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# HARMONIC POLYNOMIALS AND PEAK SETS OF REFLECTION GROUPS*

A finite reflection group fixing the origin in Euclidean space acts on any point on the unit sphere, and not in any of the reflecting hyperplanes, to generate a regular orbit. The space of restrictions of the polynomial functions to such an orbit is isomorphic to the group algebra. Delsarte [4] studied so-called discrete harmonics on certain (Johnson) association schemes, which are also finite homogeneous spaces. In these cases, however, the stabilizer group of a point is nontrivial.

The space of functions on a regular orbit can be given a 'spherical harmonic' structure (this implies the existence of a commutative set of operators analogous to $\partial/\partial x_i$ and $\Delta$, and an associated orthogonality structure), when the orbit is a 'peak set'. By this we mean the set of points on the unit sphere where the function

$$h(x) = \prod_{j=1}^{m} |\langle x, v_j \rangle|^{\alpha_j}$$

achieves its maximum (briefly, $\{v_j\}_{j=1}^{m}$ is a set of positive roots of the group, $\alpha_j > 0$, each $j$ and $\alpha_i = \alpha_j$ whenever the reflections corresponding to $v_i$ and $v_j$ are conjugate). Further, the coinvariant algebra can be represented as an algebra of operators on the harmonic functions.

This paper presents the theory and main results of the harmonic functions on peak sets. Limiting cases of results on orthogonal polynomials with respect to $h(x)^2\,\mathrm{d}\omega(x)$, established in [5] and [6], will be used here, when possible ($\mathrm{d}\omega$ is the rotation-invariant measure on the sphere). Further, the peak sets for the groups of type $A_N$, $B_N$, $D_N$ and $F_4$ will be discussed in detail. Beyond two-space, only $B_N$ and $F_4$ have one-parameter families of peak sets, rather than unique ones, among the irreducible Coxeter groups.

Because the maximization of $h$ can be considered as a maximum problem for the discriminant of a polynomial when the group is $A_N$, $B_N$ or $D_N$, the peak sets were determined long ago. Stieltjes [12] and Schur [11] found the peak sets in terms of the zero sets of Laguerre and Hermite polynomials.

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## 1. Basic Facts and Definitions

For a nonzero vector $v \in \mathbb{R}^N$ define the reflection $\sigma_v \in O(N)$ (the orthogonal group) by

$$x\sigma_v := x - 2\left(\frac{\langle x, v\rangle}{|v|^2}\right)v, \quad x \in \mathbb{R}^N,$$

where

$$\langle x, v\rangle := \sum_{j=1}^{N} x_j v_j \quad \text{and} \quad |v|^2 := \langle v, v\rangle.$$

Thus, $v\sigma_v = -v$ and $x\sigma_v = x$ if and only if $\langle x, v\rangle = 0$. A Coxeter or finite reflection group $G$ is a finite subgroup of $O(N)$ generated by reflections. Let $P^G$ denote the polynomials on $\mathbb{R}^N$ invariant under $G$, then $P^G$ is itself a polynomial algebra, with homogeneous generators $\theta_1, \theta_2, \ldots, \theta_N$ of degrees $d_1, d_2, \ldots, d_N$ respectively (see Hiller [9, Ch. II]). These degrees are structural constants of $G$; the number of reflections in $G$ is $m := \sum_{i=1}^{N} (d_i - 1)$ and the order of the group is

$$|G| = \prod_{i=1}^{N} d_i.$$

Let $\{\sigma_i\colon 1 \leqslant i \leqslant m\}$ be the set of reflections in $G$, and choose a set of vectors $\{v_i\colon 1 \leqslant i \leqslant m\} \subset \mathbb{R}^N$ such that $\sigma_i = \sigma_{v_i}$ and $|v_i| = |v_j|$ whenever $\sigma_i \sim \sigma_j$ (conjugate in $G$). (This set could be a positive root system, but the choice of signs is immaterial.)

Fix positive parameters $\alpha_i$, $1 \leqslant i \leqslant m$, such that $\alpha_i = \alpha_j$ whenever $\sigma_i \sim \sigma_j$. For the irreducible Coxeter groups, there is one conjugacy class of reflections for $A_N$, $D_N$, $H_3$, $H_4$, $E_6$, $E_7$, $E_8$, $I_2$ (odd) and two classes for $B_N$, $F_4$ and $I_2$ (even). We use the standard designations for Coxeter groups (Hiller, [9, Ch. I]).

Define $h(x) := \prod_{i=1}^{m} |\langle x, v_i\rangle|^{\alpha_i}$, a positively homogeneous $G$-invariant function of degree $\gamma := \sum_{i=1}^{m} \alpha_i$. (When the $\alpha_i$'s are all integers, the absolute value function could be omitted; if, furthermore, each $\alpha_i = 0$ or 1, the resulting polynomial is a generating relative invariant, corresponding to a linear character of $G$.) In [5] and [6] the structure of orthogonal polynomials with respect to $h(x)^2\, d\omega(x)$ on the unit sphere was developed.

Let $E$, the peak set, denote the subset of the unit sphere $S := \{x\colon |x| = 1\}$ on which $h(x)$ achieves its maximum on $S$. We shall show $E$ is a regular orbit of $G$ (this means $|E| = |G|$) and $E$ is the solution set of

$$x = \left(\frac{1}{\gamma}\right) \sum_{i=1}^{m} \left(\frac{\alpha_i}{\langle x, v_i\rangle}\right) v_i.$$

Clearly $E$ is invariant under $G$, so we must show $|E| = |G|$. The zero-set of $h$ on $S$ is the boundary of the spherical simplex associated to $G$, the number of whose connected components ('chambers') equals $|G|$. Each component contains just one maximum of $h$: indeed, suppose $x'$ and $x''$ are on the same side of the hyperplane $\langle x, v\rangle = 0$ then

$$\left|\left\langle \frac{x' + x''}{2}, v\right\rangle\right| = \frac{|\langle x', v\rangle| + |\langle x'', v\rangle|}{2} \geqslant |\langle x', v\rangle \langle x'', v\rangle|^{1/2};$$

if $x'$ and $x''$ are in the same chamber then

$$\log h\left(\frac{x' + x''}{|x' + x''|}\right) \geqslant \tfrac{1}{2}(\log h(x') + \log h(x'')) + \gamma \log\left(\frac{2}{|x' + x''|}\right).$$

1.1. PROPOSITION. *The peak set $E$ is the solution set of*

$$\text{(1.2)} \qquad \sum_{i=1}^{m} \frac{\alpha_i}{\langle x, v_i\rangle} v_i = \gamma \frac{x}{|x|^2}, \quad x \in S.$$

*(This equation is valid for peak sets on spheres of any radius.)*

*Proof.* Apply the Lagrange multiplier method to $F(x, \lambda) := \log h(x) + \lambda(\log(|x|^2) - c)$, (to maximize $h(x)$ on a sphere of radius $e^{c/2}$). The resulting equations are

$$\sum_{j=1}^{m} \frac{\alpha_j (v_j)_i}{\langle x, v_j\rangle} + \frac{2\lambda x_i}{|x|^2} = 0, \quad 1 \leqslant i \leqslant N.$$

Multiply equation #i by $x_i$ and sum over $1 \leqslant i \leqslant N$, to obtain $2\lambda = -\Sigma_j \alpha_j = -\gamma$. The second derivative test shows that each critical point of $F(x, \lambda)$ is a local maximum of $\log h(x)$. □

To solve (1.2), take the inner product of both sides with $\nabla\theta$, where $\theta$ is one of the basic invariants of $G$, and $\nabla$ denotes the gradient. Note that $\langle v_j, \nabla\theta(x)\rangle$ is divisible by $\langle v_j, x\rangle$, because $\nabla\theta(x\sigma_j) = \nabla\theta(x)\sigma_j$ and so $\langle v_j, \nabla\theta(x\sigma_j)\rangle = -\langle v_j, \nabla\theta(x)\rangle$. We obtain the equation

$$\text{(1.3)} \qquad \gamma(\deg \theta)\theta(x) = |x|^2 \sum_{j=1}^{m} \frac{\alpha_j \langle v_j, \nabla\theta(x)\rangle}{\langle v_j, x\rangle}$$

for $x \in E$; by Euler's identity $\langle x, \nabla\theta(x)\rangle = (\deg \theta)\theta(x)$. The right-hand side of (1.3) is a product of $|x|^2$ with an invariant polynomial of degree lower than $\theta$. In this way the values of all the basic invariants on $E$ can be inductively determined. We shall actually do the calculation for $A_N$, $B_N$, $D_N$ and $F_4$. Note that decomposable Coxeter groups have peak sets which are, roughly, the Cartesian products of the peak sets of the irreducible factors, with the squared norms adjusted to be proportional to the sums of $\alpha_i$ for each factor.

Note that $E$ is invariant under the central inversion ($x \mapsto -x$) even though it need not be an element of $G$; for example, $A_N$ for $N \geqslant 2$ and $D_N$ for odd $N \geqslant 5$.

## 2. ORTHOGONALITY STRUCTURE

Let $P_E$ be the space of restrictions to $E$ of the polynomial functions on $\mathbb{R}^N$, denoted by $P$. We furnish $P_E$ with the inner product

$$(f, g) \mapsto \sum_{x \in E} f(x) g(x).$$

The action of $G$ on $P_E$ is obviously an isomorphic image of the regular representation. We recall some facts about ordinary spherical harmonics, whose analogies for peak sets will be established here: let $\Delta := \Sigma_{i=1}^N (\partial/\partial x_i)^2$; a polynomial $f$ is harmonic if $\Delta f = 0$; if $f$ is harmonic then so is $\partial f/\partial x_j$, $1 \leqslant j \leqslant N$; if $f$ is harmonic and homogeneous then it is orthogonal to all polynomials of lower degree, with respect to $d\omega$ on the sphere $S$.

Some of the following are limiting cases of results for the measure $h(x)^2 \, d\omega(x)$, as $\alpha_i/\gamma$ is held constant and $\gamma \to \infty$, because the normalized measure converges to the uniform discrete measure on the peak set.

2.1. DEFINITION. The operators $T_i$, $1 \leqslant i \leqslant N$, are given by

$$T_i f(x) := \sum_{j=1}^m \alpha_j \frac{f(x) - f(x\sigma_j)}{\langle x, v_j \rangle} (v_j)_i,$$

for any polynomial $f$; each $T_i f$ is polynomial and $T_i$ is of degree $-1$ ($(v_j)_i$ is the $i$th component of $v_j$).

Indeed, each term $(f(x) - f(x\sigma_j))/\langle x, v_j \rangle$ is polynomial; these operators are used in the Schubert calculus (see Hiller [9, Ch. IV]); the actual limit taken is $\lim_{\gamma \to \infty}(T_i/\gamma)$ with $T_i$ as in [6].

2.2. THEOREM (limit of Theorem 1.9 [6]). *The set of operators* $\{T_i\colon 1 \leqslant i \leqslant N\}$ *is commutative, and* $\Sigma_{i=1}^N T_i^2 = 0$.

Although one might have expected that the Laplacian for $E$ would be a sum of squares, the fact $\Sigma_i T_i^2 = 0$ removes the obvious candidate.

2.3. DEFINITION (limit of Def. 1.1 [6]). For a polynomial $f$, let

$$\begin{aligned} Lf(x) &:= \sum_{i=1}^N \left( \frac{\partial}{\partial x_i} T_i f(x) + T_i \frac{\partial}{\partial x_i} f(x) \right) \\ &= \sum_{j=1}^m \alpha_j \left( \frac{2\langle v_j, \nabla f(x) \rangle}{\langle v_j, x \rangle} - |v_j|^2 \frac{f(x) - f(x\sigma_j)}{\langle v_j, x \rangle^2} \right). \end{aligned}$$

The operator $L$ is of degree $-2$.

2.4. THEOREM (limit of Th. 1.10, Prop. 2.2 [6]). $T_i L = LT_i$, $L(x_i f(x)) = x_i Lf(x) + 2T_i f(x)$, *for each* $i$; $L$ *commutes with the action of* $G$ (*namely* $w \mapsto R(w)f(x) := f(xw)$, $w \in G$).

2.5. THEOREM (limit of Th. 1.11 [5]). *If* $f$ *is a homogeneous polynomial of degree* $n$, *then there is a unique expansion*

$$f(x) = \sum_{j=0}^{[n/2]} |x|^{2j} f_{n-2j}(x)$$

*with* $Lf_{n-2j} = 0$ *and* $f_{n-2j}$ *is homogeneous. Further,*

$$f_{n-2j}(x) = ((4\gamma)^j j!)^{-1} \sum_{i=0}^{[n/2]-j} |x|^{2i}((-4\gamma)^i i!)^{-1} L^{i+j} f(x).$$

Henceforth, we shall call $f$ harmonic if $Lf = 0$; the projection $f \mapsto f_n$ in Theorem 2.5 will be denoted by $\pi_0$. Note that the theorem implies the set of restrictions of $P$ to $S$ agrees with the set of restrictions of harmonic polynomials.

Here is the main orthogonality result for $P_E$.

2.6. THEOREM. *Let* $f$ *and* $g$ *be harmonic homogeneous polynomials, then*

$$(\deg f\text{-}\deg g) \sum_{x \in E} f(x)g(x) = 0.$$

*Proof.* The form

$$(f, g) \mapsto \sum_{x \in E} \sum_{j=1}^{m} \frac{\alpha_j |v_j|^2 (f(x) - f(x\sigma_j))g(x)}{\langle v_j, x\rangle^2}$$

is symmetric, because $E$ is $G$-invariant and $\langle v_j, x\rangle^2 > 0$ on $E$. Further, $Lf = 0$ implies

$$\sum_{j=1}^{m} \alpha_j |v_j|^2 \frac{f(x) - f(x\sigma_j)}{\langle v_j, x\rangle^2} = 2 \sum_{j=1}^{m} \alpha_j \frac{\langle v_j, \nabla f(x)\rangle}{\langle v_j, x\rangle}$$

$$= 2\gamma \langle x, \nabla f(x)\rangle = 2\gamma(\deg f) f(x),$$

for $x \in E$ by Equation (1.2). Multiply the equation by $g(x)$ and sum over $x \in E$. By the symmetry the sum equals

$$(2\gamma \deg f) \text{ or } (2\gamma \deg g) \text{ times } \sum_{x \in E} f(x)g(x). \qquad \square$$

2.7. COROLLARY. *Under the hypotheses of the theorem*

$$\sum_{x\in E} f(x)p(x) = 0$$

for all polynomials $p$ with $(\deg p) < (\deg f)$, (this also uses 2.5).

2.8. COROLLARY. *If $\psi$ is a homogeneous G-invariant of positive degree, then* $\pi_0(\psi) \mid E = 0$.

*Proof.* The homogeneous harmonic polynomial $\pi_0(\psi)$ is also $G$-invariant (by 2.4), hence is constant on $E$. But

$$\sum_{x\in E} \pi_0(\psi)(x)1 = 0$$

by 2.6, hence,

$$\pi_0(\psi)(x) = 0 \quad \text{for } x \in E. \qquad \square$$

The adjoint $T_i^*$ of $T_i$ (on the inner product space $P_E$) is close to being multiplication by $2\gamma x_i$; it is actually multiplication by the image of $2\gamma x_i$ in the coinvariant algebra. This will be shown in the following section.

2.9. LEMMA. *$L(|x|^{2j}f(x)) = |x|^{2j} Lf(x) + 4\gamma j|x|^{2(j-1)}f(x)$, for $j = 1, 2, \ldots,$ and any polynomial $f$.*

We shall define $T_i^*$ on all polynomials, even though it is technically only defined on $P_E$, which is interpreted as the set of restrictions of harmonic polynomials to $E$.

2.10. THEOREM. *For each i, and harmonic polynomial f,*

$$T_i^* f(x) = 2\gamma x_i f(x) - |x|^2 T_i f(x).$$

*Also $LT_i^* = T_i^* L$ (considering $T_i^*$ as an operator on all polynomials).*

*Proof.* For any polynomials $f, g$,

$$\sum_{x\in E} (f(x)T_i g(x) + T_i f(x)g(x)) = 2\gamma \sum_{x\in E} x_i f(x)g(x),$$

indeed, the left-hand side equals

$$\sum_{x\in E} \sum_{j=1}^{m} \alpha_j(v_j)_i \frac{(f(x) - f(x\sigma_j))g(x) + f(x)(g(x) - g(x\sigma_j))}{\langle v_j, x\rangle}$$

$$= 2\sum_{x\in E} \sum_{j=1}^{m} \alpha_j \frac{(v_j)_i}{\langle v_j, x\rangle} f(x)g(x)$$

$$- \sum_{j=1}^{m} \alpha_j(v_j)_i \sum_{x\in E} \frac{f(x)g(x\sigma_j) + f(x\sigma_j)g(x)}{\langle v_j, x\rangle}.$$

The second sum is zero: replace $x$ by $x\sigma_j$ then $\langle v_j, x\rangle$ becomes $-\langle v_j, x\rangle$. In the first sum, for each $x \in E$,

$$\sum_{j=1}^{m} \frac{\alpha_j(v_j)_i}{\langle v_j, x\rangle} = \gamma x_i$$

by (1.2).

Further, for any polynomial $f$,

$$\begin{aligned} &L(2\gamma x_i f(x) - |x|^2 T_i f(x)) \\ &\quad = 2\gamma x_i Lf(x) + 4\gamma T_i f(x) - |x|^2 LT_i f(x) - 4\gamma T_i f(x) \\ &\quad = (2\gamma x_i - |x|^2 T_i) Lf(x), \end{aligned}$$

by 2.4 and 2.9. Thus for any harmonic polynomials $f$ and $g$, $T_i^* f$, as given in the statement of the theorem, satisfies

$$\sum_{x\in E} (T_i^* f(x))g(x) = \sum_{x\in E} f(x)T_i g(x). \qquad \square$$

## 3. The coinvariant algebra

Chevalley [3] considered the quotient of $P$ modulo the ideal $\mathscr{I}$ generated by the basic invariants $\{\theta_1, \ldots, \theta_N\}$ (that is, the invariants of positive degree). This is a finite-dimensional graded algebra, called the coinvariant algebra of $G$. (It is denoted by $S_G$ in Hiller [9, Ch. II]; also see this reference for applications of $S_G$ in the cohomology theory of Lie groups.)

The Poincaré series for $S_G$ was shown by Chevalley to be

$$\prod_{i=1}^{N} \left(\frac{1 - q^{d_i}}{1 - q}\right),$$

(where $d_i = \deg \theta_i$, and the coefficient of $q^n$ is the dimension of the component of $S_G$ with degree $n$).

We shall show that the homomorphism $P \to \operatorname{End}(P_E)$ (operators on $P_E$) given by $\pi_1 p(x_1, \ldots, x_N) = p(T_1^*/2\gamma, \ldots, T_N^*/2\gamma)$ has $\mathscr{I}$ as kernel. Then $P_E$ is itself linearly isomorphic to $S_G$ under the map induced by $p(x) \mapsto p(T_1^*/2\gamma, \ldots)1$.

This will be proven in two steps. First, let $\pi_2 p = \pi_0 p|E$ (the restriction to $E$ of the harmonic projection, see 2.5). We show that $\pi_2(p\psi) = 0$ for any $G$-invariant $\psi$ of positive degree, and $\pi_2(p) = \pi_1(p)1$ for any polynomial $p$.

Note that $|x|^2$ is $G$-invariant, hence is in $\mathscr{I}$. It suffices to consider $\pi_2(p\psi)$ for harmonic $p$, since $\pi_0(p(x)|x|^2) = 0$ for any $p$; this follows from the uniqueness of the expansion $p(x) = \Sigma_i |x|^{2i} p_i(x)$, with $Lp_i = 0$. Suppose now $Lp = 0$, then by the product rule $L(p\psi) = pL(\psi)$ and thus $\pi_0(p\psi) = p\pi_0(\psi)$ (formula of 2.5). When $\psi$ is of positive degree $\pi_0(\psi) = 0$ on $E$ (Corollary 2.8), hence $\pi_2(p\psi) = 0$.

To show $\pi_2(p) = \pi_1(p)1$ for any harmonic polynomial $p$, it suffices to show $\pi_2(x_i p(x)) = (T_i^*/2\gamma)\pi_2(p(x))$. From Theorem 2.10, $Lp = 0$ implies $LT_i^* p = 0$; further,

$$x_i p(x) = \frac{T_i^* p(x) + |x|^2 T_i p(x)}{(2\gamma)}$$

and so

$$\pi_0(x_i p(x)) = \frac{T_i^* p(x)}{(2\gamma)};$$

restriction to $E$ gives $\pi_2(x_i p(x))$.

We sketch the argument why the kernel of $\pi_2$ is contained in $\mathscr{I}$: as in [5, Th. 3.4] there is a basis of harmonic homogeneous polynomials $\{\varphi_j: 1 \leqslant j \leqslant |G|\}$ for $P$ over $P^G$ (the invariants), thus any polynomial $p$ has the expansion $p(x) = \Sigma_j \psi_j(x)\varphi_j(x)$ with $\psi_j$ $G$-invariant and $\pi_0 p = \Sigma\,(\pi_0 \psi_j)\varphi_j$. Restricted to $E$, $\pi_2 p(x) = \Sigma\, c_j \varphi_j(x)$, where $c_j$ is the constant term of $\pi_0 \psi_j$ (see 2.8). Since the dimension of $P_E$ is $|G|$, $\pi_2 p = 0$ implies each $c_j = 0$; that is, each $\psi_j \in \mathscr{I}$.

## 4. Association schemes

There is an analogy between peak sets and association schemes (for a survey see Bannai and Ito [2]): for an orbit $\Omega$ of the reflection group say that $x \in \Omega$ is adjacent to the points $x\sigma_j$, $1 \leqslant j \leqslant m$. In addition, the weight $\alpha_j$ is associated to the edge $\{x, x\sigma_j\}$. Now $E$ is the orbit $\Omega$ for which

$$\sum_{x\in\Omega} \sum_{j=1}^{m} \alpha_j \log(|x - x\sigma_j|)$$

is maximized. Indeed, let

$$\begin{aligned} s(x) &:= \sum_{j=1}^{m} \alpha_j \log|x - x\sigma_j| \\ &= \sum_{j=1}^{m} \alpha_j \log\left(\frac{2|\langle x, v_j\rangle|}{|v_j|^2}\right) \\ &= \log h(x) + \sum_{j=1}^{m} \alpha_j \log\left(\frac{2}{|v_j|^2}\right). \end{aligned}$$

Thus $s(x)$ is $G$-invariant, and the given sum is maximized at $\Omega = E$ with value $|E|$ (log $h(x)$ + constant).

The adjacency operator for $E$ is a multiple of $M := \Sigma_{i=1}^{N} (T_i T_i^* - T_i^* T_i)$; since

$$Mf(x) = 2 \sum_{j=1}^{m} \alpha_j f(x\sigma_j).$$

The eigenvalues of $M$, which are obtainable from the values of irreducible characters at the reflections, were discussed in [6, §2].

## 5. Specific Coxeter Groups

The peak sets for the groups $A_N$, $B_N$ and $D_N$ were determined by Stieltjes [12] and Schur [11]. Proofs using the differential equations satisfied by Laguerre and Hermite polynomials can be found in Szegö [14, pp. 140–142]. We shall use Equation (1.3) to directly evaluate the invariant polynomials on the peak set.

For $0 \leqslant l \leqslant N$ let $e_l(x)$ denote the elementary symmetric function of degree $l$ in $x_1, \ldots, x_N$, with generating function

$$\sum_{l=0}^{N} t^l e_l(x) = \prod_{i=1}^{N} (1 + x_i t)$$

(note that $x_1, \ldots, x_N$ is the zero-set of $p(t) = \Sigma_{j=0}^{N} t^{N-j}(-1)^j e_j(x)$). We shall use $e_l(x^2)$ to denote $e_l$ with argument $x_1^2, \ldots, x_N^2$. Also, we write $\partial_i$ for $\partial/\partial x_i$, $1 \leqslant i \leqslant N$.

### 5.1. *The Group $A_{N-1}$ (Symmetric Group $S_N$)*

We consider this as the group generated by the reflections in $\{x_i - x_j = 0: 1 \leqslant i < j \leqslant N\}$ on the $(N-1)$-dimensional subspace $\{x: \Sigma_i x_i = 0\} \subset \mathbb{R}^N$. We use $\{e_l(x): 2 \leqslant l \leqslant N\}$ as the basic invariants. There is one conjugacy class of reflections. Let

$$h(x) = \prod_{1 \leqslant i < j \leqslant N} |x_i - x_j|$$

and

$$\gamma = \frac{N(N-1)}{2}.$$

Equation (1.3) becomes

$$\sum_{i<j} \frac{\partial_i p(x) - \partial_j p(x)}{x_i - x_j} = \frac{N(N-1)}{2} (\deg p) p(x)$$

for a homogeneous invariant $p$. Now

$$\partial_i e_l(x) - \partial_j e_l(x) = e_{l-1}(\hat{x}_i) - e_{l-1}(\hat{x}_j) = (x_j - x_i)e_{l-2}(\hat{x}_i, \hat{x}_j);$$

where $e_k(\hat{x}_i, \ldots)$ denotes the elementary symmetric function of degree $k$ in $x_1, \ldots, x_N$ with $x_i, \ldots$ omitted. We substitute this in Equation (1.3) to obtain

$$N(N-1)le_l(x) = \sum_{i<j} e_{l-2}(\hat{x}_i, \hat{x}_j) = -\binom{N-l+2}{2} e_{l-2}(x)$$

for $x \in E$. (The last coefficient can be found by counting the number of times that $x_1, x_2, \ldots, x_{l-2}$ appears in the sum.) Thus we obtain a recurrence for $e_l(x), x \in E$, starting with $e_0 = 1, e_1 = 0$; indeed $e_l(x) = 0$ for $l$ odd and

$$e_{2m}(x) = \frac{(-N)_{2m}(-1)^m}{2^m m!\,(N(N-1))^m}.$$

The peak set $E$ is the $A_{N-1}$-orbit (all permutations) of $(x_1, x_2, \ldots, x_N)$, where $\{x_1, x_2, \ldots, x_N\}$ (say $x_1 < x_2 \cdots < x_N$) is the zero-set of the polynomial

$$p(t) = \sum_j t^{N-2j} e_{2j}(x) = (2N(N-1))^{-N/2} H_N\left(t\left(\frac{N(N-1)}{2}\right)^{1/2}\right);$$

and $H_N$ is the Hermite polynomial of degree $N$. Schur [11] computed the discriminant of $H_N$ (see Szegö [14, p. 143]; one can also use the formula of Stieltjes [13] for Jacobi polynomials or that of Hilbert [8] for the discriminant of the general polynomial of hypergeometric type). The result is:

$$h(x)^2 = (N(N-1))^{-N/2} \prod_{j=2}^{N} j^j \quad \text{for } x \in E.$$

### 5.2. *The Group $B_N$ (the Hyperoctahedral Group)*

For $\alpha, \beta > 0$ let

$$h(x) := \prod_{i=1}^{N} |x_i|^\alpha \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^\beta,$$

thus $\gamma = N(\alpha + (N-1)\beta)$. There are two conjugacy classes of reflections ($(x_1, x_2, \ldots) \mapsto (-x_1, x_2, \ldots)$ and $(x_1, x_2, \ldots) \mapsto (x_2, x_1, \ldots)$ are examples in the two classes). The basic invariants are $\{e_l(x^2)\colon 1 \leq l \leq N\}$. To write out Equation (1.3) we introduce the operator (which will also be used for $F_4$):

$$\delta_N p(x) := \sum_{1 \leq i < j \leq N} \left(\frac{\partial_i p(x) - \partial_j p(x)}{x_i - x_j} + \frac{\partial_i p(x) + \partial_j p(x)}{x_i + x_j}\right).$$

Then Equation (1.3) for the group $B_N$ is

$$\frac{\gamma(\deg p)p(x)}{|x|^2} = \alpha \sum_{i=1}^{N} \frac{\partial_i p(x)}{x_i} + \beta\delta_N p(x)$$

(for a homogeneous invariant $p, x \in E$). Thus

$$\begin{aligned}\frac{2l\gamma e_l(x^2)}{|x|^2} &= 2\alpha \sum_i e_{l-1}(\hat{x}_i^2) + 4\beta \sum_{i<j} e_{l-1}(\hat{x}_i^2, \hat{x}_j^2)\\ &= \left(2\alpha(N-l+1) + 4\beta\binom{N-l+1}{2}\right)e_{l-1}(x^2), \quad x \in E.\end{aligned}$$

(using the caret notation as in 5.1). Thus

$$e_l(x^2) = \frac{(-N)_l(1-N-\alpha/\beta)_l}{l!(N(N-1+\alpha/\beta))^l},$$

$1 \leqslant l \leqslant N$ for $x \in E$. The polynomial

$$\begin{aligned}p(t) &= \sum_j (-1)^j e_j(x^2)t^{N-j}\\ &= (-1)^N N!(N(N+c-1))^{-N} L_N^{(c-1)}(tN(N+c-1))\end{aligned}$$

with $c = \alpha/\beta$, in terms of the Laguerre polynomials. The peak set $E$ is the set of all permutations of $(\pm x_1, \pm x_2, \ldots, \pm x_N)$, where $x_j = (t_j/(N(N+c-1)))^{1/2}$, $0 < t_1 < t_2 \cdots < t_N$ and $L_N^{(c-1)}(t_j) = 0, 1 \leqslant j \leqslant N$. From the known discriminant of $L_N^{(c-1)}$ (see Szegö [14, p. 143]) we find that

$$h(x)^2 = \left(N\left(N-1+\frac{\alpha}{\beta}\right)\right)^{-N(\alpha+(N-1)\beta)} \prod_{j=2}^{N} j^{j\beta} \prod_{j=0}^{N-1}\left(j+\frac{\alpha}{\beta}\right)^{\alpha+j\beta}.$$

Since the sup-norm of $h^2$ is the limit of $p$-norms, the given value is the limit of $(\int_S h^{2n}\,d\omega)^{1/n}$ as $n \to \infty$; the integral is a form of Selberg's integral (see Askey [1]).

### 5.3. *The Group $D_N$*

Let $h(x) = \Pi_{i<j}|x_i^2 - x_j^2|$; this is the limiting case $\alpha \to 0, \beta = 1$ of the function $h$ in 5.2. The notable change is that $e_N(x^2) = e_N(x)^2 = 0$ for $x \in E$. The peak set is the set of permutations of $(\pm x_1, \pm x_2, \cdots \pm x_{N-1}, 0)$ where

$$x_j = \left(\frac{t_j}{N(N-1)}\right)^{1/2}, \quad 0 < t_1 < t_2 \cdots < t_{N-1}$$

and

$$L_{N-1}^{(1)}(t_j) = 0, \quad 1 \leqslant j \leqslant N-1.$$

For $x \in E$,

$$h(x)^2 = (N(N-1))^{-N(N-1)} \prod_{j=2}^{N} j^j (j-1)^{j-1};$$

(Schur [11]).

## 5.4. *The Group $F_4$*

The reflections correspond to the vectors $(1,1,0,0), (1,-1,0,0), \ldots$, denoted by $v_1, \ldots, v_{12}$ and $\sqrt{2}(1,0,0,0), \ldots, (1/\sqrt{2})(1, \pm 1, \pm 1, \pm 1)$ denoted by $v_{13}, \ldots, v_{24}$; there are two conjugacy classes of reflections $\{\sigma_1, \ldots, \sigma_{12}\}$ and $\{\sigma_{13}, \ldots, \sigma_{24}\}$. Note that

$$\prod_{i=1}^{12} \langle x, v_i \rangle = \prod_{1 \leqslant i < j \leqslant 4} (x_i^2 - x_j^2).$$

Consider the orthogonal involution $g_0$ on $\mathbb{R}^4$ given by $xg_0 = u$, where

$$u_1 := \frac{-x_1 + x_2}{\sqrt{2}}, \qquad u_2 := \frac{x_1 + x_2}{\sqrt{2}},$$

$$u_3 := \frac{-x_3 + x_4}{\sqrt{2}}, \qquad u_4 := \frac{x_3 + x_4}{\sqrt{2}}.$$

Then $w \mapsto g_0 w g_0$ is an automorphism of $F_4$ because $g_0$ maps the set $\{v_1, \ldots, v_{12}\}$ onto $\{\pm v_{13}, \ldots, \pm v_{24}\}$ (with some choice of signs). If $p$ is an $F_4$-invariant polynomial, then so is $x \mapsto p(xg_0)$. Further,

$$\prod_{i=13}^{24} |\langle x, v_i \rangle| = \prod_{1 \leqslant i < j \leqslant 4} |u_i^2 - u_j^2|.$$

Choose $\alpha, \beta > 0$ and let

$$h(x) = \prod_{1 \leqslant i < j \leqslant 4} (|u_i^2 - u_j^2|^\alpha |x_i^2 - x_j^2|^\beta);$$

thus $\gamma = 12(\alpha + \beta)$.

We proceed to find the peak set by using the method of Flatto and Wiener [7] and Ignatenko [10] who produced $F_4$-invariants by means of 'power-sums' $\Sigma_{j=1}^{12} \langle x, v_j \rangle^{2m}, m = 3, 4, 6$ (and also summed over $13 \leqslant j \leqslant 24$). Since $B_4$ is a subgroup of $F_4$ (of index 3), the $F_4$ invariants can be expressed in terms of $e_l(x^2)$, with $1 \leqslant l \leqslant 4$. The operator $\delta_4$ from 5.3, together with the action of $g_0$, will be used to expand Equation (1.3).

For $m = 1, 2, \ldots$, let

$$\psi_{2m}(x) := \sum_{i=1}^{12} \langle x, v_i \rangle^{2m} = \sum_{1 \leqslant i < j \leqslant 4} ((x_i - x_j)^{2m} + (x_i + x_j)^{2m}).$$

Then $\psi_{2m}(x) \pm \psi_{2m}(xg_0)$ are respectively invariant, skew-invariant under the $g_0$-action. The values $m = 1, 3, 4, 6$ produce basic invariants for $F_4$.

Let the homogeneous polynomials $\theta_6$, $\theta_8$, $\theta_{12}$ be defined (with $e_j$ short for $e_j(x^2)$) by:

$$\theta_6(x) := 48e_3 - 8e_2e_1 + e_1^3,$$

$$\theta_8(x) := 48e_4 - 6e_3e_1 + 4e_2^2 - e_2e_1^2,$$

$$\theta_{12}(x) := 4608e_4e_2 - 1440e_4e_1^2 - 864e_3^2 + 288e_3e_2e_1 - 128e_2^3 + 48e_2^2e_1^2 - 12e_2e_1^4 + e_1^6.$$

5.4.1. THEOREM. *$|x|^2, \theta_6(x), \theta_8(x), \theta_{12}(x)$ form a set of basic invariants of $F_4$, $\theta_6(xg_0) = -\theta_6(x)$, $\theta_8(xg_0) = \theta_8(x)$, and $\theta_{12}(xg_0) = -\theta_{12}(x)$.*

*Proof.* The following identities (established with some computer algebra assistance) imply the theorem:

$$\psi_6(x) - \psi_6(xg_0) = -3\theta_6(x);$$

$$\psi_8(x) + \psi_8(xg_0) = 20\theta_8(x) + (\tfrac{45}{2})|x|^8;$$

$$\psi_{12}(x) - \psi_{12}(xg_0) = -(\tfrac{15}{8})\theta_{12}(x) - (\tfrac{225}{4})\theta_6(x)|x|^6. \qquad \square$$

The peak-set Equation (1.3) becomes

$$24(\alpha + \beta)m\theta_{2m}(x) = \sum_{i=1}^{24} \frac{\alpha_i \langle v_i, \nabla\theta_{2m}(x)\rangle}{\langle v_i, x\rangle}$$

with $\alpha_i = \beta$ for $1 \leqslant i \leqslant 12$ and $\alpha_i = \alpha$ for $13 \leqslant i \leqslant 24$. The coefficient of $\beta$ is $\delta_4\theta_{2m}$ ($\delta_4$ was defined in 5.2). Indeed,

$$\delta_4\theta_6(x) = -24|x|^6, \qquad \delta_4\theta_8(x) = -3\theta_6(x) - 9|x|^6,$$

and

$$\delta_4\theta_{12}(x) = 36|x|^2(-8\theta_8(x) + |x|^2\theta_6(x) - |x|^8).$$

To find the coefficient of $\alpha$ we use the $g_0$-action (apply $\delta_4$ to $x \mapsto \theta_{2m}(xg_0)$, then replace $x$ by $xg_0$). Thus Equation (1.3) for $F_4$ becomes:

$$72(\alpha + \beta)\theta_6(x) = 24(\alpha - \beta);$$

$$96(\alpha + \beta)\theta_8(x) = 3((\alpha - \beta)\theta_6(x) - 3(\alpha + \beta));$$

$$144(\alpha + \beta)\theta_{12}(x) = 36((\alpha + \beta)\theta_6(x) + (\alpha - \beta)(8\theta_8(x) + 1)),$$

(note $|x|^2 = 1$ on $E$). Let $c := (\alpha - \beta)/(\alpha + \beta)$, then $\theta_6(x) = c/3$, $\theta_8(x) = (c^2 - 9)/96$, $\theta_{12}(x) = c(c^2 + 7)/48$.

Solve for $x \in E$ as follows (with $e_j$ short for $e_j(x^2)$):

(i) $e_3 = (8e_2 - 1 + c/3)/48$ (from the value of $\theta_6$);
(ii) $e_4 = (-4e_2^2 + 2e_2 + (c+7)(c-3)/96)/48$ (from the values of $\theta_6, \theta_8$);
(iii) let $q(e_2) := (4e_2 - 1)^3 - (4e_2 - 1)(c+1)(c+3)/32 + (c+1)^2(c+3)/384$ and solve $q(e_2) = 0$ for $e_2$; this equation is obtained by substituting values for $e_3$ and $e_4$ in the expression for $\theta_{12}$;
(iv) for each of the three roots of $q(e_2) = 0$ use (i) and (ii) to produce the polynomial $p(t) = \Sigma_j t^{4-j}(-1)^j e_j(x^2)$, whose zero set gives a $B_4$-orbit just as in 5.2; the peak set is the union of the three $B_4$-orbits.

Note the degeneracy in $q(e_2)$ when $c = -1$ (that is, $\alpha = 0$, $\beta > 0$) and $E$ reduces to the $D_4$ case.

We compute the value of $h(x)^2$ on $E$ as follows: express the degree 24 invariant

$$\prod_{1 \leqslant i < j \leqslant 4} (x_i^2 - x_j^2)^2$$

in terms of $e_j(x^2)$, $1 \leqslant j \leqslant 4$ (it is also possible to express it in terms of $|x|^2$, $\theta_6$, $\theta_8$, $\theta_{12}$ but computationally more tedious); use (i) and (ii) to give the value in terms of $e_2$ and $c$ (a sixth-degree polynomial); and find the remainder modulo $q(e_2)$ (with computer assistance).

The result is

$$\prod_{i<j} (x_i^2 - x_j^2)^2 = \frac{(1-c)^3(3-c)^2(3+c)}{(4!1152)^2} \quad \text{for } x \in E.$$

Use the $g_0$-action to get the other part of $h(x)$; replacing $c$ by $-c$.

For $x \in E$,

$$h(x)^2 = (3456)^{-2(\alpha+\beta)} \alpha^{3\alpha} \beta^{3\beta} (\alpha + 2\beta)^{\alpha+2\beta} \times (2\alpha + \beta)^{2\alpha+\beta} \cdot (\alpha + \beta)^{-6(\alpha+\beta)}.$$

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