Geometriae Dedicata **60**: 277–281, 1996.

Common Supports as Fixed Points

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(Received: 5 September 1994)

Abstract. A family $\mathcal{S}$ of sets in $\mathbf{R}^d$ is *sundered* if for each way of choosing a point from $r \leq d+1$ members of $\mathcal{S}$, the chosen points form the vertex-set of an $(r-1)$-simplex. Bisztriczky proved that for each sundered family $\mathcal{S}$ of d convex bodies in $\mathbf{R}^d$, and for each partition $(\mathcal{S}', \mathcal{S}'')$ of $\mathcal{S}$, there are exactly two hyperplanes each of which supports all the members of $\mathcal{S}$ and separates the members of $\mathcal{S}'$ from the members of $\mathcal{S}''$. This note provides an alternate proof by obtaining each of the desired supports as (in effect) a fixed point of a continuous self-mapping of the cartesian product of the bodies.

Mathematics Subject Classification (1991): 52A20.

Key words: common support, convex body, fixed point, orientation, transversal.

A family $\mathcal{S}$ of subsets of $\mathbf{R}^d$ is *sundered* if, for each way of choosing a point from $r \leq d+1$ members of $\mathcal{S}$, the chosen r points form a set that is affinely independent and hence is the vertex-set of an $(r-1)$-simplex. This is equivalent to being $(d-1)$*-separated* as defined in [7]. When $|\mathcal{S}| = d$ it amounts to saying that for each way of choosing a point from each set, there is a unique hyperplane that contains all the chosen points. The present note is concerned with a sundered family consisting of precisely d bodies $B_1, \ldots, B_d$ in $\mathbf{R}^d$, where a *body* is a set that is compact, convex, and has nonempty interior.

A hyperplane H in $\mathbf{R}^d$ is a *transversal* for $\mathcal{S}$ if each member of $\mathcal{S}$ intersects H. A transversal H is a *common support* (or, simply, a *support*) for $\mathcal{S}$ if each member of $\mathcal{S}$ is contained in one of the two closed halfspaces bounded by H. (We call these H*-halfspaces*.) For a partition (I, J) of the index set $\{1, \ldots, d\}$, an (I, J)*-support* of $\{B_1, \ldots, B_d\}$ is a support H such that one of the H-halfspaces contains $\bigcup_{i \in I} B_i$ and the other H-halfspace contains $\bigcup_{j \in J} B_j$. The following beautiful theorem is due to Bisztriczky [2].

THEOREM. *If $\{B_1, \ldots, B_d\}$ is a sundered family $\mathcal{S}$ of d bodies in $\mathbf{R}^d$ $(d \geq 2)$, then for each partition (I, J) of $\{1, \ldots, d\}$ there are exactly two (I, J)-supports of $\mathcal{S}$.*

The first proof of this theorem was in fact given by Cappell *et al.* [3], because Bisztriczky recently declared his own demonstration to be incomplete. [2] and [3] also consider common supports for sundered families of fewer than d bodies. The present note does not address the latter case, but it does provide an easy proof of Bisztriczky's theorem by using the elementary topological fact that each continuous self-mapping of a compact convex set must have a fixed point (Brouwer's theorem).

The proof is divided into three parts. In the first part, which shows that there are at least two (I, J)-supports, it is assumed that each of the bodies B_k is strictly convex. Using this assumption, two continuous self-mappings F and G of the product $B_1 \times \cdots \times B_d$ are defined, and it is shown that a fixed point of either mapping gives rise to an (I, J)-support of $\mathcal{S}$. The second part indicates two ways of removing the assumption of strict convexity. One is a routine limiting argument, and the other a direct application of Kakutani's fixed-point theorem rather than Brouwer's. The third part provides an alternative proof that there are at most two (I, J)-supports.

For additional information about transversal hyperplanes and common supports, see [1]–[7]. Fixed-point approaches to common supporting spheres are discussed in a second paper [9] by the present authors.

Proof

(First part) Let B denote the Cartesian product $B_1 \times \cdots \times B_d$ and let U denote the unit sphere $\{u \in \mathbf{R}^d : \|u\| = 1\}$. The proof will involve two continuous mappings,

$$\nu : B \to U \quad \text{and} \quad \beta : U \to B.$$

For each selection $b = (b_1, \ldots, b_d) \in B$, let $H(b)$ denote the transversal hyperplane containing all the points b_k, and let the unit vector $\nu(b) \in U$ be defined by the following two conditions, where $e = (1, \ldots, 1)$:

(i) $\nu(b)$ is normal to $H(b)$;

(ii) $\det \begin{bmatrix} e & 0 \\ b & \nu(b) \end{bmatrix} > 0.$

With $\langle \cdot , \cdot \rangle$ denoting the inner product, condition (i) says that $\langle \nu(b), b_k - b_d \rangle = 0$ for $1 \le k < d$ and condition (ii) says that the unit normal $\nu(b)$ is 'positively oriented' with respect to $(b_1, \ldots, b_d)$. The sign of the 'orientation determinant' in (ii) depends on the order in which the bodies B_k are listed. However, from continuity of the determinant and the fact that the bodies are convex and form a sundered family, it follows that the sign of the determinant is independent of the choice of the points b_k at which the hyperplane $H(b)$ intersects the bodies B_k. Continuity of the function $\nu : B \to U$ follows from a routine argument based on the continuity of the determinant and the compactness of the sets B_k.

The definition of the function $\beta: U \to B$ depends on the partition (I, J). For $i \in I$, let $b_i(u)$ denote the unique point of B_i at which the functional $\langle u, \cdot \rangle$ attains its B_i-maximum, and for $j \in J$ let $b_j(u)$ denote the unique point of B_j at which $\langle u, \cdot \rangle$ attains its B_j-minimum. (Uniqueness of these points follows from strict convexity.) From compactness and uniqueness it follows in a routine way that each of the functions $b_k: U \to B_k$ is continuous. That assures the continuity of the function $\beta: U \to B$ defined by setting $\beta(u) = (b_1(u), \ldots, b_d(u))$ for each $u \in U$.

Now define the continuous mappings $F: B \to B$ and $G: B \to B$ by setting (for each $b \in B$)

$$F(b) = \beta(\nu(b)) \quad \text{and} \quad G(b) = \beta(-\nu(b)).$$

Since B is a compact convex set, it follows from Brouwer's theorem that there are points r and s of B such that $F(r) = r$ and $G(s) = s$. Then the functional $\langle \nu(r), \cdot \rangle$ is constant on the set $\{r_1, \ldots, r_d\}$ and hence this set generates a support $H(r)$ of $(B_1, \ldots, B_d)$ such that one of the two $H(r)$-halfspaces contains $\bigcup_{i \in I} B_i$ and the other contains $\bigcup_{j \in J} B_j$. A similar statement applies to s.

To see that the two common supports $H(r)$ and $H(s)$ are distinct, note that otherwise the comments following (ii) would necessitate that $\nu(r) = \nu(s)$, and s would then be equal to both $F(s)$ and $G(s)$. For any body B_k with $k \in J$, this would imply that

$$\begin{aligned} \max\langle \nu(s), B_k \rangle &= \langle \nu(s), s_k \rangle = -\langle -\nu(s), s_k \rangle \\ &= -\max\langle -\nu(s), B_k \rangle = \min\langle \nu(s), B_k \rangle, \end{aligned}$$

which is impossible when B_k has nonempty interior. A similar statement applies to $k \in I$, and it follows that the family has at least two common (I, J)-supports.

(Second part) Now let us remove the assumption that each of the bodies B_k is strictly convex. For $1 \leq k \leq d$ and for each positive integer n, let B_k^n denote a strictly convex body such that $B_k^n \subseteq B_k$ and the Hausdorff distance between B_k^n and B_k is less than $1/n$. Let $\mathcal{S}_n = (B_1^n, \ldots, B_d^n)$. Then $\mathcal{S}_n$ is a sundered family to which the above reasoning applies, and passage to the limiting family $\mathcal{S}$ yields the desired conclusion.

It is also possible to deal directly with bodies that are not strictly convex. In this case, β becomes an upper-semicontinuous set-valued mapping into B, with compact convex image sets. The same is then true of the mappings $F: B \to B$ and $G: B \to B$, and thus Kakutani's extension [8] of Brouwer's theorem yields the existence of points $r, s \in B$ such that $r \in F(r)$ and $s \in G(s)$. It again turns out that $H(r)$ and $H(s)$ are two distinct (I, J)-supports.

(Third part) Finally, for the sake of completeness, we provide an alternative proof that there are at most two (I, J)-supports.

Each (I, J)-support is of the form $\{x:\langle u, x\rangle = \lambda\}$ for some $(u, \lambda) \in U \times \mathbf{R}$ such that u is positively oriented, and such that

(a) $\max\langle u, B_i\rangle = \min\langle u, B_j\rangle = \lambda$ for all $i \in I$ and $j \in J$, or

(b) $\min\langle u, B_i\rangle = \max\langle u, B_j\rangle = \lambda$ for all $i \in I$ and $j \in J$.

We will show that if the pairs (u, λ) and (v, μ) are both of type (a), then $u = v$. From this it follows that $\lambda = \mu$, and hence there is at most one support of type (a). The same argument shows that there is at most one support of type (b).

Let the points $r_k \in B_k$ be such that $\langle u, r_k\rangle = \lambda$ for all k. Then choose the points $s_k \in B_k$ so that $\langle v, s_k\rangle = \mu$ for all k, and, in addition, $s_k = r_k$ for all k such that $\langle v, r_k\rangle = \mu$. Suppose that $u \neq v$, and for each $t \in [0, 1]$ set $w(t) = (1-t)u - tv$. Note that $w(t) \neq 0$.

For each $i \in I$ it is true that

$$\begin{aligned}\langle w(t), r_i\rangle &= (1-t)\langle u, r_i\rangle - t\langle v, r_i\rangle \\ &\geq (1-t)\langle u, r_i\rangle - t\langle v, s_i\rangle = (1-t)\lambda - t\mu \\ &\geq (1-t)\langle u, s_i\rangle - t\langle v, s_i\rangle = \langle w(t), s_i\rangle\end{aligned}$$

and that either $r_i = s_i$ or $\langle v, r_i\rangle < \mu$. In the latter case, $\langle w(t), r_i\rangle > \langle w(t), s_i\rangle$ for all $t \in]0, 1[$. A similar observation is valid for $j \in J$, and we conclude that for each k and for each $t \in [0, 1]$ the segment $[r_k, s_k]$ in B_k contains a unique point $b_k(t)$ of the hyperplane $\{x:\langle w(t), x\rangle = (1-t)\lambda - t\mu\}$. Routine arguments show that the functions $w(\cdot)$ and $b(\cdot) = (b_1(\cdot), \ldots, b_d(\cdot))$ are continuous, so the orientation determinant

$$\det\begin{bmatrix} e & 0 \\ b(t) & w(t)\end{bmatrix}$$

also varies continuously with t. Since the system is sundered, this determinant is never 0. However, $D(0)$ and $D(1)$ must have opposite signs, for

$$D(0) = \det\begin{bmatrix} e & 0 \\ r & u\end{bmatrix} \quad \text{and} \quad D(1) = \det\begin{bmatrix} e & 0 \\ s & -v\end{bmatrix}.$$

This contradiction completes the proof.

References

1. Anderson, L. and Wenger, R.: Oriented matroids and hyperplane transversals (Manuscript, 1995).
2. Bisztriczky, T.: On separated families of convex bodies, *Arch. Math.* **54** (1990), 193–199.
3. Cappell, S. E., Goodman, J. E., Pach, J., Pollack, R., Sharir, M. and Wenger, R.: Common tangents and common transversals, *Adv. in Math.* **106** (1994), 198–215.
4. Dawson, R.: Common supports of families of sets, *Geom. Dedicata* **51** (1994), 1–13.
5. Dawson, R. and Edelstein, M.: Families of bodies with definite common supports, *Geom. Dedicata* **33** (1990), 195–204.

6. Goodman, J. E. and Pollack, R.: Hadwiger's transversal theorem in higher dimensions, *J. Amer. Math. Soc.* **1** (1988), 301–309.
7. Goodman, J. E., Pollack, R. and Wenger, R.: Geometric transversal theory, in J. Pach (ed.), *New Trends in Discrete and Computational Geometry*, Springer-Verlag, Berlin, 1993, pp. 163–198.
8. Kakutani, S.: A generalization of Brouwer's fixed point theorem, *Duke Math. J.* **8** (1941), 457–459.
9. Klee, V., Lewis, T. and Von Hohenbalken, B.: Apollonius revisited: supporting spheres for sundered systems, *Discrete & Computational Geometry*, to appear.