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TOROIDAL LIE ALGEBRAS AND VERTEX REPRESENTATIONS

To Professor J. Tits for his sixtieth birthday

ABSTRACT. The paper describes the theory of the toroidal Lie algebra, i.e. the Lie algebra of polynomial maps of a complex torus $\mathbb{C}^\times \times \mathbb{C}^\times$ into a finite-dimensional simple Lie algebra $\mathfrak{g}$. We describe the universal central extension $\mathfrak{t}$ of this algebra and give an abstract presentation for it in terms of generators and relations involving the extended Cartan matrix of $\mathfrak{g}$. Using this presentation and vertex operators we obtain a large class of integrable indecomposable representations of $\mathfrak{t}$ in the case that $\mathfrak{g}$ is of type A, D, or E. The submodule structure of these indecomposable modules is described in terms of the ideal structure of a suitable commutative associative algebra.

1. INTRODUCTION

The study of $\mathrm{Map}(X, G)$, the group of polynomial maps of a complex algebraic variety X into a complex simple algebraic group G, and its representations is only well developed in the case that X is a complex torus $\mathbb{C}^\times$. In this case $\mathrm{Map}(X, G)$ is a loop group and the corresponding Lie algebra $\mathrm{Map}(X, \mathfrak{g})$ is the loop algebra $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$. Here the representation theory comes to life only after one replaces $\mathrm{Map}(X, \mathfrak{g})$ by its universal central extension, the corresponding affine Lie algebra $\hat{\mathfrak{g}}$. One then obtains the well-known theory of affine highest weight modules, vertex representations, modular forms and character theory, and so on.

The next easiest case is presumably the case of a torus $X = \mathbb{C}^\times \times \mathbb{C}^\times$ and indeed there have been a couple of papers [8], [13] that describe the universal central extension $\mathfrak{t}$ of $\mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes \mathfrak{g}$. However, the theory seems to have stopped there due to the difficulty of producing any interesting representations of $\mathfrak{t}$.

In this paper we show how to construct a great number of representations of $\mathfrak{t}$ (for simply laced $\mathfrak{g}$) through the use of vertex operators. The representations that we have looked at in detail (and we have not looked at them all) are integrable to a group action and are reminiscent of highest weight representations of affine Lie algebras. However, there are considerable

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differences, the most notable being that the representations are not completely reducible and the structure of the indecomposable representations is matched by the ideal structure of a suitable ring depending on the representation in question.

In more detail this is what happens. Unlike the affine case where $\hat{\mathfrak{g}}$ is a one-dimensional central extension of the loop algebra, $\mathfrak{t}$ is an infinite-dimensional central extension of $\mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes \mathfrak{g}$. The centre has a basis that can naturally be parametrized by $(\mathbb{Z} \times \mathbb{Z}) \cup \{*\}$, where $\{*\}$ is just some singleton.

Let Q be the root lattice for the affine Lie algebra $\hat{\mathfrak{g}}$ and construct the Fock space $V(Q) := \mathbb{C}[Q] \otimes_{\mathbb{C}} S(\mathfrak{a}_-)$ where $\mathfrak{a}$ is the (degenerate) Heisenberg algebra defined by Q. The space $V(Q)$ affords a representation for the Lie algebra generated by the Fourier components $X_n(\alpha)$ of the vertex operators $X(\alpha, z)$, $\alpha \in \Delta$, where Δ is the affine root system of $\hat{\mathfrak{g}}$. This Lie algebra is a (non-faithful) homomorphic image of $\mathfrak{t}$. It can actually be made faithful by enlarging Q to a non-degenerate lattice Γ, forming the Heisenberg algebra $\mathfrak{b}$ on Γ, and using $V(\Gamma, \mathfrak{b}) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} S(\mathfrak{b}_-)$.

The centre of $\mathfrak{t}$, in terms of the operators, is the linear span of $\{X_m(k\delta)\}$, $m, k \in \mathbb{Z}$, $k \neq 0$; $\delta(m)$, $m \in \mathbb{Z}$; and $X_0(0)$, where δ is the null root of Δ. For $\lambda \in \Gamma$, $e^\lambda \otimes 1$ generates a $\mathfrak{t}$-module $V(\lambda)$ which is indecomposable. Set $V(\Gamma) := \bigoplus_{\lambda \in \Gamma} V(\lambda) = \mathbb{C}[\Gamma] \otimes S(\mathfrak{a}_-)$.

Let $(\lambda \mid \delta) =: N$ and let $\tau := X_{-N}(\delta)$. Then τ operates as an invertible (but not scalar) endomorphism on $V(\Gamma)$, commuting with the action of $\mathfrak{t}$. Let D be the symmetric algebra on the space $\Sigma_{k<0} \mathbb{C}\delta(k)$ and let $C(\lambda)$ be the ring of operators $\mathbb{C}[\tau, \tau^{-1}]D$. Then $V(\lambda)$ is a free $C(\lambda)$ module over some irreducible level 1 affine representation L (depending on λ) and the submodule structure of $V(\lambda)$ is isomorphic to the ideal structure of $C(\lambda)$.

In this paper we begin with $V(\Gamma, \mathfrak{b})$ but only study the structure of $V(\Gamma)$ in detail. The classification and structure of the remaining modules in $V(\Gamma, \mathfrak{b})$ remains to be worked out. In retrospect it is clear why good representations of $\mathfrak{t}$ are hard to find. On irreducible representations the centre should act as scalars. But this seems to reduce the representation to trivial variations of affine modules. In fact what happens in $V(\lambda)$ is that there is a rational straight line through $(0, 0)$ in the lattice $\mathbb{Z} \times \mathbb{Z}$ such that the central operators corresponding to the points above it are 0 on $V(\lambda)$, those corresponding to points on the line are the powers of τ, and those corresponding to points below it are non-invertible maps $V(\lambda) \to V(\lambda)$ whose images are submodules of $V(\lambda)$.

The structure of this paper is this. In Section 2 we discuss central extensions of $A \otimes_{\mathbb{C}} \mathfrak{g}$ as a Lie algebra over $\mathbb{C}$ where A is any commutative associative algebra over $\mathbb{C}$ and $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$. The results here are due to Kassel [8] and are valid for a very large class of commutative (associative) algebras A over commutative rings k. Here we give a much more efficient proof that works for commutative algebras over fields of characteristic 0. This is based on Wilson's proof [15] for the affine case. In Section 3 we give an abstract presentation of a toroidal Lie algebra $\mathfrak{t}$ based on an arbitrary Cartan matrix A of finite type. In Section 4 we prove that $\mathfrak{t}$ is faithfully represented by vertex operators, and in Section 5 we analyse the structure of $V(\Gamma)$.

2. Toroidal Lie algebras and central extensions

Let $\mathfrak{g}$ be a perfect (i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) Lie algebra. A *central extension* of $\mathfrak{g}$ is a Lie algebra $\hat{\mathfrak{g}}$ and a surjective homomorphism $\pi: \hat{\mathfrak{g}} \to \mathfrak{g}$ whose kernel lies in the centre of $\hat{\mathfrak{g}}$. The pair $(\hat{\mathfrak{g}}, \pi)$ is a *covering* of $\mathfrak{g}$ if in addition $\hat{\mathfrak{g}}$ is perfect. A covering $(\hat{\mathfrak{g}}, \pi)$ is a *universal covering algebra* (uca) of $\mathfrak{g}$ if for every central extension $(\mathfrak{e}, \varphi)$ of $\mathfrak{g}$ there is a unique homomorphism $\psi: \hat{\mathfrak{g}} \to \mathfrak{e}$ for which $\varphi\psi = \pi$. Every perfect Lie algebra has a uca. A good reference for this theory is [4].

LEMMA 2.1. *Let* $(\hat{\mathfrak{g}}, \pi)$ *be a covering of* $\mathfrak{g}$. *If* $\eta: \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ *is a Lie endomorphism which induces the identity map on* $\mathfrak{g}$ *then* $\eta = \mathrm{id}_{\hat{\mathfrak{g}}}$.

Proof. Let $x, y \in \mathfrak{g}$ and let $\hat{x}, \hat{y} \in \hat{\mathfrak{g}}$ with $\pi(\hat{x}) = x$, $\pi(\hat{y}) = y$. Then $[\hat{x}, \hat{y}]$ depends only on x and y. Thus $\eta([\hat{x}, \hat{y}]) = [\eta\hat{x}, \eta\hat{y}] = [\hat{x}, \hat{y}]$ and since $\hat{\mathfrak{g}}$ is perfect, $\eta = \mathrm{id}$. □

Let A be a commutative algebra over $\mathbb{C}$ and let $\dot{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Our object in this section is to describe the uca of $A \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ (as a Lie algebra over $\mathbb{C}$) and then to make the structure of this uca quite explicit in the case that $A = \mathbb{C}[s, s^{-1}, t, t^{-1}]$.

The structure of the uca of $A \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ has already been worked out by Kassel [8]. His argument is based on an argument due to Garland in [4] for the case $A = \mathbb{C}[t, t^{-1}]$. Wilson [15] gave a very elegant proof of the $\mathbb{C}[t, t^{-1}]$ case, and it generalizes easily to general A to give a more economical proof than the Garland–Kassel approach (provided one accepts some basic cohomology theory of simple Lie algebras). We wish to sketch out the Wilson approach here. For more on the history of these results see [14].

We begin by recalling a few facts about $\dot{\mathfrak{g}}$ that we shall need for the sequel. The over-dot notation is used for consistency with later sections.

Let $\dot{\mathfrak{h}}$ be a Cartan subalgebra of $\dot{\mathfrak{g}}$, let $\dot{\Delta}$ denote the corresponding root system, $\dot{\Delta} \subset \dot{\mathfrak{h}}^*$, and let $\dot{\Pi} = \{\dot{\alpha}_1, \ldots, \dot{\alpha}_l\}$ be a base for $\dot{\Delta}$. The Killing form

$(\cdot \mid \cdot)$ is non-degenerate on $\dot{\mathfrak{h}}$, and we will usually identify $\dot{\mathfrak{h}}^*$ with $\dot{\mathfrak{h}}$ by means of it. We assume that $(\cdot \mid \cdot)$ is so normalized that after this identification long roots have square length equal to 2.

For each root $\alpha \in \dot{\Delta}$ the Lie algebra $\dot{\mathfrak{g}}^\alpha + [\dot{\mathfrak{g}}^\alpha, \dot{\mathfrak{g}}^{-\alpha}] + \dot{\mathfrak{g}}^{-\alpha}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. An $\mathfrak{sl}_2$*-triplet* for this is a choice of $e_\alpha \in \dot{\mathfrak{g}}^\alpha$, $e_{-\alpha} \in \dot{\mathfrak{g}}^{-\alpha}$ for which with $h_\alpha := [e_\alpha, e_{-\alpha}]$ we have $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$. Using our identification of $\dot{\mathfrak{h}}^*$ with $\dot{\mathfrak{h}}$ we have

$$(2.1) \qquad [e_\alpha, e_{-\alpha}] = (e_\alpha \mid e_{-\alpha})\alpha$$

$$\alpha = \frac{(\alpha \mid \alpha)}{2} h_\alpha$$

$$(e_\alpha \mid e_{-\alpha}) = \frac{2}{(\alpha \mid \alpha)}.$$

Let A be any commutative algebra over $\mathbb{C}$. The *module of differentials* (Ω_A, d) of A is defined in the following way. Let $\{a_i\}$ be any basis for A over $\mathbb{C}$ and let F be the free left A-module on a basis $\{\tilde{d}a_i\}$ where $\{\tilde{d}a_i\}$ is some set equipotent with $\{a_i\}$. We treat F as a 2-sided A-module by setting $b(\tilde{d}a) = (\tilde{d}a)b$ for all $a, b \in A$. Let $\tilde{d}\colon A \to F$ be the $\mathbb{C}$-linear map $\Sigma\, c_i a_i \mapsto \Sigma\, c_i \tilde{d}a_i$ and let K be the A-submodule of F generated by the relations $\tilde{d}(ab) - ((\tilde{d}a)b + a\tilde{d}b)$, $a, b \in A$. Then $\Omega_A := F/K$ and the canonical quotient map $a \mapsto \tilde{d}a + K$ is the *differential* map $d\colon A \to \Omega_A$.

Up to evident isomorphism (Ω_A, d) is characterized by the property that for every A-module M and every derivation $D\colon A \to M$ there is a unique A-module map $f\colon \Omega_A \to M$ such that

$$\begin{array}{ccc} A & \xrightarrow{\ d\ } & \Omega_A \\ & {\scriptstyle D}\searrow \quad \swarrow{\scriptstyle f} & \\ & M & \end{array}$$

commutes. In this way $\mathrm{Der}_{\mathbb{C}}(A, M) \simeq \mathrm{Hom}_A(\Omega_A, M)$.

Let $^-\colon \Omega_A \to \Omega_A/dA$ be the canonical linear map. Observe that from $\overline{d(ab)} = 0$ we have $\overline{adb} = -\overline{(da)b} = -\overline{bda}$ for all $a, b \in A$.

Consider the vector space

$$\mathfrak{u} := A \otimes_{\mathbb{C}} \dot{\mathfrak{g}} \oplus \left(\frac{\Omega_A}{dA}\right)$$

and define a bilinear multiplication $[\cdot, \cdot]'$ on $\mathfrak{u}$ by

$$[a \otimes x, b \otimes y]' = ab \otimes [x, y] + \overline{(da)b}(x \mid y)$$

$$\left[\dot{\mathfrak{g}}, \frac{\Omega_A}{dA}\right]' = \left[\frac{\Omega_A}{dA}, \dot{\mathfrak{g}}\right]' = \left[\frac{\Omega_A}{dA}, \frac{\Omega_A}{dA}\right]' = 0.$$

This makes $\mathfrak{u}$ into a Lie algebra. Generally, we shall write $[\cdot, \cdot]$ instead of $[\cdot, \cdot]'$ in the sequel. Let $\omega: \mathfrak{u} \to A \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ be the projection with kernel Ω_A/dA.

PROPOSITION 2.2 (Kassel). *$(\mathfrak{u}, \omega)$ is the uca of $A \otimes \dot{\mathfrak{g}}$.*

Proof (following Wilson [15]). Let

$$0 \to \mathfrak{z} \hookrightarrow \hat{\mathfrak{g}} \xrightarrow{\lambda} A \otimes_{\mathbb{C}} \dot{\mathfrak{g}} \to 0$$

be a central extension. Let $\tau: A \otimes_{\mathbb{C}} \dot{\mathfrak{g}} \to \hat{\mathfrak{g}}$ be any linear map so that $\lambda \circ \tau = \mathrm{id}$. Define

$$\tau^*: A \times A \times \dot{\mathfrak{g}} \times \dot{\mathfrak{g}} \to \mathfrak{z}$$

by

$$\tau^*(a, b, x, y) = [\tau(a \otimes x), \tau(b \otimes y)] - \tau(ab \otimes [x, y]).$$

One should recall that for $u, v \in A \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$, and $\tilde{u}, \tilde{v}$ preimages of them in $\hat{\mathfrak{g}}$, $[\tilde{u}, \tilde{v}]$ depends only on u and v. One can thus prove directly that

(A) $\quad \tau^*(a, b, x, y) = -\tau^*(b, a, y, x)$

(J) $\quad \tau^*(ab, c, [x, y], z) + \tau^*(bc, a, [y, z], x) + \tau^*(ca, b, [z, x], y) = 0.$

One now proves that there is a section τ_0 for which

$$\tau_0^*(a, 1, x, y) = 0 \quad \text{for all } a \in A,\ x, y \in \dot{\mathfrak{g}}.$$

To do this define for each $a \in A$

$$f^a: \dot{\mathfrak{g}} \to \mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})$$

by $v \mapsto f_v^a$ where

$$f_v^a(u) = \tau^*(a, 1, u, v) \quad \text{for all } u \in \dot{\mathfrak{g}}.$$

Putting $b = c = 1$ in (J) we obtain

$$0 = f_z^a([x, y]) - f_{[y,z]}^a(x) + f_y^a([z, x])$$

from which

(2.2) $\quad 0 = y \cdot f_z^a - z \cdot f_y^a - f_{[y,z]}^a.$

Here $\mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})$ is given the $\dot{\mathfrak{g}}$-module structure $(x \cdot f)(y) = f(-[x, y])$.

Now (2.2) says that f^a is a 1-cocycle on $\dot{\mathfrak{g}}$ with values in $\mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})$ and since $H^1(\dot{\mathfrak{g}}, \mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})) = 0$ ([6[1]]) there is an element $g^a \in \mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})$ for which $dg^a = f^a$. Thus $f_y^a = y \cdot g^a$ and hence $f_y^a(x) = g^a([x, y])$. Since $\dot{\mathfrak{g}}$ is perfect g^a is unique and then, since f^a is linear in a, g^a is linear in a.

[1] The standard result is for finite-dimensional $\dot{\mathfrak{g}}$-modules. However, $\mathrm{Hom}_{\mathbb{C}}(\dot{\mathfrak{g}}, \mathfrak{z})$ is a sum of finite-dimensional $\dot{\mathfrak{g}}$-modules, so the result easily extends.

Define $g\colon A \otimes_{\mathbb{C}} \mathfrak{g} \to \mathfrak{z}$ by $a \otimes y \mapsto g^a(y)$ and set $\tau_0 := \tau + g$:

$$\tau_0^*(a, 1, x, y) = \tau^*(a, 1, x, y) + \{[g(a \otimes x), g(1 \otimes y)] - g(a \otimes [x, y])\}$$
$$= f_y^a(x) - g^a([x, y]) = 0.$$

Let us replace τ by τ_0. Fix $a, b \in A$ and define

$$f\colon \dot{\mathfrak{g}} \times \dot{\mathfrak{g}} \to \mathfrak{z}$$

by

$$f(u, v) := \tau^*(a, b, u, v).$$

Then with $c = 1$ in (J) we obtain

$$0 = f([z, x], y) + f(x, [z, y]) \quad \text{for all } x, y, z \in \dot{\mathfrak{g}}.$$

Thus f is invariant. Since $\dot{\mathfrak{g}}$ has a unique invariant bilinear form up to scalars, it follows that there is $z_{a,b} \in \mathfrak{z}$ such that

$$\tau^*(a, b, u, v) = f(u, v) = (u \mid v) z_{a,b} \quad \text{for all } u, v \in \dot{\mathfrak{g}}.$$

From (A) and (J) and the fact that $(\cdot \mid \cdot)$ is symmetric we have

(2.3)
(i) $z_{a,1} = 0$
(ii) $z_{a,b} = -z_{b,a}$
(iii) $z_{ab,c} + z_{bc,a} + z_{ca,b} = 0$

for all $a, b, c \in A$.

Let F be the A-module defined above and define a map

$$F \to \mathfrak{z}$$
$$\sum b_i \tilde{d} a_i \mapsto \sum z_{a_i, b_i}.$$

From 2.3(iii) K dies and hence we have an induced map

$$\Omega_A \to \mathfrak{z}.$$

Finally for $a \in A$, $da \mapsto z_{a,1} = 0$ and hence

$$\frac{\Omega_A}{dA} \to \mathfrak{z} \qquad \overline{bda} \mapsto z_{a,b}.$$

The map $\mathfrak{u} \to \hat{\mathfrak{g}}$ defined by

$$c \otimes x + \overline{bda} \mapsto \tau(c \otimes x) + z_{a,b}$$

is a homomorphism and this completes the proof of the proposition. □

We call the universal central extension of $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ a *toroidal Lie algebra* and denote it by $\mathfrak{t}_{[n]}$. The algebra $A_{[n]} := \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ is the ring of polynomial functions of the torus $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ and the Lie algebra $\mathfrak{l}_{[n]} := A_{[n]} \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ may be viewed as the Lie algebra of polynomial maps of $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times \to \dot{\mathfrak{g}}$. In the case $n = 1$ we have the well-known loop algebra $\mathfrak{l}_{[1]}$. Its uca is the corresponding affine Lie algebra. In that sense it might be more appropriate to call $\mathfrak{l}_{[n]}$ a toroidal algebra, but we felt that the uca was more important and should have a suggestive name.

In the remainder of this paper we are going to treat the case $n = 2$. We shall simply denote the Lie algebra $\mathfrak{t}_{[2]}$ by $\mathfrak{t}$ and t_1 and t_2 by s and t respectively. An explicit description of $\Omega_{A_{[n]}}/dA_{[n]}$ has been given in [13]. Since this is important to use when $n = 2$ we work it out here.

The $A_{[2]}$-module $\Omega = \Omega_{A_{[2]}}$ has generators $d(s^p t^q)$ and relations $d(ab) = adb + bda$. Thus Ω is freely generated over A by $\mathrm{d}s$ and $\mathrm{d}t$ and hence freely over $\mathbb{C}$ by $\{s^p t^q \,\mathrm{d}s\} \cup \{s^p t^q \,\mathrm{d}t\}$. In Ω/dA

$$0 = \overline{d(s^p t^q)} = \overline{p s^{p-1} t^q \,\mathrm{d}s} + \overline{q s^p t^{q-1} \,\mathrm{d}t}.$$

Thus for $q \neq 0$

$$\overline{s^p t^{q-1} \,\mathrm{d}t} = -\frac{p}{q}\, \overline{s^{p-1} t^q \,\mathrm{d}s}$$

and a set of generators for Ω/dA over $\mathbb{C}$ is

$$\begin{aligned} (2.4) \qquad & a(p, q) := \overline{s^{p-1} t^q \,\mathrm{d}s}, \quad (p, q) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) \\ & a(p, 0) := \overline{s^p t^{-1} \,\mathrm{d}t}, \quad p \in \mathbb{Z} \\ & *(0, 0) := \overline{s^{-1} \,\mathrm{d}s}. \end{aligned}$$

It is elementary to see that these elements are linearly independent over $\mathbb{C}$ and hence form a basis for Ω/dA.

3. Presentations of Toroidal Lie Algebras

Let $A = (A_{ij})_{i,j=0}^{l}$ be an indecomposable Cartan matrix of affine type $X_l^{(1)}$ $(X = A, B, \ldots, G)$. Let Q be the free $\mathbb{Z}$-module on generators $\alpha_0, \ldots, \alpha_l$ and identify the affine root system Δ defined by A with a subset of Q by identifying $\{\alpha_0, \ldots, \alpha_l\}$ with a base Π of Δ. We know [10] that there is a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot\,|\,\cdot)$ on Q for which after suitable choice of indexing, $2(\alpha_i \,|\, \alpha_j)/(\alpha_j \,|\, \alpha_j) = A_{ij}$.

Let $\delta := \sum_{i=0}^{l} n_i \alpha_i$, $n_i \in \mathbb{Z}_+$, $gcd(n_0, \ldots, n_l) = 1$, be the null root. We assume that the notation is chosen so that $n_0 = 1$, so α_0 is an 'extension node' and

$\dot{A} = (A_{ij})_{i,j=1}^{l}$ is of finite type X_l. We assume that $(\cdot|\cdot)$ is scaled so that $(\alpha_0 | \alpha_0) = 2$. In general, objects associated with $\dot{A}$ carry an over-dot, so for instance $\dot{Q} := \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_l \subset Q$, $\dot{\Pi} := \{\alpha_1, \ldots, \alpha_l\}$.

For each $i = 0, \ldots, l$ let $\alpha_i^\vee := 2\alpha_i/(\alpha_i | \alpha_i)$. Then $\{\alpha_0^\vee, \ldots, \alpha_l^\vee\}$ forms a base for the coroot system $\Delta^\vee$ whose Cartan matrix is A^T. Its null root is $\Sigma_{i=0}^{l} n_i^\vee \alpha_i^\vee = \delta$ where $n_i^\vee := n_i(\alpha_i | \alpha_i)/2$. The fact that the $n_i^\vee \in \mathbb{Z}$ can be verified by inspection, case by case.

We let $\mathfrak{t} = \mathfrak{t}(A)$ be the Lie algebra over $\mathbb{C}$ with the following presentation:

(3.1) *generators*:

$$\mathfrak{c}, \quad \alpha_i^\vee(k), \quad x_k(\pm\alpha_i) \quad i = 0, \ldots, l, \quad k \in \mathbb{Z}$$

relations:

$$\text{TA0} \quad [\mathfrak{c}, \alpha_i^\vee(k)] = [\mathfrak{c}, x_k(\pm\alpha_i)] = 0$$

$$\text{TA1} \quad [\alpha_i^\vee(k), \alpha_j^\vee(m)] = k(\alpha_i^\vee | \alpha_j^\vee)\delta_{k+m,0}\mathfrak{c}$$

$$\text{TA2} \quad [\alpha_i^\vee(k), x_m(\pm\alpha_j)] = \pm(\alpha_i^\vee | \alpha_j) x_{m+k}(\pm\alpha_j)$$

$$\text{TA3} \quad [x_m(\alpha_i), x_n(-\alpha_j)] = -\delta_{ij}\left\{\alpha_i^\vee(m+n) + \frac{2m\delta_{m+n,0}}{(\alpha_i | \alpha_i)}\mathfrak{c}\right\}$$

$$\text{TA4} \quad [x_m(\alpha_i), x_n(\alpha_i)] = 0 = [x_m(-\alpha_i), x_n(-\alpha_i)]$$

$$\left.\begin{aligned}(ad x_0(\alpha_i))^{-A_{ji}+1} x_m(\alpha_j) &= 0\\ (ad x_0(-\alpha_i))^{-A_{ji}+1} x_m(-\alpha_j) &= 0\end{aligned}\right\} i \neq j$$

for all $i, j = 0, \ldots, l, \quad k, m, n \in \mathbb{Z}$.

The elements $\{\alpha_i^\vee(k)\}$ generate an infinite-dimensional Heisenberg algebra $\hat{\mathfrak{a}}$ whose structure is made somewhat more transparent by extending the notation a bit. We set $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} Q$ and for each $k \in \mathbb{Z}$ take an isomorphic copy $\mathfrak{h}(k)$ of $\mathfrak{h}$. Denote the isomorphism by $\alpha \mapsto \alpha(k)$. Then with $\mathfrak{a}' := \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}(k) \oplus \mathbb{C}\mathfrak{c}$ and Lie bracket

$$[\mathfrak{c}, \mathfrak{a}'] := 0$$

$$[\alpha(k), \beta(m)] := k(\alpha | \beta)\delta_{k+m,0}\mathfrak{c}$$

we obtain a Lie algebra that clearly has $\hat{\mathfrak{a}}$ as a homomorphic image. In fact, this is an isomorphism as we shall see in Corollary 3.7. In any case we will find it convenient to use the notation $\alpha(k)$ for $\Sigma c_i \alpha_i^\vee(k)$ whenever $\alpha = \Sigma c_i \alpha_i^\vee$. In particular we have the elements $\delta(k) = \Sigma n_i^\vee \alpha_i^\vee(k)$ which are evidently central since $(\delta | Q) = 0$.

REMARKS 1. From TA1 and TA3 $[a_i^\vee(1), \alpha_i^\vee(-1)] = (\alpha_i^\vee | \alpha_i^\vee)\mathfrak{c}$ and $[x_0(\alpha_i), x_n(-\alpha_i)] = -\alpha_i^\vee(n)$ so $\mathfrak{t}$ is generated by the elements $x_n(\pm\alpha_i)$.

2. The choice of the sign on the right-hand side of TA3 is made for the convenience of the vertex representation in Section 4.

Let $\dot{\mathfrak{g}}$ be the finite dimensional Lie algebra with Cartan matrix $\dot{A}$, Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} \dot{Q}$, and usual set of generators e_i, f_i, h_i, $i = 1, \ldots, l$. Let $B = \mathbb{C}[s, s^{-1}, t, t^{-1}]$ be the ring of Laurent polynomials in two commuting variables. Let ξ be the highest root of $\dot{\Delta}$ relative to $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_l\}$ and let $e_0 \in \dot{\mathfrak{g}}^{-\xi}$, $f_0 \in \dot{\mathfrak{g}}^{\xi}$ be chosen so that $\{e_0, h_0, f_0\}$ is an $\mathfrak{sl}_2$-triplet where $h_0 := -\Sigma_{i=1}^{l} n_i^{\vee} h_i$. Then the mapping

$$(3.2)\qquad \begin{aligned} &¢ \mapsto 0 \\ &\alpha_i^{\vee}(k) \mapsto s^k \otimes h_i \quad i = 0, \ldots, l \\ &\left.\begin{aligned} x_m(\alpha_i) &\mapsto s^m \otimes e_i \\ x_m(-\alpha_i) &\mapsto -s^m \otimes f_i \end{aligned}\right\} i = 1, \ldots, l \\ &x_m(\alpha_0) \mapsto s^m t \otimes e_0 \\ &x_m(-\alpha_0) \mapsto -s^m t^{-1} \otimes f_0 \end{aligned}$$

defines a surjective homomorphism

$$\pi: \mathfrak{t} \to \hat{\mathfrak{g}} := \mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}.$$

We wish to prove that $\mathfrak{t}$ is the universal central extension of $\mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes \dot{\mathfrak{g}}$.

The result is similar to a result of Kassel [8, Cor. 3.4]. However the presentations are different.

We begin by introducing a grading of $\mathfrak{t}$ by $\mathbb{Z} \times Q$ by assigning degrees to the generators as follows

$$(3.3)\qquad \begin{aligned} &\deg ¢ := (0, 0) \\ &\deg \alpha_i^{\vee}(k) := (k, 0) \\ &\deg x_k(\pm\alpha_i) := (k, \pm\alpha_i) \end{aligned}$$

for all $i = 0, \ldots, l$ and for all $k \in \mathbb{Z}$. We denote the space of elements of degree (k, α) in $\mathfrak{t}$ by $\mathfrak{t}_k^{\alpha}$.

We now define

$$Q_+ := \sum_{i=0}^{l} \mathbb{Z}_{\geqslant 0}\alpha_i \backslash \{0\}, \qquad Q_- := -Q_+$$

$$\mathfrak{t}_n^{\pm} := \sum_{\alpha \in Q_{\pm}} \mathfrak{t}_n^{\alpha}, \qquad \mathfrak{t}^{\pm} := \sum_{n \in \mathbb{Z}} \mathfrak{t}_n^{\pm}$$

$$\mathfrak{t}^{\alpha} := \sum_{n \in \mathbb{Z}} \mathfrak{t}_n^{\alpha}$$

$\mathfrak{s}_n^+ :=$ linear span of all products[2] $\quad [x_{n_k}(\beta_k), \ldots, x_{n_1}(\beta_1)]$

where $\quad \beta_1, \ldots, \beta_k \in \Pi, \quad n_1, \ldots, n_k \in \mathbb{Z}, \quad \Sigma\, n_k = n$

$\mathfrak{s}^+ := \sum_{n \in \mathbb{Z}} \mathfrak{s}_n^+$

$\mathfrak{s}_n^-$ and $\mathfrak{s}^-$ similarly

$\mathfrak{s}_n^0 :=$ linear span of $\delta_{n,0}\mathfrak{c}$ and the $\alpha_i^\vee(n), \quad i = 0, \ldots, l, \quad n \in \mathbb{Z}$

$\mathfrak{s}^0 := \sum_{n \in \mathbb{Z}} \mathfrak{s}_n^0$

$\mathfrak{s} := \mathfrak{s}^+ + \mathfrak{s}^0 + \mathfrak{s}^-.$

LEMMA 3.1. (i) $\mathfrak{t} = \mathfrak{s}$, $\mathfrak{t}_n^\pm = \mathfrak{s}_n^\pm$, *and* $\mathfrak{t}^\pm = \mathfrak{s}^\pm$.

(ii) $\mathfrak{t}_n = \mathfrak{t}_n^- + \mathfrak{t}_n^0 + \mathfrak{t}_n^+$; $\mathfrak{t} = \mathfrak{t}^- + \mathfrak{t}^0 + \mathfrak{t}^+$,

Proof. Since $\mathfrak{s}$ contains all the generators of $\mathfrak{t}$, to prove that $\mathfrak{s} = \mathfrak{t}$ it suffices to prove that $\mathfrak{s}$ is closed by the action of $\mathrm{ad}(g)$ where g runs through all the generators of $\mathfrak{t}$. This is very straightforward to prove. For instance, if $y = [x_{n_k}(\beta_k), \ldots, x_{n_1}(\beta_1)]$ where $\beta_1, \ldots, \beta_k \in \Pi$ and if $(n, \beta) \in \mathbb{Z} \times \Pi$ then

$$[x_n(\beta), y] \in \mathfrak{s}^+ \quad \text{(obvious)}$$

$$[x_n(-\beta), y] \in \mathfrak{s}^+ + \mathfrak{s}^0 \quad \text{(TA2, TA3)}$$

$$[a^\vee(i), y] \in \mathfrak{s}^+ \quad \text{(TA2)}.$$

Since $\mathfrak{s} = \mathfrak{t}$ and $\mathfrak{s}_n^\pm \subset \mathfrak{t}_n^\pm$, $\mathfrak{s}_n^0 \subset \mathfrak{t}_n^0$ these inclusions must be equalities and everything follows.

PROPOSITION 3.2.

$$\dim \mathfrak{t}_n^\alpha = \begin{cases} 1 & \textit{if } \alpha \in {}^{\mathrm{re}}\Delta \\ 0 & \textit{if } \alpha \notin \Delta \end{cases}$$

where ${}^{\mathrm{re}}\Delta$ *is the set of real roots of* Δ.

Proof. The subalgebra $\mathfrak{t}_0$ of $\mathfrak{t}$ contains the elements $\alpha_i^\vee(0)$, $x_0(\pm\alpha_i)$, $i = 0, \ldots, l$ which satisfy the relations for the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$. Hence we obtain a representation φ of $\mathfrak{g}$ on $\mathfrak{t}$ by $\mathfrak{g} \to \mathfrak{t}_0 \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{t})$. Furthermore, the representation is integrable because of TA4 and $\mathfrak{t}_n$ is a submodule under this action.

Let N be the group of automorphisms on $\mathfrak{t}$ generated by the elements

$$n_i(a) := \exp(\varphi a x_0(\alpha_i)) \exp(\varphi a^{-1} x_0(-\alpha_i)) \exp(\varphi a x_0(\alpha_i))$$

[2] In a Lie algebra $[a_k, \ldots, a_1] := \mathrm{ad}\, a_k \cdots \mathrm{ad}\, a_2(a_1)$.

$i = 0, \dots, l$, $a \in \mathbb{C}^\times$. We know by standard $\mathfrak{sl}_2$-theory that $n_i(a)\mathfrak{t}_n^\alpha = \mathfrak{t}_n^{r_i\alpha}$ [12] where $r_i\alpha = \alpha - (\alpha \mid \alpha_i^\vee)\alpha_i$.

By Lemma 3.1, $\mathfrak{t}_n^{\alpha_i}$ is spanned by $x_n(\alpha_i)$ so $\dim \mathfrak{t}_n^{\alpha_i} = 1$. Now if $\alpha \in {}^{\mathrm{re}}\Delta$, $\alpha = w\alpha_i$ for some i and some $w = r_{i_p} \cdots r_{i_1}$ and we obtain $g \in N$ with $g\mathfrak{t}_n^{\alpha_i} = \mathfrak{t}_n^\alpha$ and $\dim \mathfrak{t}_n^\alpha = 1$.

Suppose now that $\alpha \in Q \backslash \Delta$. Write $\alpha = \Sigma\, c_i\alpha_i$. If the c_i are of mixed signs then $\mathfrak{t}_n^\alpha = (0)$ by Lemma 3.1. Suppose that the signs are all consistent, say, for definiteness, all positive. Then $0 = (\delta \mid \alpha) = \Sigma\, n_j(\alpha_j \mid \alpha)$ and hence for at least one j, $(\alpha_j \mid \alpha) > 0$ (for otherwise $(\alpha_j \mid \alpha) = 0$ for all j and $\alpha \in \mathbb{Z}\delta \subset \Delta$). We have $\dim \mathfrak{t}_n^\alpha = \dim \mathfrak{t}_n^{r_j\alpha}$ and $r_j\alpha = \alpha - \lambda$ for some $\lambda \in Q_+$. If $r_j\alpha \in Q_+$ we replace α by $r_j\alpha$ and repeat but with an element of reduced height. Thus we may assume that $r_j\alpha \notin Q_+$. If $r_j\alpha$ has mixed signs we are done, so we may assume that $r_j\alpha \in Q_-$. Then $\alpha = k\alpha_j$ with $k > 1$ since $\alpha \notin \Delta$. But $\mathfrak{t}_n^{k\alpha_j}$ is spanned by the products $[x_{n_k}(\alpha_j), \dots, x_{n_1}(\alpha_j)] = 0$ by TA4. □

COROLLARY 3.3.

$$\text{(TA4}') \qquad (\operatorname{ad} x_n(\alpha_i))^{-A_{ji}+1} x_m(\alpha_j) = 0$$
$$(\operatorname{ad} x_n(-\alpha_i))^{-A_{ji}+1} x_m(-\alpha_j) = 0$$

for all $i \neq j$, $m, n \in \mathbb{Z}$.

Proof. $\alpha_j + (-A_{ji} + 1)\alpha_i = r_i(\alpha_j - \alpha_i) \notin \Delta$. □

COROLLARY 3.4. *For each* $\alpha \in {}^{\mathrm{re}}\Delta$, $n \in \mathbb{Z}$,

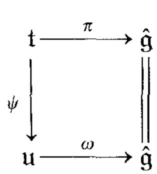
$$\mathbb{C}\mathfrak{t}_n^\alpha + \mathbb{C}[\mathfrak{t}_n^\alpha, \mathfrak{t}_{-n}^{-\alpha}] + \mathbb{C}\mathfrak{t}_{-n}^{-\alpha} \simeq \mathfrak{sl}_2(\mathbb{C}).$$

Proof. Using the Weyl group we can assume that $\alpha = \alpha_i$. Then the result follows from TA2 and TA3. □

PROPOSITION 3.5. $(\mathfrak{t}, \pi)$ *is the universal central extension of* $\hat{\mathfrak{g}} = \mathbb{C}[s, s^{-1}, t, t^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$. *The kernel of* π *is contained in* $\Sigma_{k \in \mathbb{Z}}\, \mathfrak{t}^{k\delta}$.

Proof. $\hat{\mathfrak{g}}$ is graded by $\mathbb{Z} \times Q$ by assigning degrees in (3.2) according to the degrees on the left-hand side. Then π is a graded homomorphism and, in view of Proposition 3.2, $\ker \pi \subset \Sigma_{k \in \mathbb{Z}}\, \mathfrak{t}^{k\delta}$. Since $[x_m(\pm\alpha_i), \Sigma_{k \in \mathbb{Z}}\, \mathfrak{t}^{k\delta}] \cap \Sigma_{k \in \mathbb{Z}}\, \mathfrak{t}^{k\delta} = (0)$, the kernel is central.

To prove that the extension is universal we construct a mapping ψ from $\mathfrak{t}$ to $\mathfrak{u}$ of Section 2 over $\hat{\mathfrak{g}}$



Explicitly

$$\begin{aligned}
&\not{c} \mapsto \overline{s^{-1}\,ds}\\
&\left.\begin{aligned}x_m(\alpha_i) &\mapsto s^m \otimes e_i\\ x_m(-\alpha_i) &\mapsto -s^m \otimes f_i\end{aligned}\right\} i = 1, \ldots, l\\
&x_m(\alpha_0) \mapsto s^m t \otimes e_0\\
&x_m(-\alpha_0) \mapsto -s^m t^{-1} \otimes f_0\\
&\alpha_i^\vee(k) \mapsto s^k \otimes h_i, \quad i = 1, \ldots, l\\
&\alpha_0^\vee(k) \mapsto s^k \otimes h_0 + \overline{s^k t^{-1}\,dt}.
\end{aligned}$$

One has to prove that the elements on the right-hand side satisfy relations TA. The relations TA0, TA2, TA4 are essentially trivial. Using the definition of $\mathfrak{u}$ in Section 2:

TA1: $$\begin{aligned}[s^k \otimes h_i, s^m \otimes h_j] &= \overline{(ds^k)s^m}(h_i \mid h_j)\\ &= k\delta_{k+m,0}\overline{s^{-1}\,ds}(\alpha_i^\vee \mid \alpha_j^\vee).\end{aligned}$$

TA3: We need only consider the case $i = j$ since $(e_i \mid f_j) = 0$ if $i \neq j$. Suppose $i = j \neq 0$. Then, using (2.1),

$$\begin{aligned}[s^m \otimes e_i, -s^n \otimes f_i] &= -s^{m+n} \otimes h_i - \overline{(ds^m)s^n}(e_i \mid f_i)\\ &= -s^{m+n} \otimes h_i - m\delta_{m+n,0}\overline{s^{-1}\,ds}\,\frac{2}{(\alpha_i \mid \alpha_i)}.\end{aligned}$$

If $i = j = 0$,

$$a := [s^m t \otimes e_0, -s^n t^{-1} \otimes f_0] = -s^{m+n} \otimes h_0 - \overline{(d(s^m t))(s^n t^{-1})}(e_0 \mid f_0).$$

Now

$$\begin{aligned}\overline{d(s^m t)(s^n t^{-1})} &= m\overline{s^{m+n-1}\,ds} + \overline{s^{m+n}t^{-1}\,dt}\\ &= m\delta_{m+n,0}\overline{s^{-1}\,ds} + \overline{s^{m+n}t^{-1}\,dt}\end{aligned}$$

and

$$(e_0 \mid f_0) = \frac{2}{(\alpha_0 \mid \alpha_0)} = 1 \quad \text{from (2.1).}$$

Thus together we have

$$\begin{aligned}a &= -(s^{m+n} \otimes h_0 + \overline{s^{m+n}t^{-1}\,dt}) - m\delta_{m+n,0}\overline{s^{-1}\,ds}\\ &= -\psi\left(\alpha_0^\vee(m+n) + \frac{2m\delta_{m+n,0}}{(\alpha_0 \mid \alpha_0)}\,\not{c}\right)\end{aligned}$$

which is what we want.

Thus ψ exists and $\omega\psi = \pi$. On the other hand, by the definition of $\mathfrak{u}$ there is a unique homomorphism $\lambda: \mathfrak{u} \to \mathfrak{t}$ such that $\pi\lambda = \omega$. Then $\psi\lambda$ and $\lambda\psi$ are Lie endomorphisms of $\mathfrak{u}$ and $\mathfrak{t}$ respectively over $\hat{\mathfrak{g}}$. By Lemma 2.1 they are identity maps and we conclude that $\mathfrak{u} \simeq \mathfrak{t}$. □

Let $\pi_0: Q \to \mathbb{Z}$ be the map $\Sigma c_i\alpha_i \mapsto c_0$ and let $^{-}: Q \to \dot{Q}$ be the map $\Sigma c_i\alpha_i \mapsto -c_0\xi + \Sigma_{i=1}^{l} c_i\alpha_i$. From the definition of ψ we have for all $\alpha \in {}^{\mathrm{re}}\Delta$

$$(3.4) \qquad \mathfrak{t}_n^{\alpha} \xrightarrow{\ \psi\ } s^n t^{\pi_0(\alpha)} \otimes \mathfrak{g}^{\bar{\alpha}}.$$

PROPOSITION 3.6. *Under the isomorphism ψ above we have*

(i) $\psi^{-1}(\overline{s^k t^{-1}\,\mathrm{d}t}) = \delta(k) \in \mathfrak{t}_k^0, \quad k \in \mathbb{Z}$

$\psi^{-1}(\overline{s^{k-1}t^r\,\mathrm{d}s}) \in \mathfrak{t}_k^{r\delta}, \quad k, r \in \mathbb{Z}$;

(ii) $\dim \mathfrak{t}_k^{r\delta} = \begin{cases} l+1 & \textit{if } (k, r) \neq (0, 0) \\ l+2 & \textit{if } (k, r) = (0, 0); \end{cases}$

(iii) $\dim \mathfrak{t}_k^{r\delta} \cap \mathrm{centre}(\mathfrak{t}) = \begin{cases} 1 & \textit{if } (k, r) \neq (0, 0) \\ 2 & \textit{if } (k, r) = (0, 0). \end{cases}$

Proof. (i)

$$\psi\delta(k) = \psi\left(\sum n_i^{\vee}\alpha_i^{\vee}(k)\right)$$
$$= \left(\sum_{i=0}^{l} s^k \otimes n_i^{\vee} h_i\right) + \overline{s^k t^{-1}\,\mathrm{d}t} = \overline{s^k t^{-1}\,\mathrm{d}t}.$$

Fix any long real root $\alpha \in \dot{\Delta}$ and let $\{e_\alpha, [e_\alpha, e_{-\alpha}], e_{-\alpha}\}$ be an $\mathfrak{sl}_2$-triplet with $e_\alpha \in \mathfrak{g}^{\alpha}$, $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ (for instance $\alpha = \alpha_i$, $e_\alpha = e_i$, $e_{-\alpha} = f_i$). Define

$$x_m(\pm\alpha + k\delta) = \psi^{-1}(\pm s^m t^k \otimes e_{\pm\alpha}) \quad \text{for all } m, k \in \mathbb{Z}.$$

In view (3.4) it is clear that $x_m(\pm\alpha + k\delta) \in \mathfrak{t}_m^{\pm\alpha + k\delta}$.

Now we have

$$\psi([x_m(\alpha + k\delta), x_n(-\alpha + r\delta)])$$
$$= [s^m t^k \otimes e_\alpha, \ -s^n t^r \otimes e_{-\alpha}]$$
$$= -\{s^{m+n}t^{k+r} \otimes [e_\alpha, e_{-\alpha}] + \overline{(d(s^m t^k))(s^n t^r)}\}$$
$$= -s^{m+n}t^{k+r} \otimes [e_\alpha, e_{-\alpha}] - m\overline{s^{m+n-1}t^{k+r}\,\mathrm{d}s} - k\overline{s^{m+n}t^{k+r-1}\,\mathrm{d}t}.$$

Thus

$$(3.5) \qquad \psi([x_m(\alpha + k\delta), x_n(-\alpha + r\delta)]) - \psi([x_n(\alpha + k\delta), x_m(-\alpha + r\delta)])$$
$$= (n - m)\overline{s^{m+n-1}t^{k+r}\,\mathrm{d}s}$$

and hence

$$\psi^{-1}(\overline{s^{m+n-1}t^{k+r}\,\mathrm{d}s}) \in \mathfrak{t}_{m+n}^{(k+r)\delta}.$$

(Note that (3.5) is 0 if $k + r = 0$ and $m + n \neq 0$.)

(ii), (iii) The elements $\overline{s^k t^{-1}\,\mathrm{d}t}$, $k \in \mathbb{Z}$ and $\overline{s^{k-1}t^r\,\mathrm{d}s}$, $r \neq 0$, and $\overline{s^{-1}\,\mathrm{d}s}$ form a basis for $\Omega A_{[2]}/dA_{[2]}$. Since $\dim \hat{\mathfrak{g}}_k^{r\delta} = l$ for all r, k, (ii) and (iii) follow from (i). □

COROLLARY 3.7. $\mathfrak{a}' \simeq \hat{\mathfrak{a}}$.

Proof. Let $\alpha \in \mathfrak{h} \backslash \{0\}$ and write it as $\alpha = \dot{\alpha} + a\delta$ where $\dot{\alpha} = \Sigma_{i=1}^{l} c_i \alpha_i^{\vee}$, $c_i \in \mathbb{C}$. Then for $k \in \mathbb{Z}$ the element $\alpha(k)$ of $\hat{\mathfrak{a}}$ is mapped by ψ to

$$\psi(\alpha(k)) = s^k \otimes \sum_{i=1}^{l} c_i h_i + a\overline{s^k t^{-1}\,\mathrm{d}t} \neq 0.$$

The result follows. □

COROLLARY 3.8. *Suppose that $\mathfrak{s}$ is a Lie algebra over $\mathbb{C}$ graded by $\mathbb{Z} \times Q$ and λ: $\mathfrak{t}(A) \to \mathfrak{s}$ is a surjective graded homomorphism of Lie algebras such that*

(i) *λ is injective on $\mathfrak{t}_n^{\alpha}$ for all $(n, \alpha) \in \mathbb{Z} \times {}^{\mathrm{re}}\Delta$;*
(ii) for all k, $\lambda(\delta(k)) \neq 0$ *and $\lambda|_{\mathbb{C}\delta(0)+\mathbb{C}\mathfrak{c}}$ is injective;*
(iii) *for all m, k, $m \neq 0$, $k \neq 0$,*

$$\lambda([x_m(\alpha_1 + k\delta), x_0(-\alpha_1)] - [x_0(\alpha_1 + k\delta), x_m(-\alpha_1)]) \neq 0$$

and

$$\lambda(x_1(\alpha_1 + k\delta), x_{-1}(-\alpha_1)] - [x_{-1}(\alpha_1 + k\delta), x_1(-\alpha_1)]) \neq 0.$$

Then λ is an isomorphism.

Proof. From (i), $\ker \lambda \subset \Sigma_{m,k}\, \mathfrak{t}_m^{k\delta}$ and hence, by the argument in Proposition 3.6, $\ker \lambda$ is central. Conditions (ii) and (iii) say that there is a non-zero central element in $\mathfrak{s}_m^{k\delta}$ for all $(m, k) \neq (0, 0)$ and two independent central elements for $(m\ k) = (0, 0)$. □

4. Vertex operators

From this point on we assume that the *affine Cartan matrix A of Section* 3 *is simply laced,* i.e. $A = X_l^{(1)}$ where $X = A, D, E$.

We assume that the lattice Q is a sublattice of a lattice Γ admitting a non-singular $\mathbb{Z}$-valued symmetric bilinear form $(\cdot|\cdot)$ that restricts to the previously chosen form on Q. Such a lattice Γ can be constructed in many ways. A particularly interesting example for $\dot{Q}$, Q, Γ is the series of root lattices E_8, E_9 $(= E_8^{(1)})$, E_{10} which belong to the simple Lie algebra, the affine Lie algebra, and the hyperbolic Lie algebra of the same names.

We briefly review the notation and construction of vertex representation spaces and the operators $X_m(\alpha)$ that act on it. This theory is due to Frenkel and Kac [2]. For further details one may also consult [3], [5], [11[3]].

Let $\mathfrak{k} := \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$ and define a Heisenberg algebra $\hat{\mathfrak{b}} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{k}(k) \oplus \mathbb{C}\not{c}$ from a collection of copies $\mathfrak{k}(k)$ of $\mathfrak{k}$ as was done for $\mathfrak{h}$ in Section 3. Thus multiplication in $\hat{\mathfrak{b}}$ is defined by

$$[\alpha(k), \beta(m)] := k(\alpha \mid \beta)\delta_{k+m,0}\not{c}$$

and $\hat{\mathfrak{a}}$ may be considered as a subalgebra of $\hat{\mathfrak{b}}$. We let σ be the conjugate linear involution on $\hat{\mathfrak{b}}$ defined by $\alpha(n) \mapsto \alpha(-n)$ for all $\alpha \in \mathfrak{k}_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \Gamma$. Define

$$\mathfrak{b} := \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{k}(k) + \mathbb{C}\not{c} \subset \hat{\mathfrak{b}}$$

$$\mathfrak{b}_{\pm} := \sum_{k \gtrless 0} \mathfrak{k}(k)$$

and $\mathfrak{a}$, $\mathfrak{a}_{\pm}$ similarly with $\mathfrak{h}$ replacing $\mathfrak{k}$.

The Fock space representation of $\mathfrak{b}$ is the symmetric algebra $S(\mathfrak{b}_-)$ of $\mathfrak{b}_-$ together with the action of $\mathfrak{b}$ on $S(\mathfrak{b}_-)$ defined by

$\not{c}$	acts as 1
$a(-m)$	acts as multiplication by $a(-m)$, $m > 0$
$a(m)$	acts as the unique derivation on $S(\mathfrak{b}_-)$ for which $b(-n) \mapsto \delta_{m-n,0}\, m(a \mid b)$

for all $a, b \in \mathfrak{k}$, $m, n > 0$.

$S(\mathfrak{b}_-)$ affords an irreducible representation of $\mathfrak{b}$. However, $S(\mathfrak{a}_-)$ does not afford an irreducible representation of $\mathfrak{a}$ since the form $(\cdot \mid \cdot)$ is degenerate on $\mathfrak{h}$.

Let ε: $Q \times Q \to \{\pm 1\}$ be a bimultiplicative map satisfying

CC (i) $\varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}$

(ii) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}$

(iii) $\varepsilon(\alpha, \delta) = 1$

for all $\alpha, \beta \in Q$. Condition (iii) makes it easy to determine such an ε by using a corresponding cocycle on $\dot{Q}$.

We shall assume that ε *can be extended to a bimultiplicative map*

$$\varepsilon\colon Q \times \Gamma \to \{\pm 1\}.$$

No further assumptions on ε are needed.

[3] The following errors should be corrected in this paper: p. 191, 13↓, $\delta_{m+n,0}m(x, y)c$; p. 199, 9↓, $\partial_{a(0)}\mu^n \otimes x = n(a, \mu)\mu^{n-1} \otimes x$; 3↑, $\frac{1}{2}(\mu, \mu) + n_1 + \cdots + n_k = n$; p. 208, 6↑, $\varepsilon(\alpha, \alpha) = -1$; p. 209, 8↑, $[e_m(\alpha), e_n(\beta)] = -m\delta_{m+n,0} - \alpha(m+n)$ if $(\alpha, \beta) = -2$.

For each $\gamma \in \Gamma$ let e^γ be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with basis $\{e^\gamma\}$ over $\mathbb{C}$. In particular, $\mathbb{C}[\Gamma]$ contains the subspace $\mathbb{C}[Q] := \Sigma_{\alpha \in Q} \mathbb{C}e^\alpha$. Following Borcherds [1] we define a twisted group algebra structure on $\mathbb{C}[Q]$ by

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$$

and make $\mathbb{C}[\Gamma]$ into a $\mathbb{C}[Q]$-module by defining

$$e^\alpha e^\gamma = \varepsilon(\alpha, \gamma) e^{\alpha+\gamma} \quad \text{for all } \alpha, \beta \in Q, \quad \gamma \in \Gamma.$$

Let $M \subset S(\mathfrak{b}_-)$ be any $\mathfrak{a}$-submodule (with respect to the Fock space action). We define

$$V(\Gamma, M) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} M.$$

Of particular interest in the sequel will be $V(\Gamma, S(\mathfrak{a}_-))$ and $V(\Gamma, S(\mathfrak{b}_-))$ which we simply denote by $V(\Gamma)$ and $V(\Gamma, \mathfrak{b})$ respectively.

We extend the action of $\mathfrak{a}$ on M to $\hat{\mathfrak{a}}$ on $V(\Gamma, M)$ by

$$a(m) \cdot (e^\gamma \otimes u) := e^\gamma \otimes a(m) \cdot u \quad \text{if } m \neq 0$$

$$a(0) \cdot (e^\gamma \otimes u) := (a \mid \gamma) e^\gamma \otimes u \quad \text{if } m = 0.$$

Let z be a complex-valued variable and let $\alpha \in Q$. Define

$$T_\pm(\alpha, z) := -\sum_{n \gtrless 0} \frac{1}{n} \alpha(n) z^{-n}.$$

Then the vertex operator for α on $V(\Gamma, M)$ is defined as

$$X(\alpha, z) := z^{(\alpha|\alpha)/2} \exp T(\alpha, z)$$

where

$$\exp T(\alpha, z) := \exp T_-(\alpha, z) e^\alpha z^{\alpha(0)} \exp T_+(\alpha, z)$$

and the operator $z^{\alpha(0)}$ is defined by

$$z^{\alpha(0)} e^\gamma \otimes u := z^{(\alpha|\gamma)} e^\gamma \otimes u.$$

Strictly, for each $z \in \mathbb{C}^\times$ the operator $X(\alpha, z)$ maps $V(\Gamma, M)$ into the space $C(\Gamma] \otimes \widehat{S(\mathfrak{b}_-)}$ where $\widehat{S(\mathfrak{b}_-)} = \Pi_n S(\mathfrak{b}_-)^n$ is the completion to formal power series of $S(\mathfrak{b}_-)$. However $X(\alpha, z)$ can be formally expanded in powers of z to give

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the 'moments' $X_n(\alpha)$ are indeed operators on $V(\Gamma, M)$. Moreover, for all $v \in M$, $X_n(\alpha)(e^\gamma \otimes v) = e^{\gamma+\alpha} \otimes v'$ where $v' \in M$ is obtained from v by applying some polynomial expression in the operators $a(m)$, $a \in \mathfrak{h}$, $m \in \mathbb{Z}\backslash\{0\}$.

The powers of z in $X(\alpha, z)$ are so construed that $X_n(\alpha)$ is an operator of degree $(-n, \alpha)$ relative to the $\mathbb{Z} \times \Gamma$ grading on $V(\Gamma, M)$ for which

$$\deg(e^\gamma \otimes a_k(-n_k) \cdots a_1(-n_1)) = (\tfrac{1}{2}(\gamma \mid \gamma) + n_1 + \cdots + n_k, \gamma).$$

Assigning $X_n(\alpha)$ degree (n, α) would seem more natural for what is to come but the grading on $V(\Gamma, M)$ is well established in the literature.

The basic commutation relations for the operators $\alpha(k)$ and $X_m(\alpha)$ on $V(\Gamma, \mathfrak{b})$ are these ([2], [5], [11]):

CR0 $\quad [\alpha(k), X_n(\beta)] = (\alpha \mid \beta) X_{n+k}(\beta)$

CR1 $\quad [X_m(\alpha), X_n(\beta)] = 0$ if $(\alpha \mid \beta) \geqslant 0$

CR2 $\quad [X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta) X_{m+n}(\alpha+\beta)$ if $(\alpha \mid \beta) = -1$

CR3 $\quad [X_m(\alpha), X_n(-\alpha)] = -\alpha(m+n) - m\delta_{m+n,0}\not{c}$ if $(\alpha \mid \alpha) = 2$.

In fact CR3 is a special case of

LEMMA 4.1. *Let* $\alpha, \beta \in Q$ *with* $(\alpha \mid \alpha) = (\beta \mid \beta) = -(\alpha \mid \beta) = 2$. *Then*

CR4 $\quad [X_m(\alpha), X_n(\beta)]$

$$= \varepsilon(\alpha, \beta)\{m X_{m+n}(\alpha+\beta) + \sum_{k \in \mathbb{Z}} {:}\alpha(k) X_{m+n-k}(\alpha+\beta){:}\}$$

where the normal ordering symbols : : *indicate that the operators are to be applied in the order of increasing degree. In particular here*

$$:\alpha(k) X_{m+n-k}(\beta): = \begin{cases} \alpha(k) X_{m+n-k}(\beta) & \text{if } k \leqslant m+n-k \\ X_{m+n-k}(\beta)\alpha(k) & \text{if } k > m+n-k. \end{cases}$$

This result is itself a special case of the general commutator formula (8.4.43) of [3]. However, it is also easily derivable from the technique of residue calculus so we offer a short proof if it here.

Proof. We begin with the general formula

$$\varepsilon(\alpha, \beta)[X_m(\alpha), X_n(\beta)]$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{c'} \frac{dz}{z} \int_c \frac{dw}{w} \{w^{[(\alpha|\alpha)/2]+m} z^{[(\beta|\beta)/2]+n} :\exp(T(\alpha, w) + T(\beta, z)): (z-w)^{(\alpha|\beta)}\}$$

where $:\exp(T(\alpha, w) + T(\beta, z)):$ means

$$\exp(T_-(\alpha, w) + T_-(\beta, z)) e^{\alpha+\beta} w^{\alpha(0)} z^{\beta(0)} \exp(T_+(\alpha, w) + T_+(\beta, z)).$$

This may be found in [5], [11], or [9, (15.14), (15.15)]. In these references $\mathbb{C}[Q]$ is not twisted so the left-hand side is $X_m(\alpha)X_n(\beta) - (-1)^{(\alpha|\beta)}X_n(\beta)X_m(\alpha)$. The contour c runs around z and does not have 0 in its interior. The contour c' is around 0. Since we are assuming that $(\alpha\,|\,\alpha) = (\beta\,|\,\beta) = 2$ and $(\alpha\,|\,\beta) = -2$ we can compute the inner integral by the straightforward computation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}w}\{w^m z^{n+1} :\!\exp(T(\alpha, w) + T(\beta, z))\!:\}_{w=z} \\
&\quad = mz^{m+n} :\!\exp T(\alpha+\beta, z)\!: \\
&\quad\quad + z^{m+n} :\!\left\{\sum_{k<0} \alpha(k) z^{-k} + \alpha(0) + \sum_{k>0} \alpha(k) z^{-k}\right\}\!: \exp T(\alpha+\beta, z)\!:: \\
&\quad = z^{m+n}\left\{m \sum X_k(\alpha+\beta) z^{-k} + \sum_{k,r} :\!\alpha(k) X_r(\alpha+\beta) z^{-(k+r)}\!:\right\}.
\end{aligned}$$

Applying $(1/2\pi i)\int \mathrm{d}z/z$ we obtain CR4. □

LEMMA 4.2. *Let* $\alpha, \beta \in Q$ *with* $(\alpha\,|\,\alpha) = (\beta\,|\,\beta) = -(\alpha\,|\,\beta) = 2$. *Then*

(i) *the operators* $X_m(\alpha+\beta)$ *commute with all operators* $X_n(\varphi)$, $\varphi \in Q$;

(ii) $[X_m(\alpha), X_n(\beta)] - [X_n(\alpha), X_m(\beta)] = \varepsilon(\alpha, \beta)(m-n)X_{m+n}(\alpha+\beta)$.

Proof. (i) The assumptions on α, β imply that $\alpha + \beta$ is isotropic and hence a multiple of δ. Thus (i) follows from CR1.

(ii) In Lemma 4.1 interchange m and n and subtract. □

PROPOSITION 4.3. *The Lie algebra* $\mathfrak{t}$ *of operators on* $V(\Gamma, \mathfrak{b})$ *generated by the operators* $X_m(\alpha)$, $m \in \mathbb{Z}$, $\alpha \in {}^{\mathrm{re}}\Delta$, *is isomorphic to* $\mathfrak{t}(A)$ *via the uniquely defined map* $\lambda\colon \mathfrak{t}(A) \to \mathfrak{t}$ *for which* $x_m(\pm\alpha_i) \mapsto X_m(\pm\alpha_i)$, $\mathfrak{c} \mapsto 1$.

Proof. From the relations CR it is clear that $\mathfrak{t}$ is generated by the elements $X_m(\pm\alpha_i)$, $i = 0, \ldots, l$, $m \in \mathbb{Z}$, and contains the operators $\alpha_i(k)$ $(=\alpha_k^\vee(k))$ and $\mathfrak{c}$. The relations TA are very easy to check from the relations CR. For instance in the case of TA4 the relation is trivial if $A_{ji} = 0$ since then $(\alpha_i\,|\,\alpha_j) = 0$ and we can use CR1. If $(\alpha_i\,|\,\alpha_j) = -1$ then by CR2

$$[X_m(\alpha_i), X_n(\alpha_j)] = \varepsilon(\alpha_i, \alpha_j)X_{m+n}(\alpha_i + \alpha_j).$$

Since $(\alpha_i\,|\,\alpha_i + \alpha_j) = 1 > 0$,

$$[X_m(\alpha_i), X_{m+n}(\alpha_i + \alpha_j)] = 0.$$

The remaining case is $(\alpha_i\,|\,\alpha_j) = -2$ (which occurs only in the case of $A_1^{(1)}$). After using CR4 we are left to prove that $(\mathrm{ad}\, X_0(\alpha_i))^2\{\alpha_i(k)X_n(\delta)\} = 0$ for all k, which is straightforward. This establishes a surjective homomorphism $\mathfrak{t}(A) \to \mathfrak{t}$.

We have already seen that the operators $X_m(\alpha)$ are operators of degree $(-m, \alpha)$ on $V(\Gamma, \mathfrak{b})$. Using the map $(-m, \alpha) \mapsto (m, \alpha)$ we temporarily assign $X_m(\alpha)$ degree (m, α). Then our mapping λ is a graded homomorphism. We now apply Corollary 3.8. Since the operators $\delta(k)$ and $X_m(k\delta)$ are all non-trivial and $\delta(0)$ is not a scalar map, we obtain $\mathfrak{t}(A) \simeq \mathfrak{t}$ from Lemma 4.2. □

For each $\mathfrak{a}$-submodule M of $S(\mathfrak{b}_-)$ the space $V(\Gamma, M)$ affords a subrepresentation of $\mathfrak{t}$. In general this is not faithful. In fact for the case $V(\Gamma)$ when $M = S(\mathfrak{a}_-)$ it is not faithful as we shall soon see.

The following dictionary of central elements is useful

$$\text{(4.1)} \qquad \overline{s^k t^{-1}\,\mathrm{d}t} \leftrightarrow \delta(k)$$

$$\overline{s^{k-1} t^n\,\mathrm{d}s} \leftrightarrow X_k(n\delta), \quad n \neq 0 \text{ unless } k = 0$$

(note that $\overline{s^{-1}\,\mathrm{d}s} \leftrightarrow X_0(0) = \mathrm{id} = \mathfrak{c}$ as expected). The first correspondence was given in Proposition 3.6. The second may be deduced from the comparison of (3.5) and Lemma 4.2 with α replaced by $\alpha + k\delta$, β replaced by $-\alpha + r\delta$:

$$\begin{aligned}(n-m)\overline{s^{m+n-1}t^{k+r}\,\mathrm{d}s} \leftrightarrow{} & [X_m(\alpha + k\delta), X_n(-\alpha + r\delta)] \\ & - [X_n(\alpha + k\delta), X_m(-\alpha + r\delta)] \\ & = \varepsilon(\alpha + k\delta, -\alpha + r\delta)(m-n)X_{m+n}((k+r)\delta) \\ & = (n-m)X_{m+n}((k+r)\delta).\end{aligned}$$

In the sequel we shall usually identify $\mathfrak{t}$ and $\mathfrak{t}(A)$ by the isomorphism of Proposition 4.3.

5. The structure of $V(\Gamma)$

In this section we examine how $V(\Gamma)$ decomposes as a $\mathfrak{t} = \mathfrak{t}(A)$-module and look at the structure of its indecomposable constituents.

Let $\lambda \in \Gamma$ and define

$$V(\lambda) := e^{\lambda + Q} \otimes S(\mathfrak{a}_-).$$

The operator $X(\alpha, z)$, $\alpha \in Q$, maps $V(\lambda)$ into the space $e^{\lambda+Q} \otimes \widehat{S(\mathfrak{a}_-)}$ where $\widehat{S(\mathfrak{a}_-)} = \Pi_{n=0}^{\infty} S(\mathfrak{a}_-)^n$ is the formal power series completion of $S(\mathfrak{a}_-)$. Thus for each n the component $X_n(\alpha)$ of $X(\alpha, z)$ acts as an endomorphism on $V(\lambda)$ and hence $V(\lambda)$ is a $\mathfrak{t}$-submodule of $V(\Gamma)$.

Fix $\lambda \in \Gamma$ and observe that $N := (\lambda + \alpha \mid \delta)$ is constant as $\lambda + \alpha$ runs over the coset $\lambda + Q$.

For each $\alpha \in Q$ define the polynomials $s_p(\alpha) \in S(\mathfrak{a}_-)$, $p \in \mathbb{Z}$, by

$$\exp T_-(\alpha, z) =: \sum_{p=0}^{\infty} s_p(\alpha) z^p$$

and

$$s_p(\alpha) := 0 \quad \text{if} \quad p < 0.$$

Furthermore,

$$\begin{aligned}\sum_{m \in \mathbb{Z}} X_m(\alpha) z^{-m} \cdot (e^\lambda \otimes 1) = X(\alpha, z) \cdot (e^\lambda \otimes 1) \\ = \varepsilon(\alpha, \lambda) z^{(\alpha|\alpha)/2} z^{(\alpha|\lambda)} e^{\lambda+\alpha} \otimes \exp T_-(\alpha, z) \cdot 1 \\ = \varepsilon(\alpha, \lambda) e^{\lambda+\alpha} \otimes \sum_{p=0}^{\infty} s_p(\alpha) z^{p+(\alpha|\lambda+\alpha/2)}\end{aligned}$$

and hence

$$X_m(\alpha) \cdot (e^\lambda \otimes 1) = \varepsilon(\alpha, \lambda) e^{\lambda+\alpha} \otimes s_{-m-(\alpha|\lambda+\alpha/2)}(\alpha).$$

We note two special cases of this

$$X_{-(\alpha|\lambda+\alpha/2)}(\alpha) \cdot (e^\lambda \otimes 1) = \varepsilon(\alpha, \lambda) e^{\lambda+\alpha} \otimes 1 \tag{5.1}$$

$$X_m(k\delta) \cdot (e^\lambda \otimes 1) = \varepsilon(k\delta, \lambda) e^{\lambda+k\delta} \otimes s_{-(m+kN)}(k\delta). \tag{5.2}$$

LEMMA 5.1. *$V(\lambda)$ is a cyclic $\mathfrak{t}$-module with generator $e^\lambda \otimes 1$.*

Proof. $S(\mathfrak{a}_-)$ is a cyclic $\mathfrak{a}$-module with generator 1 and the result follows at once from this and (5.1). □

Using (5.2) we obtain

LEMMA 5.2. *For all $k, m \in \mathbb{Z}$*

$$X_m(k\delta)(e^\lambda \otimes 1) = \begin{cases} \varepsilon(k\delta, \lambda) e^{\lambda+k\delta} \otimes s_{-m-kN}(k\delta) & \text{if } m + kN < 0 \\ \varepsilon(k\delta, \lambda) e^{\lambda+k\delta} \otimes 1 & \text{if } m + kN = 0 \\ 0 & \text{if } m + kN > 0. \end{cases}$$ □

Set $\tau := X_{-N}(\delta)$ (where $N = (\lambda \mid \delta)$).

PROPOSITION 5.3. *Let $k, m \in \mathbb{Z}$.*

(i) *The operators $X_m(k\delta)$ centralize the action of $\mathfrak{t}$ on $V(\lambda)$.*
(ii) *$X_{-kN}(k\delta)$ acts as multiplication by $\varepsilon(\delta, \lambda)^k e^{k\lambda}$ on $V(\lambda)$. In particular τ acts as multiplication by $\varepsilon(\delta, \lambda) e^\delta$ and $X_{-kN}(k\delta)$ acts as multiplication by τ^k on $V(\lambda)$.*
(iii) *$X_m(k\delta)$ annihilates $V(\lambda)$ iff $m + kN > 0$.*

Proof. The operators $X_m(k\delta)$ commute with $\mathfrak{t}$ because of CR1. By Lemmas 5.1 and 5.2, $X_{-kN}(k\delta)$ acts on $V(\lambda)$ by multiplication by $\varepsilon(\delta, \lambda)^k e^{k\delta}$ and the rest of (ii) follows. Similarly (iii) follows from Lemmas 5.1 and 5.2. □

PROPOSITION 5.4. *Let* $m \in \mathbb{Z}$.

(i) *The operators* $\delta(m)$ *centralize the action of* $\mathfrak{t}$ *on* $V(\lambda)$.
(ii) $\delta(0)$ *acts as scalar multiplication by* N *on* $V(\lambda)$.
(iii) $\delta(m)$ *annihilates* $V(\lambda)$ *iff* $N \neq 0$ *and* $m > 0$ *or* $N = 0$ *and* $m \geqslant 0$.

Proof. (i) is obvious from TA2 and (ii) and (iii) follow from Lemma 5.1 and the definitions of the action of $\delta(m)$ on $e^\lambda \otimes 1$. □

Figure 1 clarifies the meaning of Propositions 5.3 and 5.4. We think of the lattice point (m, k) as representing $X_m(k\delta)$ if $k \neq 0$, $\delta(m)$ if $k = 0$.

In the example $N = (\lambda \mid \delta)$ is taken to be 2. The line shown is given by $m + kN = 0$. There are two linearly independent central elements in $\mathfrak{t}_0^0$, $\delta(0)$ and $X_0(0)$, corresponding to $(0, 0)$. On $V(\lambda)$ they act (dependently) as multiplication by N and 1 respectively.

PROPOSITION 5.5. (i) *The kernel of the representation of* $\mathfrak{t}$ *on* $V(\lambda)$ *is precisely the span of the elements* $X_m(k\delta)$, $m + kN > 0$, $\delta(m)$, $m > 0$, *and* $\delta(0)$ *if* $N = 0$.

(ii) *The kernel of the representation of* $\mathfrak{t}$ *on* $V(\Gamma)$ *is the linear span of the elements* $\delta(m)$, $m > 0$.

Proof. The elements of $\mathfrak{t}_m^\alpha$ operate with degree $(-m, \alpha)$ on $V(\lambda)$. Thus the kernel $\mathfrak{c}_\lambda$ of $\mathfrak{t}$ on $V(\lambda)$ is generated by its homogeneous elements.

Now for any $\alpha \in {}^{\mathrm{re}}\Delta$ the operators $X_n(\pm\alpha)$ generate an affine Lie algebra that is faithfully represented on the subspace $e^{\lambda + \mathbb{Z}\alpha} \otimes S(\Sigma_{k<0} \mathbb{C}\alpha(k))$. In fact

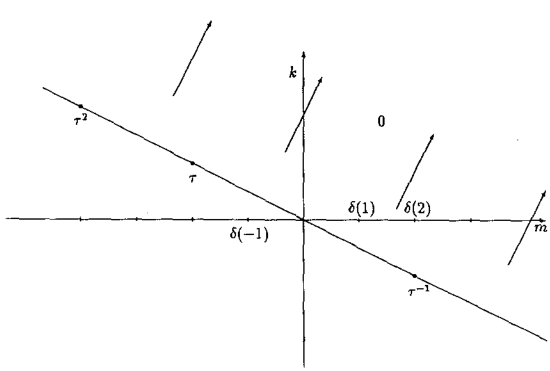


Fig. 1.

this is the Frenkel–Kac construction of the level 1 representations of affine $A_1^{(1)}$. Thus, in particular the operators $X_n(\pm\alpha)$ are faithfully represented. Hence $\mathfrak{c}_\lambda \subset \Sigma t_n^{k\delta}$ and, by the now standard argument, $\mathfrak{c}_\lambda$ is central. The result follows from Proposition 5.3 and 5.4. □

We define $C(\lambda)$ to be the algebra of endomorphisms of $V(\lambda)$ generated by the operators $\delta(m)$, $m \in \mathbb{Z}_{<0}$, and τ^k, $k \in \mathbb{Z}$. The subalgebra D of $C(\lambda)$ generated by the $\delta(m)$'s is in fact a symmetric algebra

$$D \simeq S\left(\sum_{m<0} \mathbb{C}\delta(m)\right)$$

since the action of $S(\mathfrak{a}_-)$ is faithful on $V(\lambda)$. The τ^k, $k \in \mathbb{Z}$, generate an algebra $\mathbb{C}[\tau, \tau^{-1}]$ isomorphic to the group algebra of the infinite cyclic group $\langle\tau\rangle$.

LEMMA 5.6. *The map*

$$\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}} D \to C(\lambda)$$

$$f \otimes g \mapsto fg$$

is an isomorphism of associative algebras. In particular, $C(\lambda)$ is an integral domain.

Proof. The isomorphism is obvious by looking at the actions of the τ^k and $\delta(-m)$ on $V(\lambda)$. □

Let $\dot{\mathfrak{a}}_- := \Sigma_{k<0} \Sigma_{\alpha \in \dot{Q}} \mathbb{C}\alpha(k) \subset \mathfrak{a}_-$ and let $\dot{S} := S(\dot{\mathfrak{a}}_-)$ be the symmetric algebra on $\dot{\mathfrak{a}}_-$. Then

$$S(\mathfrak{a}_-)D\dot{S} \simeq D \otimes_{\mathbb{C}} \dot{S} \tag{5.3}$$

$$\begin{aligned} V(\lambda) &= e^{\lambda+Q} \otimes_{\mathbb{C}} S(\mathfrak{a}_-) \\ &= e^{\lambda+\dot{Q}+\mathbb{Z}\delta} \otimes D\dot{S} \\ &= \mathbb{C}[\tau, \tau^{-1}]D(e^{\lambda+\dot{Q}} \otimes \dot{S}) \\ &= C(\lambda)(e^{\lambda+\dot{Q}} \otimes \dot{S}). \end{aligned} \tag{5.4}$$

PROPOSITION 5.7. *$V(\lambda)$ is a free $C(\lambda)$-module with basis $\{e^{\lambda+\dot{\alpha}} \otimes s_i\}$ where $\dot{\alpha}$ runs over $\dot{Q}$ and $\{s_i\}$ is any basis of $\dot{S}$.*

Proof.

$$\sum_{i,j,\dot{\alpha}} \tau^j d_{ij}^{(\dot{\alpha})}(e^{\lambda+\dot{\alpha}} \otimes s_i) = \sum_{i,j,\dot{\alpha}} \varepsilon(\delta, \lambda)^j e^{\lambda+\dot{\alpha}+j\delta} \otimes d_{ij}^{(\dot{\alpha})} s_i.$$

This is 0 if and only if $\Sigma_i d_{ij}^{(\dot{\alpha})} s_i = 0$ for all $\dot{\alpha}$ and j and then $d_{ij}^{(\dot{\alpha})} = 0$ from the isomorphism of (5.3). □

Consider the space $L_\lambda := e^{\lambda+\dot{Q}} \otimes_{\mathbb{C}} \dot{S}$. This is a module for the affine Lie algebra $\widehat{\mathfrak{g}(\dot{A})}$ generated by the operators $X_n(\dot{\alpha})$, $\dot{\alpha}\in\dot{Q}$. Since $\mathfrak{c}$ acts as 1 it is a level 1 representation. There is a unique $\dot{\mu}\in\dot{P}$ (= the weight lattice of $\dot{Q}$) such that $(\dot{\mu}\,|\,\alpha_i) = (\lambda\,|\,\alpha_i)$, $i = 1, \ldots, l$. From the point of view of $\widehat{\mathfrak{g}(\dot{A})}$ the representation on L_λ is indistinguishable from the standard vertex operator representation of $\widehat{\mathfrak{g}(\dot{A})}$ on the Fock space $e^{\dot{\mu}+\dot{Q}} \otimes \dot{S}$. But we know by the Frenkel–Kac theory [2] that this is an irreducible level 1 representation of $\widehat{\mathfrak{g}(\dot{A})}$.

It can be identified in the following way. Recall that the fundamental weights $\dot{\omega}_1, \ldots, \dot{\omega}_l$ of $\dot{P}$ are defined by $(\dot{\omega}_i\,|\,h_j) = \delta_{ij}$, $j = 1, \ldots, l$. There are exactly $|\dot{P}/\dot{Q}| - 1$ of these $\dot{\omega}_i$ for which $(\dot{\omega}_i\,|\,\Sigma_{j=1}^{l} n_j^\vee h_j) = 1$, and together with $\dot{\omega}_0 := 0$ they form a complete set of representatives of $\dot{P}/\dot{Q}$. For each of these (including $\dot{\omega}_0$) let ω_i be the unique linear functional on $\Sigma_{i=1}^{l} \mathbb{C}h_i + \mathbb{C}\mathfrak{c}$ such that $(\omega_i\,|\,h_j) = (\dot{\omega}_i\,|\,h_j)$, $j = 1, \ldots, l$, and $(\omega_i\,|\,\mathfrak{c}) = 1$. The coset $\dot{\mu} + \dot{Q} = \dot{\omega}_i + \dot{Q}$ for exactly one of the special $\dot{\omega}_i$. Then $e^{\dot{\mu}+\dot{Q}} \otimes \dot{S}$ is isomorphic to the irreducible highest weight module $L(\omega_i)$. Indeed $L(\omega_i)$ is the only irreducible module of level 1 which has weights whose restriction to $\dot{\mathfrak{h}}$ is in the class $\dot{\mu} + \dot{Q}$.

PROPOSITION 5.8. (i) *For every subspace E of $C(\lambda)$, $E\cdot L_\lambda$ is a $\widehat{\mathfrak{g}(\dot{A})}$ submodule of $V(\lambda)$. Every $\mathfrak{g}(\dot{A})$-submodule of $V(\lambda)$ arises in this way.*

(ii) *The $\widehat{\mathfrak{g}(\dot{A})}$-module $E\cdot L_\lambda$ is a* t*-submodule if and only if E if an ideal of $C(\lambda)$.*

Proof. (i) Let M be a $\widehat{\mathfrak{g}(\dot{A})}$-submodule of $V(\lambda)$. Let $E := \{e\in C(\lambda)\,|\,ex\in M$ for some $x\in L_\lambda\backslash\{0\}\}$. Since $U(\widehat{\mathfrak{g}(\dot{A})})\cdot ex = e\cdot U(\widehat{\mathfrak{g}(\dot{A})})\cdot x = e\cdot L_\lambda$ we see that E is a subspace of $C(\lambda)$ and $E\cdot L_\lambda \subset M$.

Let $\{e_i\}_{i\in I}$ be a basis of E over $\mathbb{C}$ and $\{e_i\}_{i\in J}$ an extension of it to a basis for $C(\lambda)$. Suppose, if possible, that $y = \Sigma_{i\in K} e_i x_i \in M\backslash(E\cdot L_\lambda)$ where the $x_i\in L_\lambda$ and assume that K is minimal for such y. Fix $k\in K\backslash I$. Evidently there is a $j\in K$, $j\neq k$. Suppose that x_j and x_k are linearly independent. Since L_λ is an irreducible highest weight module for $\widehat{\mathfrak{g}(\dot{A})}$, the centralizer of $\widehat{\mathfrak{g}(\dot{A})}$ in $\mathrm{End}_{\mathbb{C}}(L_\lambda)$ is $\mathbb{C}$. By the Jacobson density theorem [J] there is a $u\in U(\widehat{\mathfrak{g}(\dot{A})})$ such that $u\cdot x_k = x_k$, $u\cdot x_j = 0$. Then $u\cdot y = \Sigma\, e_i\cdot u\cdot x_i \in M$ contradicts the choice of y. Thus we must have $x_i = c_i x_k$, some $c_i\in\mathbb{C}$, for all $i\in K$. Then $y = (\Sigma\, c_i e_i)\cdot x_k$ from which $\Sigma_{i\in K} c_i e_i \in E$ and $K\subset I$; contradiction. This proves that $M = E\cdot L_\lambda$.

(ii) Let the $\widehat{\mathfrak{g}(\dot{A})}$-submodule $E\cdot L_\lambda$ be a t-module. Then $C(\lambda)E\cdot L_\lambda \subset E\cdot L_\lambda$. In view of Proposition 5.7, $C(\lambda)E \subset E$ so E is an ideal. Conversely, if E is an ideal then $E\cdot L_\lambda$ is closed under the action of all the operators $X_n(\dot{\alpha})$, $\dot{\alpha}\in\dot{Q}$ and all the operators $\delta(k)$, $k\in\mathbb{Z}$. Now since $(\dot{\alpha}\,|\,\delta) = 0$

$$X(\dot{\alpha}, z)X(k\delta, z) = X(\dot{\alpha} + k\delta, z)$$

from which we have the very interesting identity

$$(5.5) \qquad \sum_{n+m=r} X_n(\dot{\alpha})X_m(k\delta) = X_r(\dot{\alpha} + k\delta).$$

Although the left-hand side is an infinite sum, for any fixed element of $V(\lambda)$ only finitely many terms can act non-trivially since the operators $X_m(\beta)$ kill any element for $m \gg 0$. Now

$$X(k\delta, z) = \exp\, T_-(k\delta, z)e^{k\delta}z^{k\delta(0)}\exp T_+(k\delta, z)$$

and evidently all the operators involved stabilize $E \cdot L_\lambda$. Thus we conclude from (5.5) that $X_n(\dot{\alpha} + k\delta)$ stabilizes $E \cdot L_\lambda$ for all n, k. □

PROPOSITION 5.9. (i) $V(\Gamma) = \oplus\ V(\lambda)$ *where the sum runs over a complete set of representatives of* Γ/Q.

(ii) *For all* λ, $V(\lambda)$ *is an indecomposable* $\mathfrak{t}$-*module.*

Proof. (i) is obvious.

(ii) Suppose that $V(\lambda) = M_1 \oplus M_2$ for some $\lambda \in \Gamma$ where M_1 and M_2 are submodules. Then $M_i = E_i \cdot L_\lambda$ for some ideal E_i of $C(\lambda)$. But then $E_1E_2 \cdot L_\lambda \subset M_1 \cap M_2 = (0)$ so, by Proposition 5.7, $E_1E_2 = 0$. By Lemma 5.6, $E_1 = 0$ or $E_2 = 0$. □

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Added in proof. (1) Recently, we have become aware of Yamada's paper [16] in which a vertex representation of a central quotient of $\mathfrak{t}$ is realized.

(2) In a forthcoming paper, we [S. E. R. and R. V. M] show that the results of this paper have a natural generalization to the universal central extension $\mathfrak{t}_{[n]}$ of $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

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