal through J. By suitably choosing the unit point, we may represent X and Y by the roots of unity $-x$ and $-\bar{x}$ $(= -x^{-1})$. If P, Q, R are represented by p, q, r, then A, B, C are represented by $-qr/p$, $-rp/q$, $-pq/r$. Using condition (1) of 4.1 we find that AP, BQ, XY are concurrent if and only if

(4) $$(p + q + r + x) + (\bar{p} + \bar{q} + \bar{r} + \bar{x}) = 0.$$

Figure 17 comes from the solution

(5) $$p = i, \quad q = -\xi, \quad r = -\xi^3, \quad x = \omega,$$

where Greek letters denote roots of unity as in Table I. Figure 18 comes from combining the quadruplets obtained from two solutions (after rotating one of the quadruplets):

(6) $$p = -\omega, \quad q = \eta, \quad r = \eta^4, \quad x = \eta^2,$$

(7) $$p = -i\eta^4, \quad q = -i, \quad r = \omega i\eta^3, \quad x = -\omega i\eta^4.$$

Solution (6) depends on the fact that $\eta^7 = 1$. Solution (7) does not: it is a special case of the general solution

(8) $$p = -i\zeta, \quad q = -i, \quad r = \omega i\bar{\zeta}, \quad x = -\omega i\zeta,$$

where ζ is any root of unity.

Because of the symmetry of (4) in p, q, r, x, we see that the solution (8) gives rise to other solutions:

(9) $$p = -i\zeta, \quad q = -i, \quad r = -\omega i\zeta, \quad x = \omega i\bar{\zeta},$$

(10) $$p = -i\zeta, \quad q = -\omega i\zeta, \quad r = \omega i\bar{\zeta}, \quad x = -i.$$

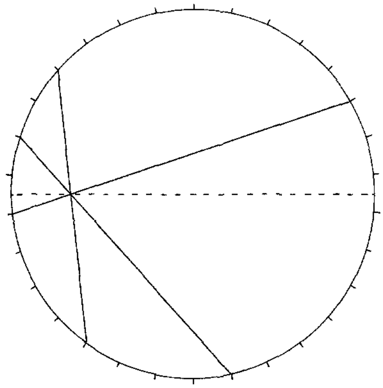


Fig. 19.

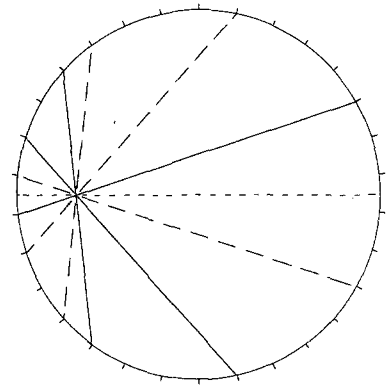


Fig. 20.

(If we take the fourth possible solution, with $x = -i\zeta$, we get nothing essentially different from (8).)

Solution (9) merely gives us a symmetric quadruplet (Bol's type IV), and solution (10) gives four diagonals, including the diameter of a symmetric quintuplet (Bol's type V).

If we have any solution of (4) and we permute the values of p, q, r and x, the various quadruplets that we obtain are connected in a geometrical manner that we shall not describe here.

By permuting the solution (5) we obtain

$$(11) \qquad p = -\xi, \quad q = -\xi^3, \quad r = \omega, \quad x = i,$$

which gives Figure 19. If we reflect this figure in the diameter we obtain a symmetric septuplet in the 60-gon (Figure 20). Removing the diameter we obtain a symmetric sextuplet in the 30-gon. There is another symmetric septuplet in the 60-gon, which cannot be obtained in this way (see 5.2).

5.2. Bol gives a complete list of intersections of more than three diagonals, *when the point of intersection lies inside the circle*. We give this list below. The concept of isomorphism and multiplication (4.2) extends to intersections of more than three diagonals, so we only list one intersection from each set of isomorphic intersections.

There exist triplets all of whose isomorphic triplets (or multiples) are external, but every external triplet is isogonally conjugate to an internal triplet, so we easily obtain all external triplets from Bol's list. However, the situation is different for quadruplets, quintuplets, etc. Isogonal conjugation no longer applies, and Figure 21 is an example of a quadruplet, all of whose multiples are external quadruplets. Such a quadruplet can be called an *essentially external* quadruplet. We have not yet tackled the problem of listing all essentially external quadruplets.

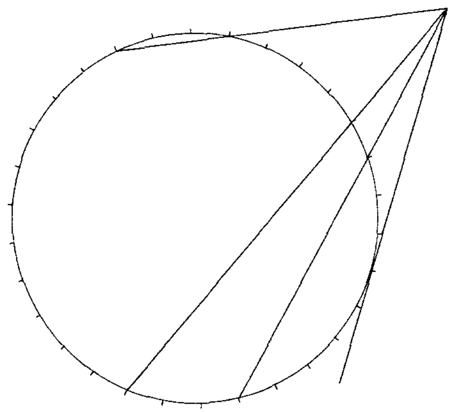


Fig. 21.

We know that certain orthic classes, and all general classes, contain triplets with two equal diagonals or with a diameter, whenever n is a multiple of 6. These triplets give rise to symmetric quadruplets and quintuplets, for which general formulae can be given (we give the ends of the diagonals, as roots of unity):

(a) $\zeta,\ \omega^2\bar{\zeta};\ \bar{\zeta},\ \omega\bar{\zeta};\ \zeta^3,\ -\omega\bar{\zeta};\ \bar{\zeta}^3,\ -\omega^2\zeta$; with or without the diameter joining 1 to -1. This corresponds to the solutions (10) and (9) of equation (4), but we must replace ζ by $-\bar{\zeta}^2$ in (9).

(b) $\zeta,\ -\zeta^3;\ \bar{\zeta},\ -\bar{\zeta}^3;\ -\omega^2\zeta,\ -\omega\zeta;\ -\omega\bar{\zeta},\ -\omega^2\bar{\zeta}$; with or without the diameter joining 1 to -1.

Solutions of type (a) correspond to Bol's IV and V; solutions of type (b) correspond to Bol's I, II, III, VI, VII and VIII. Bol always lists distances between the ends of consecutive diagonals, which must be positive, so he sometimes needs more than one formula for one algebraic type of intersection.

There is one other type of quadruplet that occurs whenever n is a multiple of 6. This is Bol's type IX, an asymmetric quadruplet, and is given by our solution (8) of equation (4).

Sporadic triplets containing two equal diagonals or a diameter give rise to *sporadic symmetric quadruplets and quintuplets*. Some of these can also be obtained from general triplets (from those general triplets that occur sporadically and produce sporadic triplets when we perform a substitution). These symmetric quadruplets and quintuplets are easily obtained from the relevant sporadic triplets, so we need not list them here.

There remain the sporadic asymmetric quadruplets and quintuplets, and the symmetric sextuplets and septuplets, which we reproduce from Bol's list, one from each isomorphic type. After each quadruplet we give the types of the four triplets contained in it; 'O' stands for 'orthic'.

$n = 30$

Symmetric septuplet

0 15, 1 19, 2 22, 3 24, 6 27, 8 28, 11 29.

Symmetric sextuplets

0 11, 1 20, 2 25, 3 27, 4 28, 6 29.
0 13, 1 18, 2 22, 4 26, 5 27, 9 29.

Asymmetric quadruplets

0 23, 1 26, 4 28, 12 29 (*O* I *D D*).
0 11, 4 12, 6 13, 9 18 (I II II II).

$n = 42$

Asymmetric quintuplet

0 18, 2 25, 5 33, 8 37, 15 41 (Figure 18).

Asymmetric quadruplet

0 8, 2 12, 4 27, 5 35 (*O* II *E E*).

$n = 60$

Symmetric septuplets: obtained by inserting diameters into the two symmetric 30-sextuplets.

Asymmetric quadruplets

0 15, 3 36, 4 44, 9 57 (*O* III *G G*) (Figure 17).
0 13, 5 34, 6 42, 9 55 (*O G F K*).
0 22, 1 25, 6 43, 9 50 (II *F G J*).
0 12, 5 18, 8 37, 9 47 (II *F G L*).
0 21, 3 37, 4 42, 10 55 (*G G J L*).[5]

The 30-quadruplet in Figure 21 contains triplets of types *O A C D*, so a complete list of essentially external sporadic quadruplets will yield types not in Bol's list; whether it will yield anything really interesting remains to be seen.

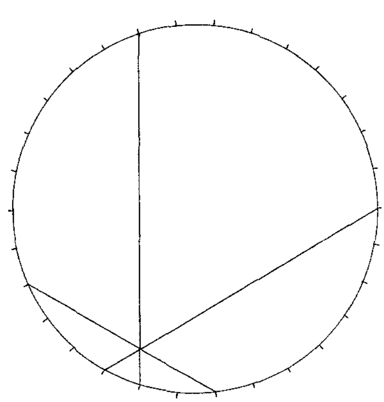


Fig. 22.

[5] This quadruplet is not in Bol's list, though he gives an isomorphic quadruplet. This appears to be a genuine omission rather than a printer's error, since the triplets that it contains are listed by Bol as triplets not contained in any quadruplet. I have noticed a number of minor misprints in Bol's paper; these are not surprising, as the calculations and the proofreading must have been a formidable task.

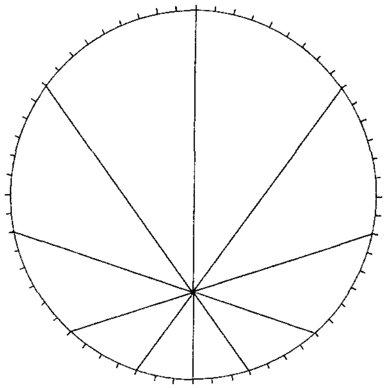


Fig. 23.

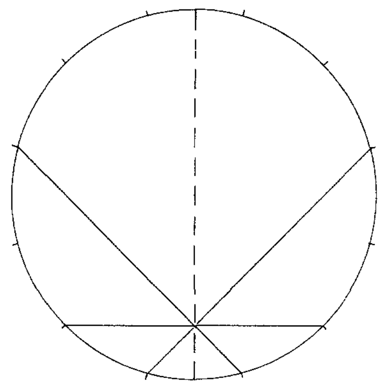


Fig. 24.

5.3. The quadruplet of Figure 17 is obtained by combining the orthic triplet indicated by unbroken lines and the general triplet 0 15, 3 36, 4 44. We can then extract two sporadic triplets of type G from the quadruplet. This is a new method of proving the existence of sporadic triplets, a generalization of the method of substitution. Unfortunately it does not help us with the inaccessible classes P and R, since no triplet in these classes belongs to a quadruplet. (No triplet in these classes is essentially external, so any such quadruplet would appear in Bol's list.)

5.4. A few multiple intersections have visual interest. Figure 23, with each diagonal inclined at 36° to the next one, was found accidentally. Figure 24, with diagonals inclined at 45°, was then obtained by adding a diameter to a symmetric triplet in the 12-gon (see frontispiece). These two figures are reminiscent of lobed or palmate leaves (Figures 25 and 26). The only asymmetric triplet with each diagonal making angles of 60° with the others is shown in Figure 22, of type B. (Using the notation of 4.1 assume, without loss of generality, that $xy = 1$. Then $zt = \omega$ and $uv = \omega^2$. Condition (1) of 4.1 then simplifies to $x + \bar{x} + \omega z + \overline{\omega z} + \omega^2 u + \overline{\omega^2 u} = 0$. The only solutions of this for $(x, \omega z, \omega^2 u)$ are (i, i, i), $(\zeta, -\zeta, i)$, $(i\zeta, i\omega\zeta, i\omega\zeta^{-1})$, $(i, i\omega, i\omega^2)$, $(i\xi, i\xi^2, i\omega)$; only the last gives an asymmetric triplet.)

5.5. What can we say about diagonals of a regular polygon in the hyperbolic plane? Let us use Beltrami's representation of points of the hyperbolic plane by points inside a circle in the Euclidean plane, and lines of the hyperbolic plane by Euclidean line segments inside the circle. We can make this representation in such a way that a given regular polygon in the hyperbolic plane is represented by a regular polygon in the Euclidean plane with its centre at the centre of the bounding circle. Intersecting diagonals in the Euclidean plane will represent intersecting diagonals in the hyperbolic plane, though the point of intersection may be infinite or ultra-infinite; hence all

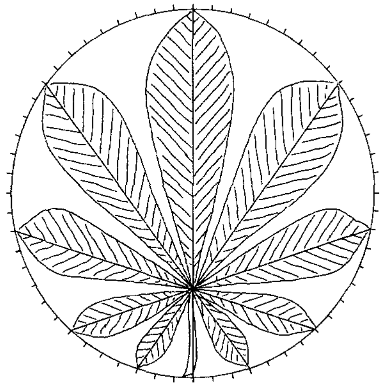


Fig. 25.

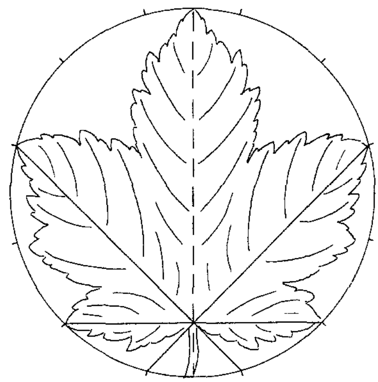


Fig. 26.

multiple intersections in the Euclidean plane give multiple intersections in the hyperbolic plane, and conversely. Beltrami's representation does not preserve angles, so orthic triplets, equally inclined diagonals etc. in the Euclidean plane are no longer orthic triplets, equally inclined diagonals etc. in the hyperbolic plane.

It can also easily be shown that multiple intersections in the Euclidean plane give multiple intersections in the elliptic plane. It should therefore be possible to prove the existence of all triplets using metric geometry only![6]

6. Conjugacy classes and quadrangles

6.1. The simple algebraic proofs of the existence of the sporadic classes P and R in the 90-gon and 210-gon can presumably be turned into geometrical proofs, but what we are still seeking is a simple geometrical proof (such as the one described in Section 2) not inspired by algebraic formulae, that will produce these two inaccessible classes as well as all the other classes. Admittedly this aim is a very subjective one.

Two possible techniques are now described:

(a) In Figure 27, BC, EF, XY is an orthic triplet and AB, DE, XY is a general triplet of type III. The Pascal line of the hexagon $ABCDEF$ is XY, so CD and FA meet on XY. Hence we have a triplet AF, CD, XY. This triplet is sporadic, of type A.

(b) In Figure 28, the angles are marked as multiples of $\pi/30$ radians; T is the incentre of triangle PQS, and R is an excentre of PTS. A calculation of angles shows that Q is an excentre of RST, so all the angles in the figure can

[6] I.e., using only the axioms common to Euclidean, hyperbolic and elliptic geometry.

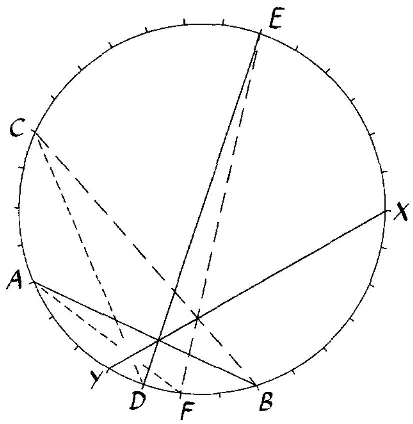


Fig. 27.

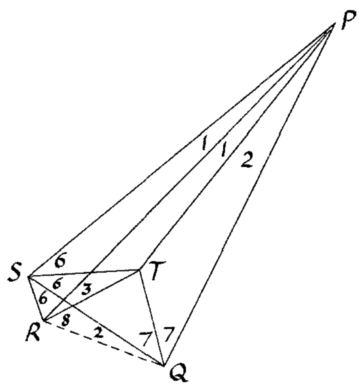


Fig. 28.

now be found. It turns out that *P* is an excentre of *QRS*, but the quadrangle *PQRT* gives rise (in the manner described in 1.2) to a sporadic triplet of type *B*.

What we have done here is fit together three 'known' adventitious quadrangles *PQST*, *PTSR* and *RSTQ* to form an *adventitious* 5-*point PQRST*, from which we then extract the adventitious quadrangle *PQRT* of type *B*.

The methods (a) and (b) can both be used to obtain sporadic triplets and 'sporadic quadrangles', but *neither method will produce the inaccessible classes P and R*. This is a negative result, but others will be saved the trouble of trying these methods if we show why they do not work for classes *P* and *R*.

6.2. First we must consider the connection between triplets and quadrangles; 'quadrangle' will mean 'non-cyclic adventitious quadrangle' unless we state otherwise.

A quadrangle yields in general four triplets (we take the circumcircle of any three vertices and use the fourth vertex as the intersection point of a triplet). Not surprisingly these triplets all lie in the same conjugacy class. A triplet yields eight quadrangles, and the fifteen triplets in a conjugacy class yield thirty distinct quadrangles. (We are concerned only with the shapes of figures, not with their relative sizes.) A quadrangle yields four triangles, and the thirty quadrangles of a conjugacy class yield only fifteen triangles.

Although a triplet, quadrangle, triangle etc. is regarded as being the same as its reflection, it should be noted that the *relative orientation* of the various figures is always the same. For instance, certain pairs of triangles fit together in three ways to form quadrilaterals; the relative sizes of the triangles differ in the three quadrangles, and one triangle will need to be rotated, but we never reflect just one of the triangles.

A neat notation can be devised for triplets, quadrangles etc., following on from Section 3. Triplets and 'lines of triplets' can be regarded as the points and lines of a configuration obtained by taking six general 3-spaces in a 4-space. Denote these 3-spaces by 1 2 3 4 5 6. If the point (or triplet) A is the intersection of the four 3-spaces 3 4 5 6, we denote it by the symbol 12, or 21. With suitable numbering of the 3-spaces, the fifteen triplets of Section 3 are

A	B	C	D	E	F	G	H	K	L	M	N	P	Q	R
12	13	23	24	14	25	15	26	16	24	35	36	46	45	56.

The 'line of triplets' ABC can be denoted by 123; this line is the intersection of the 3-spaces 4 5 6.

The five points A B E G K not in the 3-space 1 form a pentatope. The tetrahedron $ABEG$ in this pentatope consisting of the four points 12 13 14 15 can be denoted by {16}. The order of the digits here is important: {61} denotes the tetrahedron 62 63 64 65. There are six pentatopes and thirty tetrahedra.

The four triplets yielded by a quadrangle are easily seen to form one of these tetrahedra, so we denote the thirty quadrangles by {16} etc. The eight quadrangles yielded by the triplet 12 are {13} {14} {15} {16} {23} {24} {25} {26}.

The eight quadrangles derived from a triplet together yield fourteen of the fifteen triangles associated with the conjugacy class. The remaining triangle has sides parallel to the diagonals of the triplet, and so may be denoted by the same two digits as the triplet. The quadrangle {16} yields the triangles 26 36 46 56; the triangle 12 belongs to eight quadrangles, namely {31} {41} {51} {61} {32} {42} {52} {62}.

Any particular triangle combines with eight others to form quadrangles. If a pair of triangles occurs together in one quadrangle, then this pair occurs

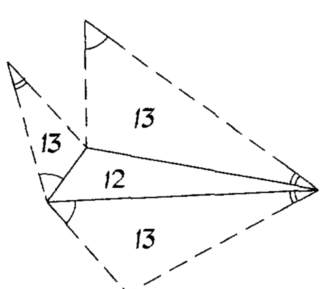


Fig. 29.

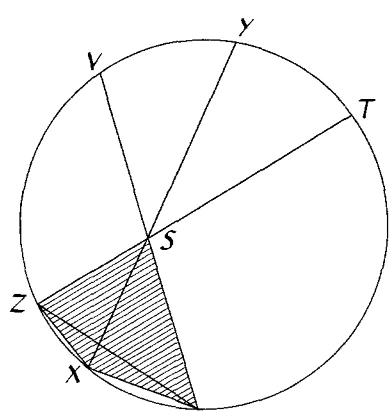


Fig. 30.

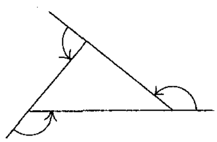


Fig. 31.

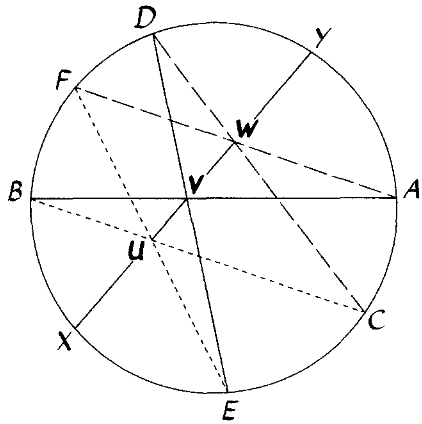


Fig. 32.

in three quadrangles (e.g. 12 and 13 occur in {41} {51} {61}) in the manner shown in Figure 29.

The concept of cyclic complementation used by Tripp in [13], to obtain one quadrangle from another, is akin to isogonal conjugation, though the two ideas were developed independently: it can be used to obtain the thirty quadrangles in a conjugacy class. In Figure 30, the quadrangle *SXZU* has cyclic complements *SYZU*, *SXTU*, *SXZV*. Since we can use the circumcircle of any three vertices of *SXZU*, a quadrangle has twelve cyclic complements. The cyclic complements of {16} are $\{ij\}$, where i and j are distinct and chosen from 2 3 4 5.

If the vertices of the triplet in Figure 30 are represented by the roots of unity x, y, z, t, u, v, the triangle with sides parallel to the three diagonals has angles s/ab, s/cd, s/ef in the notation of 4.1. (Here s/ab is a convenient shorthand for $\frac{1}{2}\arg(s/ab)$: we regard n-th roots of unity as integers modulo n, and we measure angles as multiples of π/n radians. This formula may give us the angles shown in Figure 31.) Thus we see that the fifteen triangles of a conjugacy class use only fifteen different angles, a useful check to bear in mind when we calculate the triangles of a conjugacy class.

6.3. Consider the technique described in (a) of 6.1. We are looking for triplets such that (Figure 32) *BC*, *EF*, *XY* and *BA*, *ED*, *XY* are accessible, and *CD*, *FA*, *XY* is inaccessible. Denote these triplets by **U**, **V**, **W**.

For the inaccessible triplet **W**, $n = 90$ or 210, but n need not be the same for all the triplets. Since the highest common factor of the distances *AY* and *YD* between vertices of the inaccessible triplet is coprime to n (this is a feature of the inaccessible triplets), the value of n for the triplet **V** must be 90 (or 210) or a multiple of 90 (or 210). Hence this triplet must be orthic or general.

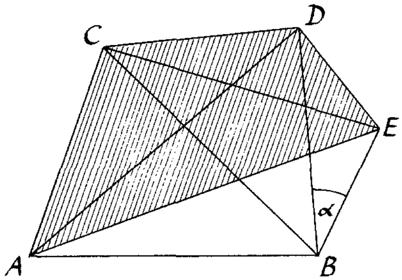


Fig. 33.

It is easily verified by geometry or algebra that, if **V** is orthic, then **W** must have two equal diagonals or a diameter, which is impossible since **W** is inaccessible. Hence **V** must be general.

Now the triplets **V** and **W** share a common triangle ADY. However, if we list the triangles in the various inaccessible classes and the general classes for $n = 90$ (or 210) we find that **V** and **W** cannot have a common triangle.[7]

The only remaining possibility is that n may be a multiple of 90 (or 210) in the general triplet **V**. We deal with this situation by listing the fifteen triangles in each of the general classes I, II and III, using Table I, for a general root of unity ζ; ζ is to be a primitive n-th root of unity, where n is a multiple of 90 (or 210), but we want a triangle whose angles are multiples of $\pi/90$ (or $\pi/210$) radians. The only possible triangles have angles $(\zeta^4, -\zeta^{-2}, -\zeta^{-2})$ in class II, and $(\omega^2\zeta^4, -\omega\zeta^2, -\zeta^{-6})$, $(\omega\zeta^4, -\omega^2\zeta^2, -\zeta^{-6})$, $(\zeta^4, -\zeta^2, -\zeta^{-6})$ in class III, where $\zeta = \theta$, a primitive 180-th (or 420-th) root of unity. But then $(-\theta^2)^{45}$ (or $(-\theta^2)^{105}) = 1$, so all the angles of the above triangles are *even* multiples of $\pi/90$ (or $\pi/210$); hence they do not coincide with any inaccessible triangle.

Thus we cannot have **V** orthic or general and **W** inaccessible (of type P or R). It is interesting to note that we have not made use of the triplet **U** in this proof.

Note that we have proved that *no triangle in any inaccessible class coincides with any triangle in a general or accessible sporadic class.*

6.4. Consider now the technique described in (b) of 6.1. We wish to fit together three accessible quadrangles $ABCD$, $ABCE$ and $ABDE$ (Figure 33) to form an adventitious 5-point, and then extract quadrangles $ACDE$ and $BCDE$, hoping that one will be inaccessible in suitable circumstances. We must be prepared to allow *cyclic* quadrangles here.

[7] We need only compare the triangles in all four isomorphic classes of type P (or R) with the triangles in one representative from each set of isomorphic general classes. If ϕ is a primitive 90-th (or 210-th) root of unity, we need only take $\zeta = \phi$ and $\zeta = \phi^2$ in type I (these values give non-isomorphic classes) and $\zeta = \phi$ in types II and III.

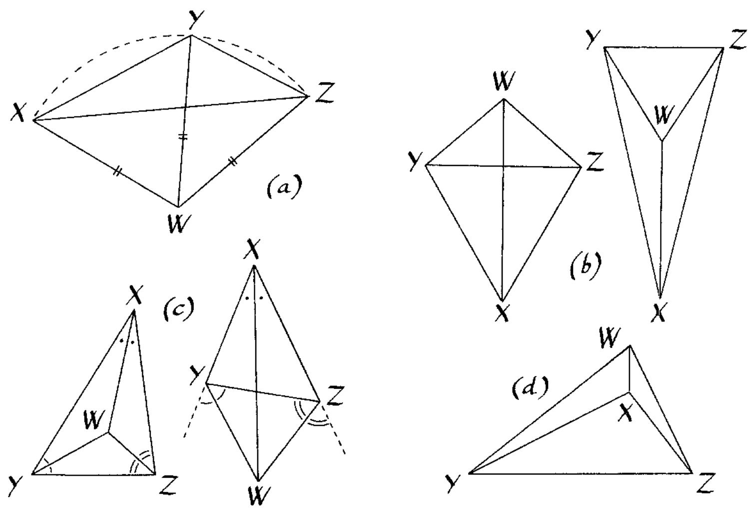


Fig. 34.

Suppose without loss of generality that $ACDE$ is inaccessible. It has the triangle ACD in common with $ABCD$, so by the remark at the end of 6.3 $ABCD$ is not general or sporadic. It must therefore be cyclic or orthic. If $ABCD$ is cyclic, then the diagonals EA, EB, EC, ED of the circle form a quadruplet containing the inaccessible triplet EA, EC, ED. As we remarked in 5.3, there are no such quadruplets. Hence $ABCD$ must be orthic, and the same applies to $ABCE$ and $ABDE$. Hence our problem is to fit together three orthic quadrangles.

The quadrangles in an orthic class are of four types: (a) fans,[8] (b) kites,[8] (c) incentric quadrangles, (d) orthocentric quadrangles (Figure 34). In (a), $WX = WY = WZ$; in (b) we have symmetry about WX; in (c) W is the incentre or an excentre of XYZ, but there is no algebraic distinction between incentres and excentres so we use the term 'incentric' in both cases; in (d) each vertex is the orthocentre of the other three. All four types of quadrangle are *orthic*.

We return now to Figure 33, in which all angles are multiples of π/n radians, where n is a multiple of 90 (or 210). Suppose $ABCD$ is a fan, with B at the centre. Then A, C, D are vertices of a regular n-gon in the circle ACD with centre B, and BE is a diameter either of this n-gon or of a $2n$-gon, depending on whether α is an even or an odd multiple of π/n. Hence EA, EC, ED, EB is a quadruplet in the n-gon or the $2n$-gon, containing the inaccessible triplet EA, EC, ED. We know that this is impossible.

Suppose $ABCD$ is a fan, with A or C or D at the centre. Then the inaccessible

[8] These names are used by Tripp in [13].

quadrangle $ACDE$ contains an isosceles triangle ACD, which is impossible.

Hence $ABCD$ is not a fan. We can say more: *Figure 33 cannot contain a fan.*

LEMMA. *If the orthic quadrangle ABCD contains an isosceles triangle, it is a fan or a kite.*

Proof. Suppose $ABCD$ contains an isosceles triangle. If it is orthocentric, it is a kite. If A is the incentre of BCD, then $ABCD$ is easily seen to be a kite. If A is an excentre as in Figure 35, and if $BC = BD$, then we have a kite by symmetry. If $CB = CD$ in Figure 35, we easily show that $CB = CD = CA$, and we have a fan. We have a kite if $AC = AD$ and a fan if $CD = CA$ or if $CB = CA$. Since $\angle BCA > 90°$, we cannot have $BC = BA$ or $AC = AB$. This completes the proof.

Suppose now that Figure 33 contains an isosceles triangle, say ABC without loss of generality. We have seen that $ABCD$ cannot be a fan, so by the lemma it must be a kite. Similarly $ABCE$ must be a kite. If $AB = AC \neq BC$, D and E (and also A) must lie on the perpendicular bisector of BC, so $ACDE$ is degenerate. If ABC is equilateral, we can also have a figure such as Figure 36. The quadrangles $ABDE$ and $BCDE$ contain isosceles triangles, so only $ACDE$ can be inaccessible. But no 'inaccessible triangle' such as ACD contains an angle $\angle ACD$ equal to 30°.

Hence Figure 33 cannot contain an isosceles triangle, so it cannot contain a kite. Moreover, it cannot contain two orthocentric quadrangles, say $ABCD$ and $ABCE$, since D and E cannot both be the orthocentre of ABC. Hence $ABCD$, $ABCE$ and $ABDE$ are incentric or orthocentric, and at most one can be orthocentric.

Before enumerating the various cases, we can eliminate some of them. (a) If D and E (for instance) are both incentres or excentres of ABC, then D and E are collinear with A, B or C, and the 5-point is degenerate; so this situation cannot occur. (b) If A is an incentre or excentre of BCD, we easily prove

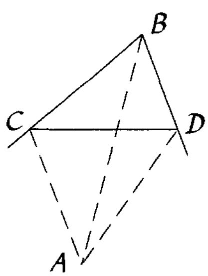


Fig. 35.

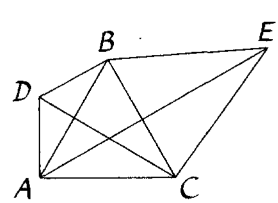


Fig. 36.

TABLE III

	ABCD	*ABCE*	*ABDE*
1	*A*	*B*	orthocentric
2	*D*	*E*	orthocentric
3	*C*	*E*	orthocentric
4	*C*	*B*	orthocentric
5	*A*	*E*	orthocentric
6	*C*	*E*	*D*
7	*C*	*E*	*A*
8	*C*	*B*	*A*

that AD is a diameter of the circle ABC (this statement is not meant to imply that D lies on the circle). Hence, if A is an incentre or excentre of both BCD and BCE, then A, D and E all lie on the same diameter of ABC through A; again the 5-point is degenerate.

Bearing in mind that permuting C, D, E or permuting A, B simply corresponds to a re-labelling of the figure, we find that the various possibilities for the three quadrangles are those given in Table III (an entry 'A' in the column $ABCD$, for instance, means that A is an incentre or excentre of BCD).

In Figure 37, the points on the circumference of the circle are represented by roots of unity as shown. The algebraic conditions for the various types of orthic quadrangles are found to be:

A incentre[9] of BCD:	$p = -a$,	$q = a^2/c$,	$r = a^2/b$;
C incentre of ABD:	$p = c^2/b$,	$q = c^2/a$,	$r = -c$;
D incentre of ABC:	$a = -qr/p$,	$b = -rp/q$,	$c = -pq/r$;
B incentre of ACE:	$x = b^2/c$,	$y = -b$,	$z = b^2/a$;
E incentre of ABC:	$a = -yz/x$,	$b = -zx/y$,	$c = -xy/z$;
$ABDE$ orthocentric:	$ax = -bq$,	$ap = -by$;	
A incentre of BDE:	$a^2 = qy$,	$ax = -py$;	
D incentre of ABE:	$p^2 = bx$,	$q^2 = ay$.	

For each of the eight possibilities in Table III, we simply have therefore to solve a set of homogeneous equations. We omit the details, but no solution gives us an inaccessible quadrangle. What we do obtain, apart from degenerate 5-points, are the 5-points illustrated in Figures 28, 38, 39, 40 and 41. Figure 39 is trivial: $OA = OB = OC = OD$, so we have a 'central quadruplet'.

The possibility of multiplying an n-triplet by k, where $(n, k) = 1$, to obtain an isomorphic triplet (4.2), extends to n-adventitious quadrangles and 5-points. Figures 28, 38 and 39 are isomorphic only to themselves, although multiplication has the effect of permuting the vertices; Figures 40 and 41 are

[9] Or excentre.

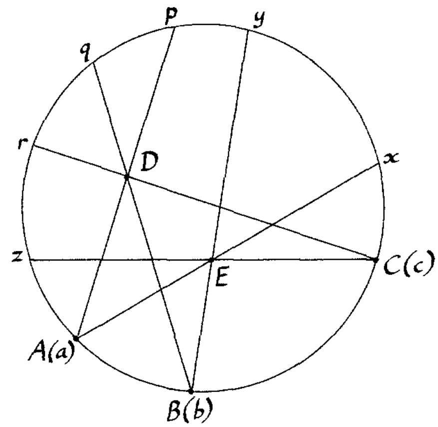


Fig. 37.

isomorphic. Of the eight possibilities in Table III, number 3 gives Figures 40 and 41, numbers 4, 5 and 8 give Figure 38, number 7 gives Figure 28, and number 6 gives Figures 38 and 39. Numbers 1 and 2 have no non-trivial solutions (as may also be seen geometrically).

Probably there are other interesting 5-points, apart from the 'trivial' ones obtained from quadruplets. We have merely found a few while looking for something that does not exist.

7. Some further remarks

7.1. The line joining the points $e^{2i\theta}$ and $e^{-i\theta}$, on the unit circle, envelops a deltoid as θ varies [10]. Hence if we label the vertices of a regular n-gon $0, 1, 2, \ldots$ going round anticlockwise and $0, -1, -2, \ldots$ going round

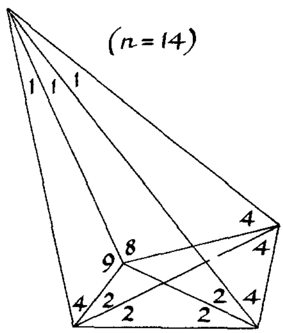


Fig. 38.

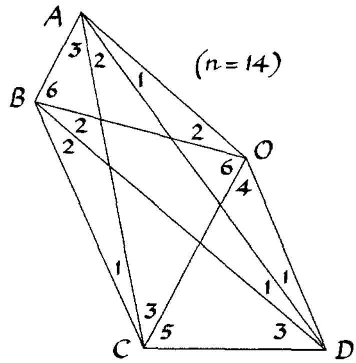


Fig. 39.

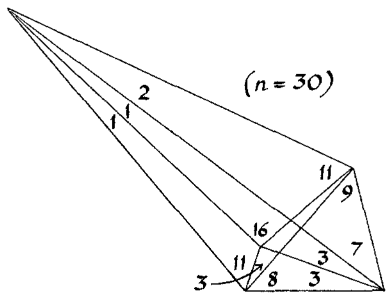


Fig. 40.

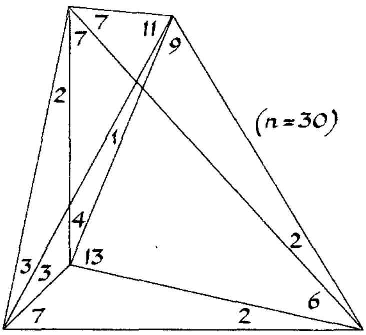


Fig. 41

clockwise, the lines $(2r, -r)$ joining $2r$ to $-r$, as r varies, are tangents of a deltoid (Figure 42). If n is even, these lines intersect in threes, as in Figure 43 where $n = 24$ and the original polygon or circle, inscribed in the deltoid, has been omitted. (We can obtain a pleasing symmetrical figure by taking n to be any multiple of 6.) This result is shown in a figure in [10], but without comment. The reason for the triple intersections is that the lines $(2r, -r)$, $(2s, -s)$, $(2t, -t)$ form an orthic triplet if and only if $r + s + t \equiv \frac{1}{2}n \pmod{n}$.

7.2. We mentioned in 6.2 that each triangle in a conjugacy class belongs to

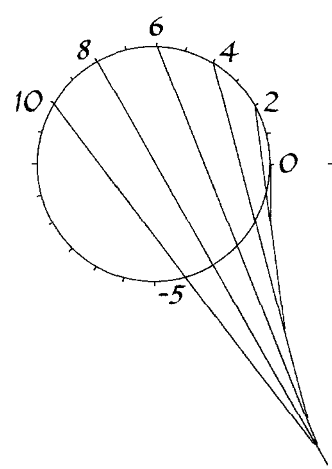


Fig. 42.

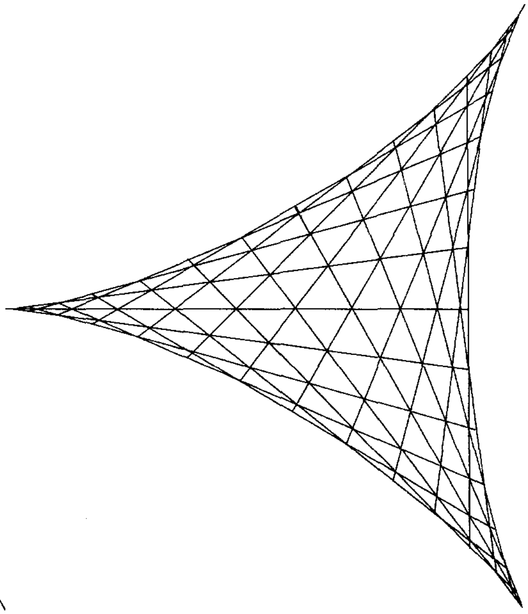


Fig. 43.

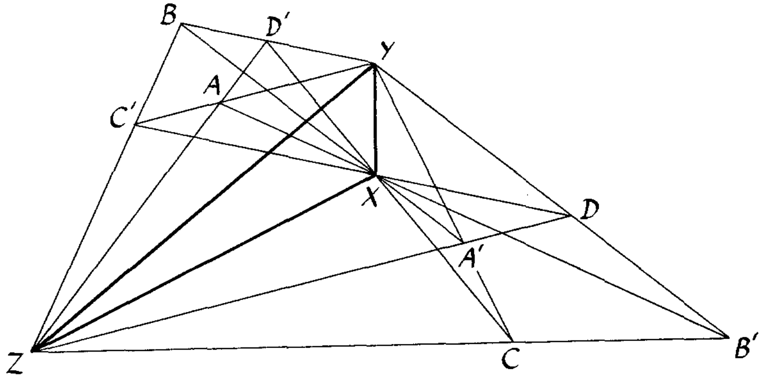


Fig. 44.

eight quadrangles of the class. In Figure 44 we start with a quadrangle $XYZA$; the eight quadrangles in the same conjugacy class containing XYZ are shown in the figure. The points A, A' are isogonal conjugates with respect to XYZ; similarly for B, B' etc. Let us use the notation $[XAB']$ to mean that X, A, B' are collinear, and $[YZAB']$ to mean that Y, Z, A, B' are concyclic. It can be shown that $[XAB']$, $[XBA']$, $[XCD']$, $[XDC']$, $[YAC']$ etc. (twelve lines as shown in the figure) and

$$[YZAB'],\quad [YZBA'],\quad [YZCD'],\quad [YZDC'],$$
$$[ZXAC'],\quad [ZXCA'],\quad [ZXBD'],\quad [ZXDB'],$$
$$[XYAD'],\quad [XYDA'],\quad [XYBC'],\quad [XYCB'].$$

If we regard the figure as being in the inversive plane, and if we denote the point at infinity by W, we have 24 circles, twelve through each of X, Y, Z, W

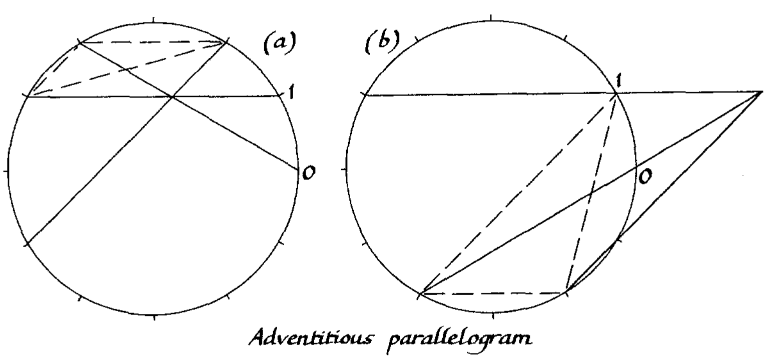


Fig. 45.

and six through each of A, B, ... Inversion with respect to X, Y or Z merely gives us the original figure again, with the points permuted.

7.3. It has been conjectured that there is only one adventitious parallelogram that is not a rhombus; see for instance [5] and [11]. The conjugacy class determined by the roots of unity $a, b, c, d, e, f; s$ (4.1) contains a parallelogram if two pairs from the set a, b, c, d, e, f are equal (e.g. $a = b$ and $c = d$). No sporadic class in Table I satisfies this condition, and the only such orthic or general class that does not produce a rhombus is the class of type II with $n = 12$ and $\zeta^{12} = 1$. One triplet in this class has vertices 0 4, 1 5, 2 7, as shown in Figure 45(a). If we multiply this triplet by 5 we obtain Figure 45(b), in the same conjugacy class, which gives the same parallelogram. Thus the conjecture is verified.

This same condition on a conjugacy class ensures that one triplet in the class contains two tangents. Thus the only asymmetric triplet with two tangents occurs when $n = 12$; it appears in the frontispiece.

7.4. The work of Bol has apparently remained virtually unknown. Partial solutions to the problem of finding intersecting diagonals were given by Harborth [6, 7] and Heineken [8, 9] in the 1960s. Bol's list in Table I was obtained by Meek and Tripp [11] using a computer, with the exception of the sporadic triplet with $n = 210$ since their program only investigated values of n up to 180. They point out that each solution can be verified trigonometrically. Bol's calculations appear to be mainly trigonometrical, whereas Pleasants worked with roots of unity. The problem of finding triplets in a regular p-gon, where p is prime, was posed by Steinhaus in 1958 [12]. Croft and Fowler showed that no such triplets exist [4].

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