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AFFINE POLAR SPACES

In honor of J. Tits on the occasion of his sixtieth birthday

ABSTRACT. Affine polar spaces are polar spaces from which a hyperplane (that is a proper subspace meeting every line of the space) has been removed. These spaces are of interest as they constitute quite natural examples of 'locally polar spaces'. A characterization of affine polar spaces (of rank at least 3) is given as locally polar spaces whose planes are affine. Moreover, the affine polar spaces are fully classified in the sense that all hyperplanes of the fully classified polar spaces (of rank at least 3) are determined.

0. INTRODUCTION

In 1959, Veldkamp [9] initiated the synthetic study of geometries induced on the set of absolute points, lines, planes, etc. with respect to a polarity, and named the subject polar geometry. After subsequent work of Tits [7], Buekenhout and Shult [2] and Buekenhout and Sprague [3] a somewhat larger class of point, line geometries emerged which could be characterized by the beautiful axiom

> *If p is a point and L a line, then the set of points incident with L and collinear with p is either a singleton or the set of all points incident with L,*

which we shall quote as the 'one or all' axiom. An incidence system $(P, \mathscr{L})$ [i.e. a pair consisting of a set P (of *points*) and set $\mathscr{L}$ (of *lines*) together with a relation between them, called *incidence*, such that each line is incident with at least two points] is called a *polar space* if the 'one or all' axiom is satisfied. An incidence system is called *nondegenerate* if no point is collinear with all others, and it is called *singular* if any two of its points are collinear. If X is a subset of the point set P of the incidence system $(P, \mathscr{L})$ and $L \in \mathscr{L}$, we denote by $X(L)$ the set of points in X incident to L, and by $\mathscr{L}(X)$ the set of all lines in $\mathscr{L}$ incident to at least two points of X. Thus, $\mathscr{L}(X) = \{L \in \mathscr{L} \mid |X(L)| > 1\}$. Restricting incidence of $(P, \mathscr{L})$, we can regard $(X, \mathscr{L}(X))$ as an incidence system. If each point incident to a line in $\mathscr{L}(X)$ belongs to X, we say that X is a *subspace* of $(P, \mathscr{L})$. A subspace of a polar space is again a polar space. The *singular rank* of an incidence system $(P, \mathscr{L})$ is the maximal number n (possibly ∞) for which there exists a chain of distinct subspaces

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$\emptyset \neq X_0 \subset X_1 \subset \cdots \subset X_n$ such that $(X_i, \mathscr{L}(X_i))$ is singular for each i $(0 \leqslant i \leqslant n)$, with the understanding that $n = -1$ if $X = \emptyset$. The *rank* of a polar space $(P, \mathscr{L})$ is the number $n + 1$ where n is its singular rank. By definition, this number is 0 if $P = \emptyset$, and 1 if $P \neq \emptyset$ but $\mathscr{L} = \emptyset$. The main characterization results hinted to above imply that if $(P, \mathscr{L})$ is a nondegenerate polar space of finite (singular) rank $\geqslant 3$, then it is one of a known list of examples (cf. Buekenhout and Sprague [3]). In this paper we shall limit ourselves to the situation in which all lines are *thick* (i.e. are incident to at least three points); the list of 'thick' examples can be found in Tits [7]. (See also Section 5 below.)

One of the main tools in Veldkamp's original approach is the notion of a *hyperplane*, a proper subspace with the property that every line is incident to (at least) one of its points; it did not recur in the subsequent papers quoted above. It is the goal of this paper to study the hyperplanes B of nondegenerate polar spaces $(P, \mathscr{L})$ of finite rank $\geqslant 3$ whose lines are thick, as well as to synthetically describe the incidence systems $(A, \mathscr{L}(A))$ induced on their complements $A = P \backslash B$.

Section 2 gives some properties of these hyperplane complements. The interest in these 'affine polar spaces' $(A\ \mathscr{L}(A))$ arose from the abundance of properties analogous to those of the usual affine spaces, i.e. the geometries induced on the complements of hyperplanes in projective spaces. Notably, the fact that (classical) affine spaces Q are locally projective spaces (in the sense that, for each point $a \in Q$, the incidence system whose points are the lines of Q on a and whose lines are the affine planes on a is a projective space), corresponds to the property of $(A, \mathscr{L}(A))$ being a locally polar space. This draws attention to the question whether all spaces that are locally polar can be classified. The analogous question for projective spaces has given rise to various characterizations (see, e.g., Teirlinck [6]). Adopting a stronger notion of locally polar spaces $(P, \mathscr{L})$, namely that $x^{\perp}$ rather than $\mathscr{L}_x := \{L \in \mathscr{L} \mid x \in L\}$ carry the structure of a polar space for each $x \in P$, Johnson and Shult [5] have obtained a satisfactory characterization without any assumptions on rank, thickness of lines, or degeneracy. (For a review of other results in this direction, see [loc. cit.].) In Section 3, we characterize affine polar spaces by an axiom system in which the locally polar space axiom is prominent (cf. 3.1.iii). Some of the proofs involved are based on ideas of J. I. Hall as displayed in the characterization of 'locally cotriangular graphs' of Hall and Shult [4]. In the remainder of Section 3, properties are derived from this axiom system, which alleviate the proof, to be found in Section 4, that the system of Section 3 is indeed a characterizing axiom system.

In view of the classification of nondegenerate polar spaces of rank at least 3 (due to Veldkamp [9] and Tits [7]) the classification of affine polar spaces comes down to the determination of all hyperplanes in well-known polar spaces. This determination is carried out in the last section (Section 5).

1. Hyperplanes

In this section, $(P, \mathscr{L})$ is a polar space all of whose lines are thick. If $X \subseteq P$, we write $X^{\perp}$ for the subset P of points collinear to each point of X, and $x^{\perp} = \{x\}^{\perp}$ if $x \in P$. Furthermore, $\langle X \rangle$ denotes the subspace of $(P, \mathscr{L})$ generated by X. (It exists since the intersection of an arbitrary collection of subspaces is again a subspace.) If X is a subspace, then so is $X \cap X^{\perp}$. The latter is called the *radical* of X, denoted by $\operatorname{rad} X$. A subspace X is called *nondegenerate* if $(X, \mathscr{L}(X))$ is nondegenerate, i.e. $\operatorname{rad} X := X \cap X^{\perp} = \varnothing$, and *degenerate* otherwise. The *quotient space* of X with respect to $\operatorname{rad} X$ is the incidence system whose points (resp. lines) are the subspaces $\langle \operatorname{rad} P \cup \{x\} \rangle$ for $x \in P \backslash \operatorname{rad} P$ (resp. $\langle \operatorname{rad} P \cup \{z \mid z \in l\} \rangle$ for $l \in \mathscr{L}$ such that no point of $\operatorname{rad} P$ is incident with l) and in which incidence is symmetrized containment. The quotient space of P with respect to $\operatorname{rad} P$ is a nondegenerate polar space. The rank of this quotient space will be referred to as the *nonsingular rank of* $(P, \mathscr{L})$. We recall that a *hyperplane* B of $(P, \mathscr{L})$ is a proper subspace such that $B(L) \neq \varnothing$ for each line $L \in \mathscr{L}$.

1.1. LEMMA. *Let B be a hyperplane of $(P, \mathscr{L})$.*

(i) *If $(P, \mathscr{L})$ has nonsingular rank at least 2, then B is a maximal proper subspace and the collinearity graph induced on $P \backslash B$ is connected of diameter at most 3.*

(ii) *If X is a subspace not contained in B, then $X \cap B$ is a hyperplane of $(X, \mathscr{L}(X))$.*

Proof. (i) Take $x, y \in P \backslash B$. We show that $y \in \langle B, x \rangle$ (the subspace generated by B and x). If x and y are collinear, say both on the line $L \in \mathscr{L}$, then $y \in \langle x, B(L) \rangle$ and we are done.

Assume that x and y are noncollinear. If $t \in \{x, y\}^{\perp} \backslash B$, then applying the above argument to x and t, and once more to t and y (instead of x and y), we are done, again.

Thus we remain with the case where $\{x, y\}^{\perp} \subseteq B$. Since the nonsingular rank of $(P, \mathscr{L})$ is at least 2, there are noncollinear v, $w \in \{x, y\}^{\perp}$ (cf. Buekenhout and Shult [2]). Let M be a line on x and v and N a line on w and y. Lines are thick, so there is a point u on M distinct from x and v. By the 'one or all' axiom, there must be a point z on N collinear with u. This point is distinct from y and w. Now x, u, z, y is a path in $P\backslash B$ and we can finish by applying the first paragraph three times. The conclusion is that $y \in \langle B, x\rangle$ for each $y \in P\backslash B$, whence $\langle B, x\rangle = P$.

(ii) Is obvious from the definition. □

1.2. REMARK. In (i), the bound on the nonsingular rank is necessary, the subspace $B = L_1$ being a counterexample in the polar space $(Q, \{L_1, L_2, L_3\})$ where L_1, L_2, L_3 are lines meeting in a fixed point of $Q = L_1 \cup L_2 \cup L_3$.

The lemma implies that, if P is nondegenerate, B can have at most one *deep point*, i.e. a point incident with no line of $\mathscr{L}(P\backslash B)$, as we shall see from the corollary below. Clearly, for each point $x \in P$, the subspace $x^\perp$ is a hyperplane with deep point x.

1.3. COROLLARY. *Assume B is a hyperplane of a polar space $(P, \mathscr{L})$ of nonsingular rank at least* 2.

(i) *If B is degenerate, then $B = b^\perp$ for some $b \in \operatorname{rad} B\backslash\operatorname{rad} P$; moreover,* $\operatorname{rad} B = \langle \operatorname{rad} P, b\rangle$ *is a point of the quotient space of $(P, \mathscr{L})$ with respect to* $\operatorname{rad} P$.

(ii) *Any deep point of B is in* $\operatorname{rad} B\backslash\operatorname{rad} P$; *in particular, if $(P, \mathscr{L})$ is nondegenerate, there is at most one deep point in B.*

(iii) *The nonsingular rank of B is at least one less than the nonsingular rank of P.*

Proof. (i) Suppose $b \in \operatorname{rad} B$. Then $B \subseteq b^\perp$. But $b^\perp$ is a hyperplane, so by (i) of the above lemma, $B = b^\perp$. Now $b \notin \operatorname{rad} P$ as B is a proper subspace of $(P, \mathscr{L})$. Moreover, $B = b^\perp$ contains $\operatorname{rad} P$ and the image of B in the quotient space of $(P, \mathscr{L})$ with respect to $\operatorname{rad} P$ is again a hyperplane of the quotient space. Thus, for the proof of the second assertion of (i), we may assume $(P, \mathscr{L})$ is nondegenerate. But then $b^{\perp\perp} = \{b\}$, whence $\operatorname{rad} B = B \cap b^{\perp\perp} = \{b\}$.

(ii) Suppose $d \in B$ is incident with no line of $\mathscr{L}(P\backslash B)$. Then by thickness every line on d must have another point in B, and so all of its points lie in B. Thus $d^\perp \subseteq B$. By maximality of $d^\perp$ (cf. Lemma 1.1(i)) and $B \neq P$, we obtain $B = d^\perp$, and we can finish as before.

(iii) Is obvious. □

We now recall the construction of a linear space – i.e. a space in which every pair of points lie on a unique line – on the set $\mathscr{H}$ of all hyperplanes of $(P, \mathscr{L})$ in which $(P, \mathscr{L})$ can be embedded. The basic idea is caught in the following lemma.

1.4. LEMMA. *Suppose* $(P, \mathscr{L})$ *is a polar space of nonsingular rank* $\geqslant 3$, *and* B_1, B_2 *are distinct hyperplanes.*

(i) *If* $x \in B_1 \backslash B_2$, *then* $B_1 = \langle x, B_1 \cap B_2 \rangle$.
(ii) *If* $p \in P \backslash (B_1 \cap B_2)$, *then there is at most one hyperplane containing* p *and* $B_1 \cap B_2$.

Proof. Observe that B_1 and B_2 are polar spaces.

(i) In view of Corollary 1.3(iii) and Lemma 1.1(i), $B_1 \cap B_2$ is a maximal subspace of B_1. Hence (i).

(ii) Suppose $p \in P \backslash (B_1 \cap B_2)$ and B is a hyperplane containing $B_1 \cap B_2$ and p. By (i), we may assume $p \notin B_i$ for $i = 1, 2$. Then $B_1 \supset B_1 \cap B \supseteq B_1 \cap B_2$, and so, by (i), we must have $B_1 \cap B = B_1 \cap B_2$. Also, (i) applied to B_1 and B gives $B = \langle B_1 \cap B, p \rangle$. We conclude that $B = \langle B_1 \cap B_2, p \rangle$, whence B is the unique hyperplane containing $B_1 \cap B_2$ and p. □

This lemma implies that the pair $(\mathscr{H}, \mathscr{S})$ where $\mathscr{S}$ is the collection of all intersections $B_1 \cap B_2$ with B_1, $B_2 \in \mathscr{H}$, $B_1 \neq B_2$, becomes a linear incidence system if incidence of $B \in \mathscr{H}$ and $S \in \mathscr{S}$ is defined by $S \subseteq B$. This incidence system will be called the *Veldkamp space* of $(P, \mathscr{L})$.

1.5. LEMMA. *Suppose* $(P, \mathscr{L})$ *is a nonsingular polar space of rank at least* 3. *Then the map* $x \mapsto x^{\perp}$ *from* P *to* $\mathscr{H}$ *is an injective morphism from* $(P, \mathscr{L})$ *to* $(\mathscr{H}, \mathscr{S})$ *mapping lines onto lines.*

Proof. In view of Corollary 1.3 the map is injective. Now, let $L \in \mathscr{L}$, take two points x, y incident with L, and let B be a hyperplane containing $x^{\perp} \cap y^{\perp}$. We have to show $B = z^{\perp}$ for some point z of the line L. By Lemma 1.1 there exists $b \in B \backslash \{x, y\}^{\perp}$. Then, by the 'one or all' axiom, there is a unique point $z \in b^{\perp}$ on L. According to the previous lemma there is at most one hyperplane containing b and $\{x, y\}^{\perp}$. But $z^{\perp}$ and B are such hyperplanes. Therefore $B = z^{\perp}$ as required. □

2. Hyperplane complements

Throughout this section, $(P, \mathscr{L})$ is a non-degenerate polar space all of whose lines are thick. Since there is at most one line incident with any two points (cf. Buekenhout and Shult [2]), a line is uniquely determined by the set of all points incident to it. We shall thus frequently view members of $\mathscr{L}$ as subsets of P. Also, if x, y are collinear and distinct, we shall write xy to denote the line containing them.

Let B be a hyperplane of $(P, \mathscr{L})$ and set $A = P \backslash B$. We may define a derived incidence system $(A, \mathscr{L}(A))$ where incidence is that of $(P, \mathscr{L})$. We wish to examine some of the properties of $(A, \mathscr{L}(A))$ – enough to show that $(A, \mathscr{L}(A))$ carries with it sufficient information to recover $(P, \mathscr{L})$. More precisely we shall show that if $(A, \mathscr{L}(A))$ is embedded in a second polar space so that its complement there is also a hyperplane, then the embedding extends to an isomorphism of $(P, \mathscr{L})$ onto the second polar space.

First observe that, for each line $L \in \mathscr{L}(A)$, the set $P(L)\backslash A(L)$ is a singleton. If $(P, \mathscr{L})$ has rank at least 3, each line of $(A, \mathscr{L}(A))$ lies on at least two affine planes. Any three pairwise collinear points of $(A, \mathscr{L}(A))$ lie on an affine plane. This implies $(A, \mathscr{L}(A))$ is a gamma space – i.e. a space in which, for every point p and line L, none, one or all points of L are collinear with p.

For each $L \in \mathscr{L}(A)$, set $\Delta(L) = \{a \in A \mid a^{\perp} \cap A(L) = \varnothing \text{ or } A(L)\}$, where $A(L)$ denotes the set of all points in A incident to L. We define an equivalence relation on $\mathscr{L}(A)$ as follows: L_1 is *parallel to* L_2 if and only if $\Delta(L_1) = \Delta(L_2)$. We denote this relation by $L_1 \parallel L_2$. The symbol $[L]$ will denote the equivalence class of all lines in $\mathscr{L}(A)$ parallel to L.

2.1. LEMMA. *Two lines of $\mathscr{L}(A)$ are parallel if and only if their intersections with B coincide. Moreover, if $L \in \mathscr{L}(A)$, each point of $\Delta(L)$ lies on a unique member of $[L]$. Thus $\Delta(L)$ is the disjoint union of the lines of $[L]$, regarded as point sets of A.*

Proof. Suppose $x \in \Delta(L)$. Then x is collinear with the unique point b of B incident with L. Set $M = xb$. Then $M \in \mathscr{L}(A)$, as $x \in A$. Now, $\Delta(L) = \{y \in A \mid y \perp b\} = \Delta(M)$. Thus $M \in [L]$. This proves the second assertion.

If $N \in \mathscr{L}(A)$ is a line on x distinct from M, it meets B in a point $c \neq b$ so the choice of a point $y \in b^{\perp} \cap A\backslash(x^{\perp} \cup c^{\perp})$ (possible since lines are thick and $(P, \mathscr{L})$ is nondegenerate!) leads to $y \in \Delta(M)\backslash\Delta(N)$. Hence M is the unique member of $[L]$ on x. This establishes one implication in the first assertion. The other implication is a direct consequence of the 'one or all' property of polar spaces. □

Let $\mathscr{L}(A)/\parallel$ denote the collection of parallel classes $[L]$ on $\mathscr{L}(A)$. A direct consequence of the previous lemma is

2.2. COROLLARY. *There is a 1–1 correspondence $f: B\backslash \mathrm{rad}\, B \to \mathscr{L}(A)/\parallel$, which takes each point $b \in B\backslash \mathrm{rad}\, B$ to the parallel class $f(b) := \{L \in \mathscr{L}(A) \mid b \in B(L)\} = [M]$, where $M \in \mathscr{L}(A)$ with $b \in B(M)$.* □

2.3. LEMMA. *Assume $(P, \mathscr{L})$ has rank 3 or more. If $L \in \mathscr{L}(B)$ does not contain a point of* rad B, *then there exists an affine plane π in A such that the restriction $f|_L: L \to \{[M] \mid M \in \mathscr{L}(\pi)\}$ of f as given in the previous corollary is a 1–1* *correspondence. Conversely, for each affine plane π in A, the elements $f^{-1}([M])$ for $M \in \mathscr{L}(\pi)$ comprise a line of B not containing a point of* rad *B.*

Proof. First assume $L \in \mathscr{L}(B)$ and $B(L) \cap \text{rad } B = \varnothing$. Since $(P, \mathscr{L})$ has rank at least 3, there exists a projective plane T on L such that $T \nsubseteq B$ (to see this, use Corollary 1.3(ii) to find $x \in L \backslash \text{rad } B$, $y \in x^{\perp} \backslash B$, and use that the rank of the quotient space of $x^{\perp}$ by its radical, denoted by P_x, has rank $\geqslant 2$ to find $z \in \{x, y\}^{\perp} \backslash B$ such that xz and L span a singular subspace of P_x). Then $(T \backslash B, \mathscr{L}(T \backslash B))$ is an affine plane π whose parallel classes are (via f) in 1–1 correspondence with the points of the line $T \cap B \in \mathscr{L}(B)$, the so-called 'line at infinity'. Thus if $L \in \mathscr{L}(\pi)$ and b is the 'point at infinity' in $B(L)$, then $f(b) = [L]$ by the previous corollary.

Conversely if π is an affine plane in A then we see that the union T of $P(L)$ over all $L \in \mathscr{L}(\pi)$ is a projective plane (since it is clearly generated by any two of the lines L). It follows that $\{f^{-1}[L] \mid L \in \mathscr{L}(\pi)\}$ coincides with the line $B(T)$. This line clearly contains no point of rad B since no point of $B(T)$ is in rad B. □

2.4. LEMMA. *Again assume $(P, \mathscr{L})$ has rank at least 3. Then two points b_1 and b_2 of $B \backslash$*rad *B are collinear by a line disjoint from* rad *B if and only if there exist lines $L, M \in \mathscr{L}(A)$ such that $A(L) \subseteq \Delta(M)$, $L \in f(b_1)$, and $M \in f(b_2)$.*

Proof. Suppose first that b_1 and b_2 are collinear by a line R in $\mathscr{L}(B)$ disjoint from rad B. Then by the previous lemma, there exists an affine plane π containing two lines L and M which are not parallel and for which $f(b_1) = [L]$ and $f(b_2) = [M]$. It follows that $A(L) \subseteq \Delta(M)$.

Conversely, assume lines L and M exist with $L \in f(b_1)$, $M \in f(b_2)$, and $A(L) \subseteq \Delta(M)$. Then for each point p of A lying in L, we have $p^{\perp} \cap A(M) = \varnothing$ or $A(M)$. In either case p is collinear with the point b_2 comprising $B(M)$. Thus $A(L) \subseteq b_2^{\perp}$. Since L is thick, $|A(L)| \geqslant 2$, so $b_2^{\perp}$ contains the point b_1 of $B(L)$. Thus b_2 is collinear with b_1. Finally if the line R on b_1 and b_2 contains a deep point of B, no point of A could be collinear with both b_1 and b_2. But we have just seen that the points of $A(L)$ are collinear with both b_1 and b_2. Thus R contains no point of rad B. □

For L, $M \in \mathscr{L}(A)$, set $[L] \sim [M]$ if there are L', $M' \in \mathscr{L}(A)$ with $L \parallel L'$, $M \parallel M'$, and $\langle L', M' \rangle$ a projective plane in $(P, \mathscr{L})$. By the above lemma, we have $f(b_1) \sim f(b_2)$ if and only if there are lines $L \in f(b_1)$ and $M \in f(b_2)$ such that $A(L) \subseteq \Delta(M)$.

Let $\equiv$ be the relation on the set of parallel classes of $\mathscr{L}(A)$ defined by $[L] \equiv [M]$ if and only if $\Delta(L) \cap \Delta(M) = \varnothing$.

2.5. LEMMA. *Suppose, for two lines L, M of $\mathscr{L}(A)$, we have $[L] \equiv [M]$.*

Then there is a unique partition $A = \bigcup_{N \in X} \Delta(N)$ *with* $L, M \in X \subseteq \mathscr{L}$. *The points* $f^{-1}([L])$ *and* $f^{-1}([M])$ *of* B *are collinear by a line* R *and the set* $\{f^{-1}([N]) \mid N \in X\}$ *coincides with* $B(R) \backslash \mathrm{rad}\, B$. *Moreover, if* $(P, \mathscr{L})$ *has rank* 3 *or more, then* $\mathrm{rad}\, B \subseteq B(R)$.

Proof. Suppose $\Delta(L) \cap \Delta(M) = \varnothing$ for $L, M \in \mathscr{L}(A)$ and set $B(L) = \{b\}$, $B(M) = \{c\}$. Then each point of $A(L)$ is collinear with exactly one point of $A(M)$ and vice versa, forcing a 1–1 correspondence $A(L) \to A(M)$ defined by collinearity. Since $(P, \mathscr{L})$ is a polar space, $b \neq c$ and b and c are a collinear pair of points. Let R be the line in $\mathscr{L}$ on b and c. Clearly $R \in \mathscr{L}(B)$. Suppose a point x in A were collinear with two points of R. Then $x \in b^{\perp} \cap c^{\perp}$ and as $f(b) = [L]$ and $f(c) = [M]$ this means $x \in \Delta(L) \cap \Delta(M)$, a contradiction.

On the other hand, the polar space property forces $x^{\perp} \cap B(R) \neq \varnothing$ for $x \in A$ and so we see that each point of A is collinear with exactly one point of $B(R) \backslash \mathrm{rad}\, B$. Since collinearity of $x \in A$ with $r \in B(R) \backslash \mathrm{rad}\, B$ means $x \in \Delta(f(r))$, we have a partition

$$(2.1) \qquad A = \dot{\bigcup_{r \in B(R) \backslash \mathrm{rad}(B)}} \Delta(f(r)).$$

If $(P, \mathscr{L})$ has rank at least 3, then by the proof of Lemma 1.5, R must contain a deep point of B. It remains to show that this partition of A is the unique such one containing $\Delta(L)$ and $\Delta(M)$ as components. Suppose instead there were a second partition

$$A = \Delta(L) \cup \Delta(M) \cup \dot{\bigcup_{N \in Y}} \Delta(N).$$

Since this partition is assumed to differ from that in (2.1), there exists at least one line $N \in Y$ such that $\Delta(N)$ is not one of the components of (2.1). Set $\{y\} = f^{-1}([N]) = N \cap B$, $y \notin R$. As argued for b and c alone, $\Delta(N) \cap \Delta(L) = \varnothing$ implies b is collinear with y via some line R' distinct from R. But similarly $\Delta(N) \cap \Delta(M) = \varnothing$ implies y collinear with c whence $R' \subseteq R^{\perp}$, so $\langle R, R' \rangle$ is a plane. This means $(P, \mathscr{L})$ has rank at least 3. Then by the above, R' and R both contain a deep point. But by Corollary 1.3, there is only one deep point. Thus $R \cap R' \supseteq \{b, d\}$ with $b \neq d$. This implies $(P, \mathscr{L})$ is not linear, defying the nondegeneracy of $(P, \mathscr{L})$ by well-known arguments. □

As an immediate consequence, we have

2.6. COROLLARY. *Suppose* $(P, \mathscr{L})$ *has rank at least* 3 *or* $\mathrm{rad}\, B \neq \varnothing$. *Then the reflexive closure of* $\equiv$ *is an equivalence relation on* $\mathscr{L}(A)/\|$. *If* X *is an* $\equiv$-*class on* $\mathscr{L}(A)/\|$ *of size at least* 2, *then there is a line* $R \in \mathscr{L}(B)$ *with* $B(R) = \mathrm{rad}\, B \cup \{f^{-1}([N]) \mid [N] \in X\}$.

2.7. PROPOSITION. *For $i = 1,2$, let $(P_i, \mathscr{L}_i)$ be a thick nondegenerate polar space of rank at least 2. Let B_i be a hyperplane of $(P_i, \mathscr{L}_i)$. Set $A_i = P_i \backslash B_i$. Suppose $\phi:(A_1, \mathscr{L}_1(A_1)) \to (A_2, \mathscr{L}_2(A_2))$ is an isomorphism of incidence systems. Then ϕ can be uniquely extended to an isomorphism $\phi:(P_1, \mathscr{L}_1) \to (P_2, \mathscr{L}_2)$.*

Proof. For $L \in \mathscr{L}_i(A_i)$, let $[L]$ be the parallel class in $(A_i, \mathscr{L}_i(A_i))$ containing L– i.e. all lines L' of $\mathscr{L}_i(A_i)$ such that $\Delta(L') = \Delta(L)$. Then, for each $i = 1,2$, there are bijective mappings $f_i: B_i \backslash \mathrm{rad}\, B_i \to \mathscr{L}_i(A_i)/\|$ from the set of non-deep points of B_i to the set of parallel classes on $\mathscr{L}_i(A_i)$, $i = 1,2$.

Obviously ϕ, being an isomorphism, maps parallel classes on $\mathscr{L}_1(A_1)$ to parallel classes on $\mathscr{L}_2(A_2)$, and commutes with the 'functor' $\Delta: \mathscr{L}_i(A_i) \to \mathscr{P}(A_i)$, the power set of A_i, $i = 1,2$. Since the property of $(P_i, \mathscr{L}_i)$ having rank at least 3 can be recognized in $(A_1, \mathscr{L}_1(A_1))$ by the property that each line lies in an affine plane, it follows that

(2.2) *$(P_1, \mathscr{L}_1)$ has rank at least 3 if and only if $(P_2, \mathscr{L}_2)$ does.*

If $(P_1, \mathscr{L}_1)$ is a generalized quadrangle, then the presence of a deep point in B_1 can be recognized by the fact that the reflexive closure of the relation $\equiv$ (defined for $(P_i, \mathscr{L}_i)$ as above for $(P, \mathscr{L})$) is an equivalence relation on the set of parallel classes $\mathscr{L}_1(A_1)/\|$.

On the other hand, if $(P_1, \mathscr{L}_1)$ has rank at least 3, the presence of a deep point in B_1 is indicated by the appearance of two lines L, M in $\mathscr{L}_1(A_1)$ with $[L] \equiv [M]$. Thus

(2.3) *B_1 has a (unique) deep point if and only if B_2 does.*

We now extend $\phi: A_1 \to A_2$ to $\hat{\phi}: P_1 \to P_2$ as follows. First $\hat{\phi}$ restricted to A_1 is ϕ. If $b \in B_1 \backslash \mathrm{rad}\, B_1$ is a nondeep point of B_1, set $\hat{\phi}(b) = f_2^{-1}([\phi(L_b)])$ where L_b is any representative of the parallel class $f_1(b)$. Put another way, since Δ is 'functional', there is an induced map $\bar{\phi}: \mathscr{L}_1(A_1)/\| \to \mathscr{L}_2(A_2)/\|$. Then $\hat{\phi}(b) = f_2^{-1} \cdot \bar{\phi} \cdot f_1(b)$. Finally, if d_1 is a deep point of B_1, then d_1 is unique via Corollary 1.3 and, by (2.3), B_2 has a unique deep point d_2, and we write $\hat{\phi}(d_1) = d_2$.

From the above it is clear that, if there is an extension of ϕ as stated, it must coincide with $\hat{\phi}$. Thus uniqueness follows and it remains to show that $\hat{\phi}: P_1 \to P_2$ induces a mapping $\hat{\phi}: \mathscr{L}_1 \to \mathscr{L}_2$ via incidence. We already have that $\hat{\phi}: \mathscr{L}_1(A_1) \to \mathscr{L}_2(A_2)$ is a bijective mapping-preserving incidence since ϕ was an isomorphism $(A_1, \mathscr{L}_1(A_1)) \to (A_2, \mathscr{L}_2(A_2))$.

Suppose, first, $(P_1, \mathscr{L}_1)$ is a generalized quadrangle and B_1 has a deep point d_1. Then the reflexive closure of the relation $\equiv$ on $\mathscr{L}_1(A_1)$ is an equivalence relation, and so the reflexive closure of $\equiv$ on $\mathscr{L}_2(A_2)/\|$ is also an equivalence relation. Since each line of $\mathscr{L}_i(B_i)$ is formed by taking d_i together with

$f_i^{-1}([L])$ where $[L]$ ranges over a fixed $\equiv$-class on $\mathscr{L}_i(A_i)/\|$, we see $\hat{\phi}$ induces a bijection $\mathscr{L}_1(B) \to \mathscr{L}_2(B)$ and we are done. Next, suppose $(P_1, \mathscr{L}_1)$ and $(P_2, \mathscr{L}_2)$ are still both generalized quadrangles but the reflexive closure of $\equiv$ is not an equivalence relation on $\mathscr{L}_i(A_i)/\|$. Let $R \in \mathscr{L}_1(B)$ be a line containing no deep point of B_1. Then still it is true that whenever $b, c \in R$ we have $\Delta(L) \cap \Delta(M) = \varnothing$ L, M lines with $f_1^{-1}([L]) = b$ and $f_1^{-1}([M]) = c$. Thus a unique partition results:

$$A_1 = \dot{\bigcup_{r \in R}} \Delta(f_1(r)).$$

Then

$$A_2 = \phi(A_1) = \dot{\bigcup_{r \in R}} \phi(\Delta(f_1(r)) = \dot{\bigcup_{r \in R}} \Delta(\bar{\phi}(f_1(r)) = \dot{\bigcup_{r \in R}} \Delta(f_2(\hat{\phi}(r)) \tag{2.4}$$

is a partition on A_2 containing $\Delta(f_2(\hat{\phi}(b)))$ and $\Delta(f_2(\hat{\phi}(c)))$ as components. By Lemma 2.5, $\hat{\phi}(b)$ and $\hat{\phi}(c)$ are collinear by a line R_2 in $\mathscr{L}_2(B)$ and there is a partition

$$A_2 = \dot{\bigcup_{r' \in R_2}} \Delta(f_2(r')) \tag{2.5}$$

also containing $\Delta(f_2(\hat{\phi}(b)))$ and $\Delta(f_2(\hat{\phi}(c)))$ as components. But by Lemma 2.5 such a partition is unique subject to containing these two components and so the right side of (2.5) is the same partition as the one in the expression after the last equal sign of (2.4). This means $\hat{\phi}(R) = R_2$.

Now assume $(P_1, \mathscr{L}_1)$ has rank at least 3. Then the reflexive closure of $\equiv$ is an equivalence relation whose classes of size at least 2 represent lines in $\mathscr{L}_1(B_1)$ on a deep point d_1 of B_1. Then, just as in the first part of the proof when $(P_1, \mathscr{L}_1)$ was a generalized quadrangle, ϕ takes the point-shadows of lines of $\mathscr{L}_1(B_1)$ lying on a deep point of B_1 to the point-shadows of lines of $\mathscr{L}_2(B_2)$ lying on a deep point of $\mathscr{L}_2(B_2)$–i.e. $\hat{\phi}$ induces a 1–1 mapping of all lines on d_1 in $\mathscr{L}_1$ to all lines of $\mathscr{L}_2$ on a deep point d_2.

There remain the lines of $\mathscr{L}_1(B_1)$ contained in $B \backslash \mathrm{rad}\, B$. Since $(P_1, \mathscr{L}_1)$ has rank at least 3, such a line R has as its points the set $\{f_1^{-1}([L]) \mid L \in \mathscr{L}(\pi)\}$ for some affine plane π of $(A_1, \mathscr{L}_1(A_1))$ (cf. Lemma 2.3). Then $\hat{\phi}(f_1^{-1}([L])) = f_2^{-1}([\phi L])$, so $\hat{\phi}(R) = \{f_2^{-1}([L_2]) \mid L_2 \in \mathscr{L}(\phi\pi)\}$ where $\phi\pi$ is an affine plane of $(A_2, \mathscr{L}_2(A_2))$. By Lemma 2.3 once more, $f_2^{-1}([\phi(L)])$, $\phi(L) \in \phi(\pi)$, now ranges over all points of a line R_2 of $\mathscr{L}_2(B_2)$ not containing a deep point of B_2. This means $\hat{\phi}(R) = R_2$. As $(P_2, \mathscr{L}_2)$ has rank 3, this procedure is reversible and so $\hat{\phi}$ induces a bijective mapping of the lines of B_1 not containing a deep point to the lines of B_2 not containing a deep point. It is now clear the $\hat{\phi}$ induces a complete bijective mapping $\mathscr{L}_1 \to \mathscr{L}_2$ via point shadows and so $\hat{\phi}$ is an isomorphism $(P_1, \mathscr{L}_1) \to (P_2, \mathscr{L}_2)$ extending ϕ. □

3. Affine polar spaces

We consider here the following axioms concerning an incidence system $(P, \mathscr{L})$:

(3.1) The incidence system $(P, \mathscr{L})$ is a connected gamma space supplied with a nonempty collection Π of subspaces (called affine planes) such that

(3.1.i) any two collinear points x, y lie on a unique line (thus, $\mathscr{L}$ can be viewed as a collection of subsets of P), denoted by xy; any three pairwise collinear points x, y, z not on a line lie in a unique member of Π;

(3.1.ii) for each $\pi \in \Pi$, the incidence system $(\pi, \mathscr{L}(\pi))$ is an affine plane;

(3.1.iii) if $p \in P$ and $\pi \in \Pi$, then $p^\perp \cap \pi$ is either empty, is the set of points on a line or coincides with the set of all points in π;

(3.1.iv) $x^\perp \subseteq y^\perp$ implies $x = y$ for any two points x and y.

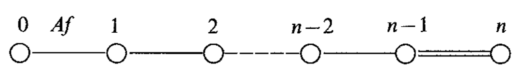
The introduction of Π is not needed if all lines have length > 2. For then we can take Π to be the set of all singular subspaces of singular rank 2. Condition (3.1.ii) means that $(P, \mathscr{L})$ is locally a polar space. By (3.1.iv) the polar space P_x is nondegenerate for each $x \in P$.

Any line lies in a member of Π. For, if $L \in \mathscr{L}$, then by connectedness of the incidence system, we may assume there is $M \in \mathscr{L}$ with $L \cap M \neq \emptyset$ and a plane $\pi \in \Pi$ with $M \subset \pi$; invoking (3.1.iii), we see that there is a line M' contained in $\pi \cap L^\perp$; thus, according to (3.1.i), there is a unique member of Π containing $\langle L, M' \rangle$.

3.1. REMARK. The triple $(P, \mathscr{L}, \Pi)$ satisfying (3.1) is a residually connected geometry of points, lines, and planes. This leads to an alternative description of the geometry in terms of a diagram: For incidence systems $(P, \mathscr{L})$ with finite singular rank n, hypothesis (3.1) is equivalent to the following:

(3.2) $P, \mathscr{L}, \Pi$ is the triple of the sets of objects of respective types 0, 1, 2 of a residually connected geometry Γ over $\{0, 1, \ldots, n\}$ $(n \geqslant 2)$ such that

(3.2.i) Γ has diagram



(3.2.ii) the residue of an object of type 0 is a building – i.e. a non-degenerate polar space of rank n;

(3.2.iii) the incidence system induced on $(P, \mathscr{L})$ is a gamma space;

(3.2.iv) two points lie on at most one line; three collinear points not on a line belong to a member of Π.

It is straightforward that (3.2) implies (3.1).

Assume (3.1). We first argue that each point lies on some affine plane (member of Π) and that each such plane has a fixed universal order $q \geqslant 2$ independent of the plane or the point. Since $(P, \mathscr{L})$ is connected and Π is not empty it suffices to show that if a point x lies on an affine plane of order q and y is a point collinear with x then y lies on an affine plane of order q. Thus, suppose x lies on an affine plane π. If $y \in \pi$ we are done, so suppose $y \notin \pi$. Since $y^\perp \cap \pi$ contains x, by (3.1.iii), $y^\perp \cap \pi$ is a line or is π. In any event, there is a line L in $y^\perp \cap \pi$ lying on x. Then by (3.1.i) $\langle L, y \rangle$ lies in an affine plane on y. Since L has cardinality $q + 1$, the order of the new affine plane is q as well. Consequently, every point lies on a plane of order q.

Now consider the sets of lines and of planes on a point p–i.e. the *residue* geometry at p. Thus we regard the lines on p as Points and the planes on p as Lines. Then by (3.1.iii), two Points L and M on p are collinear if and only if $M \subseteq L^\perp$ (so, by (3.1.i), $\langle L, M \rangle$ lies in an affine plane). Moreover, if π is a plane on p and L is a line on p with $L \cap \pi = \{p\}$ it follows that $L^\perp \cap \pi$ is either a line on p, or includes all lines on p within π – i.e. one or all Points incident with the Line π. Thus the residue geometry at a point obeys the fundamental 'one or all' polar space axiom. Moreover, (3.1.iv) implies this residue geometry has no radical. Thus by the theorem of Buekenhout and Shult [2], the residue geometry is a nondegenerate polar space of rank at least 2, and so is a building. Since $(P, \mathscr{L})$ is a gamma space, the subspaces of the residue geometry of a point corresponds to singular 'affine' subspaces (that is, closed with respect to taking affine planes on triples of collinear points not on a line) whose point residues are projective. Hence, if the singular rank of $(P, \mathscr{L})$ equals $n < \infty$, including singular affine subspaces of rank i $(0 \leqslant i \leqslant n)$ as objects of our geometry, we obtain a diagram geometry with diagram that of (3.2.i). It is clearly residually connected, and all parts of (3.2) hold.

Note that if Γ satisfies (3.2), the polar space comprising the residue of a point has thick lines. This is because for each Line of $\text{Res}(x) := (\mathscr{L}_x, \Pi_x)$ there is $\pi \in \Pi_x$ such that all lines through x incident with π have $q + 1$ points. Since $q \geqslant 2$, any Line of $\text{Res}(x)$ must have at least $1 + q \geqslant 3$ points. Thus all Lines of $\text{Res}(x)$ are thick. This is important in applying the results of Section 2.

For the remainder of this section, we assume that $(P, \mathscr{L})$ satisfies (3.1). We shall now derive properties of the affine polar space $(P, \mathscr{L})$.

3.2. LEMMA. *For any two points x and y, the polar spaces* $\text{Res}(x)$ *and* $\text{Res}(y)$ *are isomorphic.*

Proof. First, assume $d(x, y) = 2$, that is x and y are at distance 2. The lines on x are of two types: the set $\mathscr{L}_A(x, y)$ of lines which meet $y^\perp$ and the set $\mathscr{L}_B(x, y)$ of lines which do not meet $y^\perp$.

Similarly, by (3.1.iii), there are two sets of planes on x, the set $\Pi_A(x, y)$ of planes meeting $y^\perp$ at a line and the set $\Pi_B(x, y)$ of planes which do not contain a point of $y^\perp$. Now each plane in $\Pi_A(x, y)$ meets $y^\perp$ at a line L and carries exactly one line on x parallel to L. As $\mathscr{L}_A(x, y)$ is not empty (recall $d(x, y) = 2$), this means, $(\mathscr{L}_B(x, y), \Pi_B(x, y))$ is a hyperplane of the polar space $\mathrm{Res}(x)$. Similarly $(\mathscr{L}_B(y, x), \Pi_B(y, x))$ is a hyperplane of the polar space $\mathrm{Res}(y)$.

For every line M in $\mathscr{L}_A(x, y)$, there is a corresponding line $\phi(M) = \langle M \cap y^\perp, y\rangle$ in $\mathscr{L}_A(y, x)$. Moreover, for each plane π in $\Pi_A(x, y)$, there is a corresponding plane $\phi(\pi)$ containing $\langle \pi \cap y^\perp, y\rangle$. These mappings preserve incidence and have both left and right inverses. Thus we have an isomorphism

$$\phi\colon (\mathscr{L}_A(x, y), \Pi_A(x, y)) \to (\mathscr{L}_A(y, x), \Pi_A(y, x)).$$

Since the polar spaces $\mathrm{Res}(x)$ and $\mathrm{Res}(y)$ are both thick nondegenerate and of rank at least 2, by Proposition 2.7, ϕ can be extended to an isomorphism $\hat{\phi}$: $\mathrm{Res}(x) \to \mathrm{Res}(y)$. Thus the conclusion of the lemma holds when x and y are at distance 2 from one another.

Next suppose x and y are collinear, and set $L = xy$. Since $\mathrm{Res}(x)$ is a nondegenerate polar space of rank at least 2 there is a plane π on L. Then choose $z \in \pi\backslash L$. Again since $\mathrm{Res}(z)$ is a nondegenerate polar space there is a plane π_1 on z not lying in $\pi^\perp$ and intersecting π at the line L' on z parallel to L. Then, for any point $w \in \pi_1\backslash L'$, we have $d(w, x) = 2 = d(w, y)$. Thus, from the argument of the previous case, $\mathrm{Res}(x) \cong \mathrm{Res}(w) \cong \mathrm{Res}(y)$. We see that if x and y are collinear, then $\mathrm{Res}(x) \cong \mathrm{Res}(y)$.

Finally, since $(P, \mathscr{L})$ is connected, the isomorphism holds for any x, y in P. □

3.3. LEMMA. *The collinearity graph of $(P, \mathscr{L})$ has diameter at most 3. If x and y are at distance 3, all lines on x contain a point at distance 2 from y.*

Proof. Assume $d(x, y) = 3$. Then there exists a point z collinear with x and at distance 2 from y. The lines and planes on z which meet $y^\perp$ at a point or line respectively, form the incidence system $(\mathscr{L}_A(z, y), \Pi_A(z, y))$, and complements the hyperplane $\bar{B} = (\mathscr{L}_B(z, y), \Pi_B(z, y))$ of $\mathrm{Res}(z)$ for which the line $L = zx$ is a deep point. By Corollary 1.3, L is the unique deep point of $\bar{B}$ and $\bar{B} = (\bar{L})^\perp$. Thus $\mathscr{L}_B(z, y)$ is simply the set of all lines on z lying within $x^\perp$, and every point of $\bar{B}$ distinct from L is adjacent to a line of $\mathscr{L}_A(z, y)$. This means, that each point $r \in x^\perp \cap z^\perp$ not on the line Lr is collinear with a point of $y^\perp$ (since zr lies in a plane with a line of $\mathscr{L}_A(z, y)$ carrying a point of $y^\perp$). Thus we see that if L in Res(x) contains a point z at distance 2 from y, then the same holds for any line M of Res(x) with $M \subseteq L^\perp$. Since Res(x), being a nondegenerate polar space of rank at least 2, is connected, we see that *every* line of Res(x) carries a point at distance 2 from y. It follows that no pair of points of P are at distance 4, and, since $(P, \mathscr{L})$ is connected, it has diameter at most 3. □

3.4. DEFINITION. We define an equivalence relation '$\|$' on $\mathscr{L}$ as follows. For each line L in $\mathscr{L}$, set

$$\Delta(L) = \{p \in P \mid p^\perp \cap L = L \text{ or } \varnothing\}.$$

We write $L_1 \| L_2$ if and only if $\Delta(L_1) = \Delta(L_2)$ and say L_1 is *parallel* to L_2, for any two lines L_1 and L_2 of $\mathscr{L}$. Manifestly, '$\|$' is an equivalence relation on $\mathscr{L}$. For each line L of $\mathscr{L}$, we let $[L]$ be the equivalence class of $\mathscr{L}$ containing the line L – i.e. the set of all lines parallel to L.

Note that if π is a plane, and L_1 and L_2 belong to the same parallel class in the ordinary sense of parallelism for an affine plane, then $\Delta(L_1) = \Delta(L_2)$, and so L_1 and L_2 are parallel in the sense of the previous paragraph. For, if $p \in \Delta(L_1)$ then by (3.1.iii), $p^\perp \cap \pi$ is either empty, or is a line, or is π. In the first and last cases $p \in \Delta(L_2)$. If $p^\perp \cap \pi = M \in \mathscr{L}$, then clearly, as $p \in \Delta(L_1)$, either $M = p^\perp \cap L_1 = L_1$ or $M \cap L_1 = p^\perp \cap L_1 = \varnothing$. In any case, M is parallel (in the ordinary sense) to L_1 and hence to L_2. Thus, in all cases, $p \in \Delta(L_2)$.

This shows that '$\|$' contains at least the transitive extension on $\mathscr{L}$ of the relation of being parallel lines within an affine plane. In the next two lemmas and Corollary 3.9 it will be seen that parallelism is *precisely* this extension.

3.5. LEMMA. *Suppose $y \in P$ and $L \in \mathscr{L}$. Then, for at least one line $L_0 \in [L]$, the intersection $y^\perp \cap L_0$ is nonempty.*

Proof. Let y, L be such that there is no plane π on L with $y^\perp \cap \pi \neq \varnothing$. By the previous lemma, there is a path $y \perp t \perp x$ with $x \in L$.

Take a plane ρ_1 on tx. As $H_1 = y^\perp \cap \rho_1$ contains t but not x, it is a line, so there exists an affine plane σ_1 containing $\langle H_1, y\rangle$. Similarly $K_1 = L^\perp \cap \rho_1$ is a line (observe that $t \notin L^\perp$ since otherwise the affine plane containing $\langle t, L\rangle$ defies the hypothesis) and there is $\pi_1 \in \Pi$ containing $\langle K_1, L\rangle$. If $H_1 \cap K_1$ contains a point, say w, then $w \in \pi_1 \cap y^\perp$, contradicting the hypothesis. Thus $H_1 \cap K_1 = \varnothing$, and so $H_1 \| K_1$.

Now choose $x_1 \in K_1 \backslash \{x\}$. Take t_1 to be the point on H_1 and on the line in ρ_1, through x_1 parallel to tx, and take y_1 in σ_1 such that $N_1 = yy_1$ is parallel to H_1 and $t_1 y_1$ is parallel to ty. Denote by L_1 the line in π_1 on x_1 parallel to L. If $x_1 \perp y_1$, then $x_1 \in (t_1 y_1)^\perp$, so $x_1 \in \Delta(t_1 y_1) = \Delta(ty)$ whence, as $x_1 \perp t$, we have $x_1 \in (ty)^\perp$, and so $x_1 \in y^\perp \cap \pi_1$, a contradiction. Thus $d(x_1, y_1) = 2$.

From now on assume that there is no line $L_0 \parallel L$ with $y^\perp \cap L_0 \neq \emptyset$. This implies the earlier assumption on L that there is no plane π on L with $y^\perp \cap \pi \neq \emptyset$. Suppose μ is a plane on L_1 with $\mu \cap y_1^\perp \neq \emptyset$. Then, as $x_1 \in \mu \backslash y_1^\perp$, we have that $\mu \cap y_1^\perp$ is a line. If this line is a parallel of L_1, then the affine plane on $\langle y_1, \mu \cap y_1^\perp \rangle$ contains a parallel L_0 of L with $y_1 \in y^\perp \cap L_0$, a contradiction. So $\mu \cap y_1^\perp$ is a line of μ meeting L_1 in a point, say z_1. Now $z_1 \in \Delta(K_1) = \Delta(N_1)$ and $z_1 \in y_1^\perp$ imply $z_1 \in N_1^\perp$ so $z_1 \perp y$, again a contradiction. Hence there is no plane v on L_1 such that $y_1^\perp \cap v \neq \emptyset$. We repeat the construction of the previous paragraph to get H_2, K_2, N_2, L_2 as H_1, K_1, N_1, L_1, this time starting from L_1, x_1, y_1, t_1 instead of L, x, y, t. We choose an affine plane ρ_2 on $\langle H_2, K_2 \rangle$ and t_1x_1 in such a way that $y_1^\perp \cap \rho_2$ and $y^\perp \cap \rho_2$ are distinct lines. (In Res(t_1), this simply means that $H_1 \notin H_2^\perp$.) Then $N_2 \parallel H_2 \parallel K_2$ (as before) and $N_1 \nsubseteq N_2^\perp$. Therefore, $y^\perp \cap N_2 = \{y_1\}$, whence $y \notin \Delta(N_2) = \Delta(K_2)$. Consequently, $y^\perp \cap K_2$ is a point, and so $y^\perp \cap \pi_2 \neq \emptyset$. But now there is a line $L_3 \parallel L_1 \parallel L$ in π_2 with $L_3 \cap y^\perp \neq \emptyset$, the final contradiction. □

3.6. COROLLARY. *If $y \in \Delta(L)$, then y lies on a line of $[L]$. In other words, $\Delta(L)$ is the point-set union of the lines of $[L]$.*

Proof. Suppose $L_0 \in [L]$ and $L_0 \subseteq y^\perp$. If $y \in L_0$ we are done. Otherwise we may form the affine plane on $\langle y, L_0 \rangle$ and find a parallel of L_0 in y and again we are done. So we may assume no line of $[L]$ lies in $y^\perp$. But since $y \in \Delta(L)$, this means $y^\perp \cap L_0 = \emptyset$ for each $L_0 \in [L]$, contrary to the previous lemma. Hence the corollary. □

3.7. LEMMA. *$\Delta(L)$ is a subspace and each point is on a unique line of $[L]$.*

Proof. Suppose L_1, $L_2 \in [L]$ and $p \in L_1 \cap L_2$. Then, in Res(p), we have $L_1^\perp = L_2^\perp$, so $L_1 = L_2$ by nondegeneracy of Res(p). In view of the above corollary, this gives that each point of $\Delta(L)$ lies on a unique line in $[L]$.

Suppose $x, y \in \Delta(L)$ are distinct points of the line $M \in \mathscr{L}$, and let z be a point of M. In order to show that $\Delta(L)$ is a subspace, we derive that z belongs to $\Delta(L)$. If $M \in [L]$ this is obvious, so assume the contrary. Let L_0 be the member of $[L]$ containing y. Then $\langle M, L_0 \rangle$ is a singular subspace generated by three points not on a line, so it lies in an affine plane. In this plane, there is a line $L_1 \parallel L_0$ on z. As $L_1 \in [L]$, we have $z \in \Delta(L)$. The proof is complete. □

We see from Corollary 3.6 and Lemma 3.7 that $\Delta(L)$ is the union of disjoint lines from $[L]$. Moreover, if L_1 and L_2 are two distinct lines of $[L]$, either $L_1^\perp \cap L_2 = \emptyset$ or $L_1 \subseteq L_2^\perp$. In the latter case L_1 and L_2 are parallel lines within the affine plane containing $\langle L_1, L_2 \rangle$. For, if $L_1 \subseteq L_2^\perp$ and $x \in L_2$ there exists $\pi \in \Pi$ on $\langle L_1, x \rangle$, so x carries a line L' in π parallel to L_1. But then L_2 and L'

are lines of $[L]$ lying on x, so $L' = L_2$ by Lemma 3.7 again. Thus $\langle L_1, L_2 \rangle$ is the plane π.

From this it is clear that, for $L \in \mathscr{L}$, we may form a linear incidence system whose points are the lines of $[L]$ and whose lines are the sets of lines of $[L]$ lying in planes generated by two mutually perpendicular members of $[L]$. We denote this incidence system by the symbol $\Delta(L)/L$.

3.8. LEMMA. *Assume $L \in \mathscr{L}$ and x is a point not in $\Delta(L)$. Then the lines on x which do not meet $\Delta(L)$ form a hyperplane $B(x, [L])$ of* Res(x).

Proof. Suppose π is an affine plane on x containing a point z of $\Delta(L)$. Then, by Corollary 3.6, z lies on a line L_1 in $[L]$. Since x is not in $\Delta(L) = \Delta(L_1)$, $x^{\perp} \cap L_1 = \{z\}$ and so $L_1^{\perp} \cap \pi$ is a line M not on x. But $M \subseteq \Delta(L)$. Any further point of $\pi \cap \Delta(L_1)$, since it lies in $z^{\perp}$, must lie in $L_1^{\perp}$ whence $\Delta(L) \cap \pi = \Delta(L_1) \cap \pi = L_1^{\perp} \cap \pi = M$.

Thus every line on x lying in π meets $\Delta(L)$ (at a point of M) except one, namely, the line M_1 on x parallel to M (in π). As, obviously, $M_1 \cap \Delta(L_1) = \emptyset$, we have thus seen that every Line of Res(x) has exactly one or all of its Points represented by lines on x not meeting $\Delta(L)$. Furthermore, $B(x, [L]) \neq$ Res(x). It follows that the lines on x not meeting $\Delta(L)$ represent a hyperplane $B(x, [L])$ of Res(x). □

For $x \in \Delta(L)$, we also consider the incidence system $A(x, [L]) = (\mathscr{L}_A(x, [L]), \Pi_A(x, [L]))$ of all lines on x which meet $\Delta(L)$ and all planes on x which meet $\Delta(L)$ at a line. Then $A(x, [L])$ and $B(x, [L])$ together form Res(x).

3.9. COROLLARY. *If x and y are two points not lying in $\Delta(L)$, then $B(x, [L]) \cong B(y, [L])$ as incidence systems. Moreover, $A(x, [L]) \cong \Delta(L)/L$.*

Proof. For each line L_1 of $[L]$ there is a unique point $x(L_1)$ with $\{x(L_1)\} = x^{\perp} \cap L_1$ and a unique line $\phi_x(L_1) := x(L_1)x$ on x meeting $\Delta(L)$ at a point of L_1. If L_1 and L_2 lie in $[L]$, then $L_1 \subseteq L_2^{\perp}$ if and only if $x(L_1) \in x(L_2)^{\perp}$ if and only if $\phi_x(L_1) \subseteq (L_2)^{\perp}$. Also, for $M \in \mathscr{L}_A(x, [L])$, Lemma 3.7 yields that there is a unique point $y \in M \cap \Delta(L)$ and a unique line $L_0 \in [L]$ so that $\phi_x^{-1}(M) = L_0$. Thus $\phi_x \colon [L] \to \mathscr{L}_A(x, [L])$ is a 1–1 correspondence preserving collinearity so ϕ_x is an isomorphism $\phi \colon \Delta(L)/L \to A(x, [L])$ of linear incidence systems.

Similarly, $\Delta(L)/L \cong A(y, [L])$ and so $f = \phi_y \phi_x^{-1}$ is an isomorphism $f\colon A(x, [L]) \to A(y, [L])$. By Proposition 2.7, f extends to an isomorphism Res$(x) \to$ Res(y), whose restriction to the complementary hyperplane $B(x, [L])$ defines the required isomorphism $B(x, [L]) \to B(y, [L])$ between hyperplanes. □

3.10. LEMMA. *Let L be a line and x a point in $P\backslash\Delta(L)$. Suppose M is a line on x, representing a deep point of $B(x, [L])$ in* Res(x). *Then*

(i) $\Delta(L) \cap \Delta(M) = \varnothing$;

(ii) *there is no line in $[M]$ on which there is a plane meeting $\Delta(L)$.*

Proof. By hypothesis, no plane on M meets $\Delta(L)$ (nontrivially). Let $[M]_0$ represent those lines M' of $[M]$ for which no plane on M' meets $\Delta(L)$. Thus $M \in [M]_0$.

If $u \in \Delta(L) \cap \Delta(M)$, then u carries a line $M' \in [M]$, and any plane on M' will contain u, so meets $\Delta(L)$ nontrivially, whence $[M]\backslash[M]_0$ is nonempty. Thus (i) will be a consequence of (ii).

As for (ii), suppose, by way of contradiction, that $[M]\backslash[M]_0$ is nonempty. Since $\Delta(M)/M$ is connected (cf. Corollary 3.9, Lemma 3.8, and Lemma 1.1(i)) and $[M]_0 \neq \varnothing$, there is a line $M_1 \in [M]_0$ lying in a plane π_1 with a line $M_2 \in [M]\backslash[M]_0$. Then M_2 lies in a plane π_2 which meets $\Delta(L)$ nontrivially, whence at a line N. Since $\pi_1 \cap \Delta(L) = \varnothing$, by hypothesis, we see N is parallel to $M_2 = \pi_1 \cap \pi_2$ in π_2. Since N lies in $\Delta(L)$ and $N \notin [L]$, it corresponds to a plane ρ in $\Delta(L)$ representing a line of $\Delta(L)/L$. (Observe that ρ is the affine plane on $\langle N, L_1 \rangle$ for each $L_1 \in [L]$ meeting N at a point.) Take a point $y \in M_1$. The fact $M_1 \cap \Delta(L) = \varnothing$ implies that $y^\perp \cap \rho$ is a line M_y not in $[L]$. Moreover, if $q \in M_y \cap N$, then $y^\perp \cap \pi_2$ contains M_2 and q and hence all of π_2. Thus $N \subseteq y^\perp$ and so $\langle y, N \rangle$ lies on an affine plane π_3 containing a line N' on y parallel to N. From the above, $N \in [M]$, so $N' \in [M]$, and, by Lemma 3.7, $N' = M_1$. But then π_3 is a plane containing M_1 and meeting $\Delta(L)$ in N, contradicting $M_1 \in [M]_0$.

Thus M_y must be parallel to N. Now choose any point z on M_y. Then $z^\perp \cap M_1$ is not M_1 as $M_1 \in [M]_0$, so $z^\perp \cap M_1 = \{y\}$ and $z^\perp \cap \pi_1$ is a line S not parallel with M_1. Thus $S \cap M_2 = \{y_2\}$. Then $y_2^\perp \cap \rho$ includes both N and z not lying on M. Thus $y_2^\perp$ contains all of ρ. But since ρ contains lines from $[L]$ we have $y_2 \in \Delta(L)$, so π_1 meets $\Delta(L)$ nontrivially against our choice of M_1 as lying in $[M]_0$. This contradiction completes the proof. □

3.11. REMARK. Quite clearly the converse of Lemma 3.10 holds – that is

> *If $\Delta(L) \cap \Delta(M) = \varnothing$ and $x \in \Delta(M)$ and M' is the unique* (see Lemma 3.7) *line of $[M]$ lying on x, then M' represents a deep point of the hyperplane $B(x, [L])$ of* Res(x).

Proof. Clearly as $M' \cap \Delta(L) = \varnothing$, the line M' represents a point of $B(x, [L])$. But if M' were not a deep point, there would be a plane π on M' meeting $\Delta(L)$ at a point y. But then y carries a line M'' parallel to M' so $y \in \Delta(M) \cap \Delta(L)$, a contradiction. □

3.12. DEFINITION. We denote by $\mathscr{L}/\|$ the collection of parallel classes of lines of $\mathscr{L}$. We define the relation $\equiv$ on $\mathscr{L}/\|$ by asserting $[L] \equiv [M]$ if and only if $\Delta(L) \cap \Delta(M) = \varnothing$. Considering that $L' \in [L]$ if and only if $\Delta(L) = \Delta(L')$, this relation is certainly well defined.

3.13. LEMMA. *The reflexive closure of the relation* $\equiv$ *is an equivalence relation on* $\mathscr{L}/\|$.

Proof. Assume $\Delta(L) \cap \Delta(M) = \varnothing = \Delta(M) \cap \Delta(N)$ for lines L, M, N, and that $\Delta(L) \cap \Delta(N)$ contains a point u. Then there exist lines L' and N' on u belonging to $[L]$ and $[N]$, respectively, and, by the above remark, L' and N' both represent deep points of the hyperplane $B(u, [M])$ of $\mathrm{Res}(u)$. As $\mathrm{Res}(u)$ is a nondegenerate thick polar space of rank at least 2, Corollary 1.3 forces $L' = N'$ and hence $\Delta(L) = \Delta(N)$. This proves the assertion. □

3.14. COROLLARY. *Suppose* $\Delta(L_1) \cap \Delta(L_2) = \varnothing$ *for two lines* L_1 *and* L_2. *Then the sets* $\Delta(L)$ *for* $[L]$ *running over the members of the* $\equiv$*-class of* $[L_1]$, *form a partition of* P.

Proof. Denote by X the $\equiv$-class of $[L_1]$. Clearly, for distinct $[M], [N] \in X$, we have $\Delta(M) \cap \Delta(N) = \varnothing$ so it only remains to show that each $p \in P$ lies in some $\Delta(L)$ for $[L] \equiv [L_1]$. We may (and shall) assume $p \notin \Delta(L_1)$. Choose $x \in \Delta(L_1)$ and $y \in \Delta(L_2)$. Then $B(y, [L_1])$ contains a deep point – some line parallel to L_2 – see Remark 3.11. But, by Corollary 3.9, $B(y, [L_1]) \cong B(p, [L_1])$ and so the latter contains a deep point in $\mathrm{Res}(p)$ represented by the line L. By the above lemma, $\Delta(L) \cap \Delta(L_1) = \varnothing$ and so $[L] \equiv [L_1]$ as required. □

We now wish to define a second relationship on $\mathscr{L}/\|$. Write $[L] \sim [M]$ if some line L' of $[L]$ and some line M' of $[L]$ lie together in a plane. Note that in this case $\Delta(L) \cap \Delta(M)$ is not empty (it contains every point of the aforementioned plane, for example). The next several lemmas concern this relation $\sim$.

3.15. LEMMA. *For any two lines* L *and* M *with* $L \subseteq \Delta(M)$, *we have* $[L] \sim [M]$.

Proof. Without loss of generality, $[L] \neq [M]$. Take distinct points u, v on L and $M' \in \Delta(M)$ on u (cf. Corollary 3.6). Now $v \in \Delta(M')$ and $u \in v^\perp \cap M'$ so $M' \subseteq \{u, v\}^\perp = L^\perp$ and there is an affine plane containing L and M'. □

3.16. LEMMA. *Let* L *and* M *be lines such that* $L \cap \Delta(M) = \varnothing$ *and* $\Delta(L) \cap \Delta(M) \neq \varnothing$. *Then* $[L] \sim [M]$.

Proof. There exists a point $u \in \Delta(L) \cap \Delta(M)$ and by Corollary 3.6 u lies on a line M' of $[M]$ and a line L' of $[L]$. Since $\mathrm{diam}(\Delta(L)/L) \leqslant 3$ (for it is a polar space minus a hyperplane, see Corollary 3.9, Lemma 3.8, and Lemma 1.1(i)), there exist planes π_1, π_2, and π_3 such that π_1 contains L', $\pi_1 \cap \pi_2 = L_1 \in [L] \backslash \{L'\}$, $\pi_2 \cap \pi_3 = L_2 \in [L]$, $L_3 \subseteq \pi_3$, $L_3 \neq L_2$, $L_3 \in [L]$ so that the line L is either L_3, L_2 or L_1.

Now $(M')^\perp \cap \pi_1$ is a line N_1. If $N_1 \in [L]$ we are done since $\langle N_1, M' \rangle$ is contained in an affine plane. So we may assume N_1 is not parallel to L_1 and so $N_1 \cap L_1$ is a point $p_1 \in L_1$. Now in the affine plane on $\langle N_1, M' \rangle$ and on the point p_1 there is a line M_1 parallel to M'. As $p_1 \in \Delta(M)$, the assumption $L \cap \Delta(M) = \emptyset$ yields $L \neq L_1$. So $L = L_3$ or L_2. Now $(M_1)^\perp \cap \pi_2 = N_2$ is a line. Again if $N_2 \in [L]$ we are done since $\langle N_2, M_1 \rangle$ is on an affine plane and $M_1 \in [M]$. Thus we may assume N_2 is not parallel to L_2 and hence $N_2 \cap L_2$ is a point p_2 which lies in $\Delta(M)$. Again $L \cap \Delta(M) = \emptyset$ yields $L \neq L_2$, whence $L = L_3$. On p_2 there is a line M_2 parallel to M_1, so $M_2 \in [M]$, and again $(M_2)^\perp \cap \pi_3$ is a line N_3. If $N_3 \in [L]$ we are done as $\langle N_3, M_2 \rangle$ is contained in an affine plane. Thus N_3 is not parallel to $L_3 = L$ and so $\emptyset \neq N_3 \cap L_3 \subseteq L \cap \Delta(M) = \emptyset$, a contradiction. This completes the proof. □

3.17. LEMMA. *Let L and M be lines. Assume $|L \cap \Delta(M)| \geqslant 1$ and $[L] \sim [M]$. Then $L \subseteq \Delta(M)$.*

Proof. Since $[L] \sim [M]$ there exists a plane $\pi \in \Pi$ containing $L_1 \in [L]$ and $M_1 \in [M]$. Suppose, by way of contradiction, that L does not lie in $\Delta(M)$. Then $|L \cap \Delta(M)| = 1$ by hypothesis and Lemma 3.7. Choose $w \in L \backslash \Delta(M)$. Then $w^\perp \cap M_1$ is a point p, and so $w^\perp \cap \pi$ is a line N not parallel to M_1. If N were not parallel to L_1 we would have $|w^\perp \cap L_1| = 1$ against $w \in L \subseteq \Delta(L_1)$. Thus N is parallel to L_1. Then there is a line N' on w in the affine plane on $\langle w, N \rangle$ parallel to N. Thus $N' \in [L]$ and since it lies on w, $L = N'$. Thus $L^\perp \cap \pi = N$. Recall there is a point $v \in L \cap \Delta(M)$. Now $v^\perp \cap \pi$ contains N, whence the point p of M_1, and so, as $v \in \Delta(M)$, we must have $M \subseteq v^\perp$. This means $v^\perp \cap \pi \supseteq \langle N, M_1 \rangle$. Taking $q \in M_1 \backslash \{p\}$, we see that $q \in \Delta(L)$ and $v \in q^\perp \cap L$, whence $q \in L^\perp \subseteq w^\perp$. Therefore, $w^\perp \supseteq M_1$, so $w \in \Delta(M)$, contrary to assumption. This completes the proof. □

The next lemma combines both relations $\equiv$ and $\sim$ on $\mathscr{L}/\|$.

3.18. LEMMA. *Let L, M, and R be lines with $[R] \sim [L] \equiv [M]$. Then either $[R] = [L]$ or $[R] \sim [M]$.*

Proof. We may assume that R and L intersect at a point x in an affine plane π. Without loss of generality, $[L] \neq [M]$. Then $R \subseteq \Delta(L)$, whence (as $[L] \equiv [M]$) we have $R \cap \Delta(M) = \emptyset$, and in view of Lemma 3.16 we may assume $[R] \equiv [M]$ (otherwise the proof is complete). But then, by Lemma 3.13, $[R] \equiv [L]$, so, by Lemma 3.7, $R = L$. □

3.19. LEMMA. *Suppose L and M are (distinct) intersecting lines of an affine plane* $\pi \in \Pi$. *If N is a line in* π *which is not parallel to L then* $\Delta(L) \cap \Delta(M) = \Delta(L) \cap \Delta(N)$. *Moreover,* $[N] \cap \mathscr{L}(\rho) \neq \emptyset$ *for each plane* ρ *containing members of both* $[L]$ *and* $[M]$.

Proof. Let $y \in \Delta(L) \cap \Delta(M)$. If $y^\perp \cap \pi = \emptyset$, then $y^\perp \cap N = \emptyset$. Otherwise, $\pi \subseteq y^\perp$ (as $y^\perp \cap X \neq \emptyset$ for at least one $X \in \{L, M\}$) so $N \subseteq y^\perp$. Thus, $y \in \Delta(N)$ in each case. This establishes $\Delta(L) \cap \Delta(M) \subseteq \Delta(L) \cap \Delta(N)$, and the other inclusion follows by symmetry in M and N. Hence the first assertion.

Now suppose ρ is a plane containing the lines L' parallel to L and M' parallel to M, and take $x \in L' \cap M'$. As $x \in \Delta(N)$, there is a line N' on x parallel to N. Let U be any line on x contained in $\{L, M\}^\perp$. Then $U \subseteq \Delta(L) \cap \Delta(M) \subseteq \Delta(N)$, so U is coplanar with N'. This shows that the point N' of $\mathrm{Res}(x)$ belongs to $\{L, M\}^{\perp\perp} = \mathscr{L}(\pi)_x$. Hence the lemma. □

4. CHARACTERIZATION OF AFFINE POLAR SPACES

Starting with an affine polar space, i.e. an incidence system $(P, \mathscr{L})$ satisfying (3.1), we form a new geometry $(\underline{P}, \underline{\mathscr{L}})$.

The points of $\underline{P}$ are of three kinds

$\underline{P}_1$: the points of P;
$\underline{P}_2$: the elements of $\mathscr{L}/\|$, that is, the parallel classes $[L]$, $L \in \mathscr{L}$;
$\underline{P}_3$: the symbol ∞.

The lines also, are of three kinds

$\underline{\mathscr{L}}_1$: for each $L \in \mathscr{L}$, the set $L \cup \{[L]\}$;
$\underline{\mathscr{L}}_2$: the set $\{[L] \mid L \in \mathscr{L}(\pi)\}$ for each plane π;
$\underline{\mathscr{L}}_3$: the sets $\{\infty\} \cup X$ for each $\equiv$-class X of $\underline{\mathscr{L}}$.

Incidence on $(\underline{P}, \underline{\mathscr{L}})$ is defined by containment.

4.1. THEOREM. *Each line of the incidence system* $(\underline{P}, \underline{\mathscr{L}})$ *not containing* ∞ *is thick. Moreover,*

(i) $\infty^\perp = \underline{P}_2 \cup \underline{P}_3$ *is a subspace of* $(\underline{P}, \underline{\mathscr{L}})$; *the induced incidence system* $(\infty^\perp, \underline{\mathscr{L}}(\infty^\perp))$ *is a polar space of rank at least* 3;
(ii) *if some line containing* ∞ *is thick, then all members of* $\underline{\mathscr{L}}_3$ *are thick lines, and* $(\underline{P}, \underline{\mathscr{L}})$ *is a nondegenerate polar space of rank at least* 3 *with hyperplane* $\infty^\perp$ *and* $P = \underline{P} \backslash \infty^\perp$;
(iii) *if none of the lines containing* ∞ *is thick, then* $(\underline{P}_1 \cup \underline{P}_2, \underline{\mathscr{L}}_1 \cup \underline{\mathscr{L}}_2)$ *is a nondegenerate polar space of rank at least* 3, *in which* $\underline{P}_2$ *is a nondegenerate hyperplane.*

Proof. To show that a subspace of $(\underline{P}, \underline{\mathscr{L}})$ is a polar space it suffices to verify the basic 'one or all' axiom for all relevant nonincident point–line pairs. This task will be easier if we start by reviewing what $\underline{\mathscr{L}}$-collinearity means for points.

First, two points p, q of $\underline{P}_1$ are collinear if they are collinear in $(P, \mathscr{L})$. Second, a point $p \in \underline{P}_1$ and a point $[L]$ of $\underline{P}_2$ are collinear if and only if $p \in \Delta(L)$. Third, two elements $[L]$ and $[M]$ of $\mathscr{L}/\|$ are collinear if and only if $[L] \equiv [M]$ or $[L] \sim [M]$. Fourth, ∞ is collinear with all elements of $\underline{\mathscr{L}}_2 \cup \underline{\mathscr{L}}_3$.

Clearly, lines not containing ∞ have at least three points, $\infty^\perp = \underline{P}_2 \cup \underline{P}_3$ is a subspace of $(\underline{P}, \underline{\mathscr{L}})$ with lines $\underline{\mathscr{L}}(\infty^\perp) = \underline{\mathscr{L}}_2 \cup \underline{\mathscr{L}}_3$.

(i) We verify the 'one or all' axiom for $\infty^\perp$ in four cases.

CASE 2,2. *point* $[N] \in \underline{P}_2$; *line* $\underline{A} = \{[L] \mid L \in \mathscr{L}(\pi)\} \in \underline{\mathscr{L}}_2$. If a line $N' \in [N]$ meets π nontrivially then $(N')^\perp \cap \pi$ is either a line R or all of π. In the former case $N' \subseteq \Delta(R)$ and $[R]$ is the unique point $[L]$ of $\underline{A}$ with $N' \subseteq \Delta(L)$. Therefore $[N] \sim [R]$ and, for $L \in \mathscr{L}(\pi)$ with $[L] \neq [R]$, neither $[N] \equiv [L]$ nor $[N] \sim [L]$ holds (cf. Lemma 3.17). Thus $[N]$ is collinear with exactly one member of $\underline{A}$, namely $[R]$. In the latter case, $N' \subseteq \Delta(L)$ for all $L \in \mathscr{L}(\pi)$ and, by 3.15, $[N]$ is collinear with all members of $\underline{A}$.

Thus we may assume no member of $[N]$ meets π nontrivially – i.e. $\pi \cap \Delta(N) = \varnothing$. This means each point of π is collinear with exactly one point of N' for each $N' \in [N]$. There are thus only two situations which can arise

(a) There exists a unique $x_0 \in N$ with $x_0^\perp \supseteq \pi$ and $x^\perp \cap \pi = \varnothing$ for all $x \in N \backslash \{x_0\}$.

(b) For each $x \in N$, the intersection $x^\perp \cap \pi = M_x$ is either empty or a line, and the lines M_x all belong to one parallel class of π.

Consider first case (a). Take two nonparallel lines L, $M \in \mathscr{L}(\pi)$. Then $x_0 \in \Delta(L) \cap \Delta(M)$, so, according to Lemma 3.7, there are lines $L' \in [L]$ and $M' \in [M]$ on x_0. By Lemma 3.17, $L' \perp M'$, and by Lemma 3.19, the affine plane π' containing L and M satisfies $\underline{A} = \{[T] \mid T \in \mathscr{L}(\pi')\}$. Thus, upon replacing π by π', we are back in the case where N meets π nontrivially which was covered at the beginning of this case, so we are done.

Next we consider case (b). Clearly, for each $x \in N$, we have $[M_x] \in \underline{A}$ whenever M_x is a line. Consider a line M_x. If $y \in N$, then $y^\perp \cap M_x = M_x$ or $\varnothing$. Thus, $N \subseteq \Delta(M_x)$, whence, by Lemma 3.15, $[N] \sim [M_x]$. Consequently, $[N]$ is $\underline{\mathscr{L}}$-collinear with $[M_x]$.

Next assume $[N] \equiv [L_0]$ for some $L_0 \in \mathscr{L}(\pi)$ with $[L_0] \neq [M_x]$. Then, for any $L \in \mathscr{L}(\pi)$, we have $[L] \sim [L_0] \equiv [N]$, and so, by Lemma 3.18, $[N] \sim [L]$. Thus $[N]$ is collinear to all points of $\underline{A}$.

Therefore we may assume $\Delta(L) \cap \Delta(N) \neq \emptyset$ for each $L \in \underline{\mathscr{L}}(\pi)$. Suppose $L_0 \in \mathscr{L}(\pi)$ is such that $[L_0]$ is $\underline{\mathscr{L}}$-collinear with $[N]$ and distinct from $[M_x]$. Then $[L_0) \sim [N]$ and, for $L \in \mathscr{L}(\pi)$ with $[L] \neq [M_x]$, we have $L \cap \Delta(N) = \emptyset$ (as each point of π is collinear with exactly one point of N), it follows from Lemma 3.16 that $[N] \sim [L]$. Thus $[N]$ is collinear with each member of $\underline{A}$. This completes Case 2,2.

CASE 2,3. *point* $[N] \in \underline{P}_2$; *line* $\underline{A} = \{\infty\} \cup X \in \underline{\mathscr{L}}_3$. Assume $[N] \sim [L]$ for some $[L] \in X$. Then, for any $[M] \in X$ different from $[L]$, we have $[N] \sim [L] \equiv [M]$, and so, by Lemma 3.18, $[N] \sim [M]$. Thus, as $[M]$ was arbitrary in $X \backslash \{[L]\}$, we see that $[N]$ is $\underline{\mathscr{L}}$-collinear with all points on the line $\underline{A}$.

Otherwise, $[N] \not\sim [L]$ for any $[L] \in X$, which means – as $[N] \not\equiv [L]$ for any $[L] \in X$ by the hypothesis $[N] \notin \underline{A}$ – that $[N]$ is $\underline{\mathscr{L}}$-collinear only with ∞ on $\underline{A}$. This finishes Case 2,3.

CASE 3,2. *point* ∞; *line* $\{[L] \mid L \in \mathscr{L}(\pi)\} \in \underline{\mathscr{L}}_2$. The point ∞ is collinear to all $[L]$.

CASE 3,3. *point* ∞; *line* $(\{\infty\} \cup X) \in \underline{\mathscr{L}}_3$. In this case ∞ is incident with the line, so there is nothing to prove.

So far, we have shown that $\infty^\perp$ is a polar space. Since there are lines disjoint from ∞ in $\infty^\perp$, the rank of $\infty^\perp$ exceeds 2. For each $L \in \mathscr{L}$, taking $x \in L$ and $M \in \mathscr{L}$ such that L, M are not collinear in $\mathrm{Res}(x)$, we find that $[M]$ is a point of $\infty^\perp$ not collinear to $[L]$. Thus, rad $\infty^\perp = \{\infty\}$. Therefore, the quotient space, denoted by $\infty^\perp/\infty$, of $\infty^\perp$ with respect to its radical has rank at least 2.

(ii) Suppose that $\infty^\perp$ contains a thick line $\{\infty\} \cup X \in \underline{L}_3$ on ∞. Then X is an $\equiv$-class of $L/\|$ containing at least two members $[X_1]$ and $[X_2]$, say. If $\{\infty\} \cup Y$ is another line on ∞ such that $X \perp Y$, take $Y_0 \in Y$. There are lines L_1 and L_2 in $\infty^\perp$ on X_1, Y_0 and X_2, Y_0, respectively. Since $\infty^\perp/\infty$ is nondegenerate, it is a partial linear space, so $\langle L_1, \infty\rangle$ and $\langle L_2, \infty\rangle$ (having both $\{\infty\} \cup X$ and $\{\infty\} \cup Y$ in common) coincide. Thus, if $\{\infty\} \cup Z \in \underline{\mathscr{L}}_3$ is a third line on ∞ in $\langle\infty, X, Y\rangle$ (such a Z exists as L_1 has cardinality at least 3), it meets both L_1 and L_2. Due to Lemma 3.19 and the assumption $X_1 \neq X_2$, the points $Z \cap L_1$ and $Z \cap L_2$ are distinct, proving that all lines on ∞ lying in the singular subspace $\langle\infty, X, Y\rangle$, with the possible exception of $\{\infty\} \cup Y$, are thick. It readily follows (by repetition of the same argument) that $\{\infty\} \cup Y$ is also thick, and (by connectedness of $\infty^\perp \backslash \infty$) that each line on ∞ is thick.

It remains to show that $(\underline{P}, \underline{\mathscr{L}})$ is a polar space. We verify the 'one or all' axiom for the cases involving points from P or lines from $\mathscr{L}$.

CASE 1,1. *point* $p \in P$; *line* $M \cup \{[M]\} \in \underline{\mathscr{L}}_1$. If $p \in \Delta(M)$ then, by Corollary 3.6, p lies on a line $M' \in [M]$ and $M' \cup \{[M]\}$ is a line on p, so p is collinear with $[M]$. But as $p \in \Delta(M)$, by definition, $p^\perp \cap M = \emptyset$ or M. Thus p is $\underline{\mathscr{L}}$-collinear to just $[M]$ or to all of $M \cup \{[M]\}$.

If $p \notin \Delta(M)$, then $|p^\perp \cap M| = 1$ and p is not collinear with $[M]$ so p is $\underline{\mathscr{L}}$-collinear with exactly one point of $M \cup \{[M]\}$.

CASE 1,2. *point* $p \in P$; *line* $\{[L] \mid L \in \mathscr{L}(\pi)\} \in \underline{\mathscr{L}}_2$. First assume $p^\perp \cap \pi = \emptyset$. Then, for all $L \in \mathscr{L}(\pi)$, we have $p^\perp \cap L = \emptyset$ so $p \in \Delta(L)$, whence p is $\underline{\mathscr{L}}$-collinear with $[L]$. Similarly, if $p^\perp \supseteq \pi$, then $p \in \Delta(L)$ for all $L \in \mathscr{L}(\pi)$ and the same conclusion holds.

In the remaining case, $p^\perp \cap \pi$ is a line N. Then, by definition of $\Delta(L)$, we have, for $L \in \mathscr{L}(\pi)$, that $p \in \Delta(L)$ holds if and only if $[L] = [N]$, so p is $\underline{\mathscr{L}}$-collinear with exactly one point of the line.

CASE 1,3. *point* $p \in P$; *line* $\underline{A} = \{\infty\} \cup X \in \underline{\mathscr{L}}_3$. Here, by Corollary 3.14, the sets $\Delta(L)$, $[L] \in X$, partition P, *hence* p is $\underline{\mathscr{L}}$-collinear with precisely one point of $\underline{A}$.

CASE 2,1. *point* $[N] \in \underline{P}_2$; *line* $M \cup \{[M]\} \in \underline{\mathscr{L}}_1$. Since the point and line are not incident, we have $[N] \neq [M]$.

Assume $[N]$ is not $\underline{\mathscr{L}}$-collinear with any point of M. Then $M \cap \Delta(N) = \emptyset$. If $\Delta(M) \cap \Delta(N) = \emptyset$, then $[M]$ and $[N]$ are collinear by a line of $\underline{\mathscr{L}}_3$. So assume $\Delta(M) \cap \Delta(N) \neq \emptyset$. We have now attained the hypotheses of Lemma 3.16 and so $[M] \sim [N]$ – i.e., they are $\underline{\mathscr{L}}_2$-collinear. So far, we have shown that

(4.1) *If* $[N]$ *is collinear with no point of* M, *it is collinear with* $[M]$.

Next assume $[N]$ is collinear with two points of M. They by use of Lemma 3.7, $M \subseteq \Delta(N)$, so each point of M is collinear with $[N]$, and, by Lemma 3.15, $[N] \sim [M]$. This means that

(4.2) *If* $[N]$ *is collinear with two points of* M, *it is collinear with all points of* $M \cup \{[M]\}$.

It remains to consider the case where $[N]$ is collinear with $[M]$ and a point of M. Then $M \cap \Delta(N) \neq \emptyset$. In particular $\Delta(M) \cap \Delta(N) \neq \emptyset$ and so $[N] \sim [M]$. We now have the hypotheses of Lemma 3.17 and so $M \subseteq \Delta(N)$. This shows

(4.3) *If* $[N]$ *is collinear with* $[M]$ *and a point of* M, *then it is collinear with all points of* $M \cup \{[M]\}$.

The assertions (4.1), (4.2), and (4.3), put together, complete Case 4.

CASE 3,1. *point* ∞; *line* $M \cup \{[M]\} \in \underline{\mathscr{L}}_1$. Then ∞ is adjacent only to $[M]$ so the 'one or all' rule holds.

This establishes that $(\underline{P}, \underline{\mathscr{L}})$ is a polar space. Nondegeneracy of $(\underline{P}, \underline{\mathscr{L}})$

follows from the fact that, for each $x \in P$, the space Res(x) is nondegenerate. Finally, $\infty^{\perp}$ is clearly a hyperplane of $(\underline{P}, \underline{\mathscr{L}})$.

(iii) Assume that no line on ∞ is thick. Then the arguments of all cases but 1,3 of the proof of (ii) still prevail: after discarding the point ∞ and the lines in $\underline{\mathscr{L}}_3$, we obtain the final assertions of the theorem. □

4.2. COROLLARY. *If $(P, \mathscr{L})$ is an affine polar space, then it is isomorphic to the derived incidence system $(A, \underline{\mathscr{L}}(A))$ on the complement A of a hyperplane of a nondegenerate polar space $(\underline{P}, \underline{\mathscr{L}})$ of rank at least* 3.

Proof. Immediate from assertions (ii) and (iii) of the above theorem. □

5. Classification of hyperplanes

According to the fundamental result of Tits and Veldkamp (cf. Tits [7, Th. 8.22]), a polar space of rank $\geqslant 3$ whose planes are Desarguesian is isomorphic to the polar space associated with a nondegenerate polarity, the polar space associated with a nondegenerate pseudo-quadratic form or the Grassmannian whose points are the lines of a 3-dimensional projective space $\mathbb{P}^3$ and whose lines are the pencils (X, π), where X is a projective point of the projective plane π, consisting of all projective lines on X in π. We shall deal with the latter case first.

But before dealing with these three cases, we briefly discuss hyperplanes in generalized quadrangles (nondegenerate polar spaces of rank 2). Casting aside the nondegenerate hyperplanes, we remain with *ovoids* (those of rank 1) and hyperplanes which are subgeneralized-quadrangles (those of rank 2). If the generalized quadrangle is finite of order (s, t), an ovoid is characterized as a set of $1 + st$ point no two of which are collinear, and a subgeneralized-quadrangle is a hyperplane if and only if it has order $(s, t/s)$. Many examples are known, and there is no hope for a classification of ovoids. The subgeneralized-quadrangles of the embeddable generalized quadrangles, however are known by Tits [7, Lemma 8.10]; the hyperplanes among them are easily recognized.

The Grassmannian of lines in $\mathbb{P}^3$. Let $\mathbb{P}^3$ denote the projective space of rank 3 over some (skew) field. Setting P for the set of lines of $\mathbb{P}^3$ and $\mathscr{L}$ for the set of pencils (X, π), where X is a point of $\mathbb{P}^3$ incident to the plane π of $\mathbb{P}^3$, we obtain a polar space $(P, \mathscr{L})$ of rank 3 if incidence of the line $l \in P$ with the pencil (X, π) is given by $X \in l \subseteq \pi$.

5.1. PROPOSITION. *If B is a hyperplane of $(P, \mathscr{L})$, then either $B = l^{\perp}$ for some $l \in P$, or there is a symplectic polarity on $\mathbb{P}^3$ (i.e. a polarity with the* *property that all points of $\mathbb{P}^3$ are absolute) such that B coincides with its absolute lines.*

Proof. Suppose B contains a plane of $(P, \mathscr{L})$. Then, up to duality, we may assume B contains all lines $l \in P$ contained in a plane π of $\mathbb{P}^3$. Take a point X of $\mathbb{P}^3$ outside π, and consider the plane of $(P, \mathscr{L})$ consisting of all lines of $\mathbb{P}^3$ on X. There must be a line $(X, \pi') \in \mathscr{L}$ all of whose P-points belong to B. The plane π' meets π in a line, say l. Now any $\mathbb{P}^3$-line meeting l belongs to B. For, if m is such a line meeting l in Z, say, consider the plane π'' on Z containing both X and m. Since the P-points $\pi'' \cap \pi$ and $\pi'' \cap \pi'$ both belong to $B \cap (Z, \pi'')$, the whole P-line (Z, π''), whence m, belongs to B. Thus, $B = l^{\perp}$.

Next, suppose B has rank 2. Then, in each plane π of $\mathbb{P}^3$, there is a unique point $\sigma(\pi)$ of $\mathbb{P}^3$ such that $(\sigma(\pi), \pi) \in \mathscr{L}(B)$. Also, each point X of $\mathbb{P}^3$, lies in a unique plane $\sigma(X)$ of $\mathbb{P}^3$ such that $(X, \sigma(X)) \in \mathscr{L}(B)$. It is readily seen that σ defines a symplectic polarity, and that $B = \{l \in P \mid \sigma(l) = l\}$, where $\sigma(l)$ is the line $\sigma(X_1) \cap \sigma(X_2)$ whenever $l = X_1 X_2$ for X_1, X_2 distinct points of $\mathbb{P}^3$. □

The embedding of $(P, \mathscr{L})$ in the Veldkamp space is a 'synthetic version' of the well-known Plücker embedding of the polar space $(P, \mathscr{L})$ in the Klein quadric. In the finite case, over $\mathbf{F}_q$, there are $(q^2 + 1)(q^2 + q + 1)$ points in P and $q^2(q^3 - 1)$ hyperplanes corresponding to symplectic polarities, together accounting for the $(q^6 - 1)/(q - 1)$ points of the Veldkamp space.

Projective embeddings. Now we suppose $(P, \mathscr{L})$ has an embedding (W, π, ϕ), that is, a thick projective space W with polarity π such that $\phi\colon P \to W \backslash W^{\perp\pi}$ is an injection mapping lines to lines of W_π, the polar space of totally isotropic points and lines with respect to π, and such that $W = [\phi P]$. (Here we adopt the terminology and much of the notation of Tits [7, §8.5].) We recall that a *morphism* $\mu\colon (\bar{W}, \bar{\pi}, \bar{\phi}) \to (W, \pi, \phi)$ of embeddings of $(P, \mathscr{L})$ is a morphism $\mu\colon \bar{W} \to W$ of projective spaces such that $\bar{\pi} = \mu^*\pi$ and $\phi = \mu\bar{\phi}$ (where $\mu^*\pi(x, y) = \pi(\mu x, \mu y)$ for $x, y \in \bar{W}$), and that an embedding (W, π, ϕ) is called *dominant* if every morphism of embeddings to it is an isomorphism.

By Tits [7, §§8.6 and 8.7], the embeddable polar space $(P, \mathscr{L})$ has a dominant embedding (W, π, ϕ), and $(P, \mathscr{L}) \cong W_\pi$ or W_κ, where κ is a projective pseudo-quadratic form in W with associated polarity π. In particular, up to polar space isomorphism, we have that $(P, \mathscr{L})$ is one of W_π or W_κ, and $\phi = \mathrm{id}_P$.

5.2. PROPOSITION. *Let $(P, \mathscr{L})$ be a nondegenerate polar space of finite rank $\geqslant 3$ of the form W_π or W_κ, where W is a projective space and π is a polarity of W and κ is a projective pseudo-quadratic form with associated polarity π. Suppose that (W, π, id_P) is a dominant embedding. If H is a hyperplane of*

$(P, \mathscr{L})$, then $[H]$, the projective subspace of W spanned by H, is a hyperplane of W, and $H = [H] \cap P$.

Proof. If $[H] \neq W$, then, for $x \in P \backslash H$, by Lemma 2.1, $\langle H, x \rangle = P$, so $[H, x] = [P] = W$, showing that $[H]$ is a hyperplane. Furthermore, $[H] \cap P$ is proper subspace of P containing H, and so, by the same lemma, $H = [H] \cap P$, as required.

Therefore, assume $[H] = W$. Then (W, π, id_H) is an embedding of H. By Tits [7, §8.6] (and the observation rk $H \geqslant$ rk $P - 1 \geqslant 2$), there exists a morphism μ of embeddings of H from a dominant embedding $(\bar{W}, \bar{\pi}, \bar{\phi})$ to (W, π, id_H). (Thus $\mu\bar{\phi} = \mathrm{id}_H$ and $\mu^*\pi = \bar{\pi}$.) Since, by assumption $H \neq P$, Theorem 8.6 and Corollary 8.7 of [7] show that $P = W_\kappa$, and $\phi H = \bar{W}_{\bar{\kappa}}$, where $\bar{\kappa}$ is a projective pseudo-quadratic form with associated polarity $\bar{\pi}$. Now $\mu^*\kappa$ (cf. [7, 8.4.1]) and $\bar{\kappa}$ are both projective pseudo-quadratic forms with associated polarity $\bar{\pi} = \mu^*\pi$, and if $\bar{x} \in \bar{W}$ satisfies $\bar{\kappa}(\bar{x}) = 0$ (in the obvious interpretation that $\bar{q}(\bar{x}) = 0$ for any pseudo-quadratic form $\bar{q}$ representing $\bar{\kappa}$), then $\bar{x} \in \bar{\phi} H$, so $\mu(\bar{x}) \in H \subseteq P$, whence $\mu^*\kappa(\bar{x}) = \kappa(\mu(\bar{x})) = 0$. Thus, as $[\bar{H}] = \bar{W}$, (by [7, 8.2.5]), we have $\bar{\kappa} = \mu^*\kappa$. If $x \in P$, taking $\bar{x} \in \bar{W}$ with $x = \mu(\bar{x})$, we get $\kappa(\bar{x}) = \mu^*\kappa(\bar{x}) = \kappa(x) = 0$, and so $x = \mu(\bar{x}) \in \mu(\phi H) = H$, showing $P = H$, a contradiction. Hence the proposition. □

Non-embeddable polar spaces. The classification of nondegenerate polar spaces possessing non-Desarguesian planes has also been completed by Tits [7, §9.1]. In this case, the planes are defined over a division Cayley algebra C. Conversely, for each division Cayley algebra C there is a unique nondegenerate polar space of rank at least 3 whose planes are defined over C. It has rank 3 and is not embeddable in a projective space.

Throughout the remainder of this section, we let $(P, \mathscr{L})$ be a nondegenerate polar space of finite rank 3 whose planes are defined over a division Cayley algebra C. Denote by n the norm map from C to k, the center of C. The only properties of C that we shall need are that n is an anisotropic quadratic form on C and that $\dim_k C = 8$. By [loc. cit.], $(P, \mathscr{L})$ has rank 3, is uniquely determined up to isomorphism, and for any two noncollinear x, $y \in P$ the subspace $\{x, y\}^\perp$ is isomorphic to the dual Q^* of the generalized quadrangle Q associated with the quadratic form $C \oplus k^4 \to k$ defined by

$$(x_0; x_1, x_2, x_3, x_4) \mapsto n(x_0) - x_1x_3 + x_2x_4$$

where $x_0 \in C$ and $x_1, x_2, x_3, x_4 \in k$.

Let E be the algebraic k-group of linear transformations of $C \oplus k^4$ that is the direct product of the anisotropic orthogonal group (of type D_4) over k on C (acting trivially on the direct summand k^4) and the group GL$(2, k)$ (an algebraic k-group of type A_1T_1, where T_1 indicates a 1-dimensional torus) acting trivially on C and on k^4 via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x_1, x_2, x_3, x_4) = \left(ax_1 + bx_2, cx_1 + dx_2, \frac{(dx_3 + cx_4)}{(ad - bc)}, \frac{(bx_3 + ax_4)}{(ad - bc)}\right).$$

Then E induces a group of automorphisms of Q, whence of Q^*, of type $D_4A_1T_1$, fixing the lines $(0; *, *, 0, 0) := \{(0; x_1, x_2, 0, 0) \mid x_1, x_2 \in k\}$ and $(0; 0, 0, *, *)$ of Q.

The following result, derived by study of Q, establishes nonexistence of certain kinds of hyperplane of Q^*.

5.3. LEMMA. *Let Q^* be as above.*

(i) *Each hyperplane of Q^* of rank 2 is of the form $p^\perp$ for some point p of Q^*.*
(ii) *There is no E-invariant ovoid on Q^* containing $\{x, y\}^\perp$, where $x = (0; *, *, 0, 0)$ and $y = (0; 0, 0, *, *)$.*

Proof. Let H^* be a hyperplane of Q^*. Then its dual H is a set of points and lines of Q with the following properties:

(a) If l, m are lines of H meeting in a point $p \in Q$, then p belongs to H.
(b) If p is a point of H, then any line of Q on p belongs to H.
(c) Every point of Q belongs to at least one line of H.

As a direct consequence of (a) and (b), we obtain:

(d) If p, q are noncollinear points of H, then $p^\perp \cap q^\perp$ is entirely contained in H.

(i) Suppose H^* has rank 2. Then it contains a line, and so H contains a point. Assume that H^* contains no deep point. Then, for each line l of H, there is $m \in H$ with $l \cap m = \varnothing$. We claim that H contains a quadrangle. For take a point p_1 of H and a line l_1 containing p_1. Then, by (b), l_1 belongs to H. As we have just seen, there is a line l_3 in H with $l_1 \cap l_3 = \varnothing$. By the 'one or all' axiom, there is a line l_2 on p_1 with $l_2 \cap l_3 = \{p_2\}$ for some point p_2. Then l_2 and p_2 belong to H in view of (b) and (a). Again by the above, there is a line l_4 in H disjoint from l_2. Now, letting p_3, p_4 be the unique points of $p_2^\perp \cap l_4$, $p_1^\perp \cap l_4$, respectively, we obtain the quadrangle with points p_1, p_2, p_3, p_4 fully contained in H, as claimed. Since the automorphism group of Q is transitive on the set of quadrangles in Q (use that the group is Moufang or Witt's theorem), there is no loss in assuming that $p_1 = (0; 1, 0, 0, 0)k$, $p_2 = (0; 0, 1, 0, 0)k$, $p_3 = (0; 0, 0, 1, 0)k$, and $p_4 = (0; 0, 0, 0, 1)k$ belong to H. Take a, $b \in C$ with $a \neq b$ and $n(a) = n(b) = 1$. (This choice is possible.) Then $p_a = (a; 0, -1, 0, 1)k$ and $p_b = (b; 0, -1, 0, 1)k$ are distinct points of Q contained in $p_1^\perp \cap p_3^\perp$ and so, by (d), belong to H. Moreover, they are noncollinear, so for each $\lambda \in k$, the point $q_\lambda = (0; 1, \lambda, \lambda^2, \lambda)k$, being contained in $p_a^\perp \cap p_b^\perp$, also belongs to H. Finally, p_3 and q_λ are noncollinear, so $(0; 0, 0, \lambda, 1)k$ being in $p_3^\perp \cap q_\lambda^\perp$, belongs to H for each $\lambda \in k$. We conclude that all the points of the line $p_3p_4 = \{(0; 0, 0, \lambda, \mu)k \mid \lambda, \mu \in k\}$ of Q are in H, so that p_3p_4 is a deep point of H^*. This contradicts the assumption, and so ends the proof of (i).

(ii) Suppose H^* is an E-invariant ovoid containing $\{x, y\}^\perp$. Then H is a *spread* of Q, that is, every point of Q belongs to exactly one line of H. Notice that the part of the spread corresponding to the subset $\{x, y\}^\perp$ of H^* covers all points $(x_0; x_1, x_2, x_3, x_4)$ of Q having $x_0 = 0$. Suppose z were the line of H on the point $p := (1; 1, 0, 1, 0)$ of Q. Let $q = (q_0; q_1, q_2, q_3, q_4)$ be a second point of Q on z. Now $p_0 = 1$ and q_0 are linearly independent over k (for otherwise, z would contain a Q-point of $(0; *, *, *, *)$, and so meet a member of $\{x, y\}^\perp$). Replacing q by a $q + \lambda p$ for a suitable $\lambda \in k$, we may assume that q_0 is perpendicular to 1 (with respect to the bilinear form associated with n), that is, $n(q_0 + 1) = n(q_0) + 1$. The stabilizer E_p of p in E contains the orthogonal group of type B_3, so there is an E_p-conjugate $q' = (q_0'; q_1, q_2, q_3, q_4)$ of q with $q_0' \neq q_0$. Thus, if H^* is E-invariant, the line on p and q' also belongs to the spread H, contains p and is distinct from z. This is absurd. Hence the lemma. □

5.4. LEMMA. *Let H be a nondegenerate hyperplane of $(P, \mathscr{L})$. Then H has rank 2. For each $a \in H$, the residue H_a of H at a is an ovoid in the generalized quadrangle P_a with the property that, if xa and ya are noncollinear points of P_a (where $x, y \in a^\perp \backslash \{a\}$) with $|\{xa, ya\}^\perp \cap H_a| > 1$, then $\{xa, ya\}^\perp \subseteq H_a$. Moreover, either $\{x, y\}^\perp \subseteq H$ or $\{x, y\}^\perp \cap H = \{x, y, a\}^\perp$.*

Proof. It is immediate from part (i) of the above lemma that if H is a hyperplane of $(P, \mathscr{L})$ of rank 3, then $H = x^\perp$ for some $x \in P$. Hence the first assertion. Also, the assertion about the ovoid is clear.

For x, y, a as indicated, take u, v to be points of $H \cap a^\perp \backslash \{a\}$ such that ua, va are distinct points of $\{xa, ya\}^\perp \cap H_a$. Since H has rank 2, H_a contains no lines. In particular, the points u, v are noncollinear. The subspace $\{x, y\}^\perp \cap H$ is either a hyperplane of the generalized quadrangle $\{x, y\}^\perp$ or coincides with $\{x, y\}^\perp$. In the latter case, we have $\{x, y\}^\perp \subseteq H$ and we are done, so assume the former. Then $\{x, y\}^\perp \cap H$ is of rank 2 (as it contains the line ua) and, by the previous lemma, has shape $\{x, y, b\}^\perp$ for some $b \in \{x, y\}^\perp$. But $b \in \{x, y, u, a, v\}^\perp = \{a\}$, so $\{x, y\}^\perp \cap H = \{x, y, a\}^\perp$, as required. □

5.5. REMARK. The condition on an ovoid $\mathcal{O}$ of a generalized quadrangle Q^* that $|\{x, y\}^\perp \cap \mathcal{O}| > 1$ for noncollinear points x, y implies $\{x, y\}^\perp \subseteq \mathcal{O}$

seems very strong, but does not suffice for a nonexistence proof as the following construction shows. For the special case $k = \mathbb{R}$, a spread O^* in Q will be given, whose dual has the indicated property. Consider the quadratic form f on $\mathbb{C}^{2n}$ given by

$$f(x) = x_2x_4 - x_1x_3 + \sum_{i=5}^{2n} x_i^2 \quad \left(x = \sum_{i=1}^{2n} x_i e_i\right)$$

with respect to the standard basis $e_1, \dots, e_{2n}$. Then $U = \mathbb{C}(ie_1 + e_2) + \mathbb{C}(e_3 + ie_4) + \mathbb{C}(e_5 + ie_6) + \cdots + \mathbb{C}(e_{2n-1} + ie_{2n})$ is a maximal singular subspace of the polar space $O(f, \mathbb{C}^{2n})$ of (Dynkin) type D_n. Let σ be the semilinear transformation on $\mathbb{C}^{2n}$ performing complex conjugation with respect to the standard basis. Then σ is an automorphism of $O(f, \mathbb{C}^{2n})$ and $U \cap U^\sigma = 0$, while the σ-fixed points of $O(f, \mathbb{C}^{2n})$ form a generalized quadrangle, Q_n say. Observe that Q is isomorphic to Q_6. If $p \in Q_{2n}$, then $p \not\subseteq U^\sigma$, so $\dim_{\mathbb{C}}(U^\sigma + \mathbb{C}p) = n + 1$, whence there exists a point p_U contained in $U \cap (U^\sigma + \mathbb{C}p)$. Moreover, U^σ is a hyperplane of $U^\sigma + \mathbb{C}p_U$, so the line pp_U meets U^σ in a point, q say. As pp_U contains three singular points of $O(f, \mathbb{C}^{2n})$ (namely p, p_U, and q), it is a singular line of the polar space. We claim it is the unique line through p meeting both U and U^σ. (For, if L would be another such line, L and pp_U would span a projective plane meeting both U and U^σ in a line, so the intersection of the latter two lines would be a point of $U \cap U^\sigma$, contradicting $U \cap U^\sigma = \emptyset$.) But pp_U is a line on p meeting both U (in q^σ) and U^σ (in p^σ), so coincides with pp_U. In particular, $q = p^\sigma$. The line pp_U has more than one point of Q_n. (For each point of shape $\lambda a + \bar{\lambda} a_U$ with $\lambda \in \mathbb{C}$, $p = \mathbb{R}a$, $p_U = \mathbb{R}a_U$ for some a, $a_U \in \mathbb{C}^{2n}$ belongs to it.) Thus the σ-fixed points of pp_U form a line of Q_n, which we denote by L_p. By what we have just seen, $O^* = \{L_p \mid p \in Q_n\}$ is a spread of Q_n. If M, N are disjoint lines of Q_n and L_x, L_y distinct members of the spread meeting both M and N, then every line of Q_n meeting both M and N is also of shape L_z for some point z, whence a member of O^*. It follows that O^* provides an ovoid $\mathcal{O}$ of Q_n^* satisfying the property described in Lemma 5.4.

The following lemma states that every point of the Veldkamp space lies on a *secant*, that is a line with at least two points of the form $x^\perp$ for some $x \in P$.

5.6. LEMMA. *Let H be a nondegenerate hyperplane of $(P, \mathscr{L})$. For each quadrangle $V \subset H$, and any two $x, y \in V^\perp \backslash H$, we have $\{x, y\}^\perp = x^\perp \cap H = y^\perp \cap H$.*

Proof. Clearly, x and y are noncollinear. Let $a \perp u \perp b \perp v \perp a$ be the circuit in V. The points xa, ya of P_a are noncollinear and satisfy $\{xa, ya\}^\perp \cap H_a \supseteq \{ua, va\}$. By the previous lemma, $\{x, y\}^\perp \not\subseteq H$ implies

$\{x, y\}^\perp \cap H = \{x, y, a\}^\perp$. But then, likewise we have $\{x, y\}^\perp \cap H = \{x, y, b\}^\perp$, contradicting $a \in \{x, y, a\}^\perp \backslash b^\perp$. Hence $\{x, y\}^\perp \subseteq H$.

Suppose $h \in x^\perp \cap H$. If $z \in y^\perp \cap hx \backslash \{h\}$, then $z \in \{x, y\}^\perp \subseteq H$ and $h \in H$, so $x \in hz \subseteq H$, contradicting $x \notin H$ (for $x \in H$ would imply $\operatorname{rk} H = 3$). Hence $h \in \{x, y\}^\perp$, proving $\{x, y\}^\perp = x^\perp \cap H$. The remainder follows by symmetry in x and y. □

5.7. LEMMA. *Let H be a nondegenerate hyperplane of $(P, \mathscr{L})$. If $a, b \in H$ are distinct, then $\{a, b\}^{\perp\perp} \subseteq H$.*

Proof. If $a \perp b$, then $\{a, b\}^{\perp\perp}$ is the line on a and b, and so belongs to H by the definition of subspace. Otherwise, let $c \in \{a, b\}^{\perp\perp}$ and take distinct $u, v \in \{a, b\}^\perp \cap H$. There are $x, y \in \{a, b, u, v\}^\perp \backslash H$. By the above lemma, $\{x, y\}^\perp \subseteq H$. Thus, $c \in \{a, b\}^{\perp\perp} \subseteq \{x, y\}^\perp \subseteq H$, as required. □

We shall exploit the following description of $(P, \mathscr{L})$. Recall that k is the center of the Cayley division algebra C and that $n\colon C \to k$ is the norm map. By Proposition 7 of Tits [8], there is an algebraic k-group $\mathscr{G}$ of adjoint type E_7 whose anisotropic kernel is isogenous to $\mathrm{SO}(C, n)'$ (the commutator subgroup of the group of all linear transformations the 8-dimensional k-vector space C that leave invariant the norm n). By G we denote the group of all k-rational points of $\mathscr{G}$. We adopt the labeling of Bourbaki [1] for the nodes of the Dynkin diagram of type E_7. According to Tits [7] the group G has a Tits system (B, N, W, R) of type C_3 in which the maximal parabolic subgroups of G containing B are P_1, P_6, and P_7, the indices indicating the nodes of the diagram to which they correspond. The group $B \cap N$ contains the anisotropic kernel. The polar space $(P, \mathscr{L})$ can be viewed as the rank 3 geometry whose points, lines, and planes are the conjugates of respectively P_1, P_6, and P_7 in G, and in which two elements of type $i, j \in \{1, 6, 7\}$ are incident if and only if their intersection contains a conjugate of $P_i \cap P_j$.

A *root group of* G is a subgroup G-conjugate to the center of the unipotent radical of P_1. It is isomorphic to k^+. (The full unipotent radical R is unipotent of dimension 33 over k; its commutator subgroup coincides with the root group.) Denote by P_o the G-class of all root groups. For any two $x, y \in P_o$, we have either $[x, y] = 1$ or $\langle x, y\rangle \cong \mathrm{SL}(2, k)$. We shall write $x \perp y$, and say x and y are collinear if $[x, y] = 1$. This definition is justified by the choice of lines in the following lemma.

5.8. LEMMA. *The polar space $(P, \mathscr{L})$ of rank* 3 *defined over C is isomorphic to the space $(P_o, \mathscr{L}_o)$ whose lines are the sets $\{x, y\}^{\perp\perp}$ for any two distinct $x, y \in P_o$ with $[x, y] = 1$.*

Proof. By what has been said above, P can be viewed as $\{P_1^g \mid g \in G\}$. Now if

$x \in P_o$, then $x = x_1^g$ for some $g \in G$, where x_1 is the center of the unipotent radical of P_1. Thus $N_G(x) \supseteq P_1^g$ and, by maximality of P_1 as a subgroup of G, we have $N_G(x) = P_1^g$; setting $f(x) = P_1^g$, we obtain a well-defined bijection $f: P_o \to P$.

Let $n \in N$ correspond to the fundamental reflection associated with the first node of the Dynkin diagram. Then, by the definition of Tits system (or a direct computation verifying that $P_1 \cap P_1^n$ contains a conjugate of $P_1 \cap P_6$), for any $g \in G$, the point $P_1 \in P$ is collinear with the point P_1^g if and only if $g \in P_1 n P_1$. A direct computation also shows that x and x^n commute. It follows that x is collinear with x^g if and only if the intersection of $f(x) = P_1$ and $f(x^g) = P_1^g$ contains a conjugate of $P_1 \cap P_6$, that is, if and only if P_1 is collinear with P_1^g. Hence the collinearity graphs of $(P_o, \mathscr{L}_o)$ and $(P, \mathscr{L})$ are isomorphic. Since $(P_o, \mathscr{L}_o)$ is built from its collinearity graph in the same way that $(P, \mathscr{L})$ can be obtained from its collinearity graph (namely by letting lines be the double $\perp$s of distinct collinear points), the lemma follows. □

We shall now identify $(P, \mathscr{L})$ and $(P_o, \mathscr{L}_o)$. This enables us to consider P as a collection of (root) subgroups of G, and to consider G as a group of automorphisms of $(P, \mathscr{L})$.

Some more notation: for $S \subseteq G$ and $X \subseteq P$, we write $S \cap X := \{x \in X \mid x \subseteq S\}$ and $G(X)$ for the group generated by all $x \in X$.

5.9. LEMMA. *Let X be a subspace of $(P, \mathscr{L})$. Then*

(i) *$G(X) \cap P$ is a union of $G(X)$-orbits; any two points from different orbits commute.*

(ii) *Suppose $\{u, v\}^{\perp\perp} \subseteq X$ for any two non-collinear $u, v \in X$. Then $y \in X$ implies that the full $G(X)$-orbit $y^{G(X)}$ of y belongs to X.*

Proof. (i) If for $x, y \in G(X) \cap P$, we have $[x, y] \neq 1$, then, as the subgroup of G generated by x and y is isomorphic to $\mathrm{SL}(2, k)$, we have $x \in y^{G(\{x,y\})}$. In particular, $x \in y^{G(X)}$. Thus, any two elements of $G(X) \cap P$ either commute or are $G(X)$-conjugate, whence (i).

(ii) Suppose $y \in X$. By induction on the length of an element of $G(X)$ expressed as a product of elements from root groups, it suffices for the proof of the last statement of the lemma to show that $y^\xi \in X$ for any $\xi \in x \in X$. If $[\xi, y] = 1$, this is trivial, so suppose the contrary. Then $\{x, y\}^{\perp\perp} \subseteq X$ by the assumption on X. On the other hand, $G(\{x, y\})$ stabilizes this subspace (for, it stabilizes $(\langle x, y\rangle \cap P)^\perp$ which, by Lemma 5.8, coincides with $\{x, y\}^\perp$), so $y^\xi \in \{x, y\}^{\perp\perp}$, whence $y^\xi \in X$. □

5.10. PROPOSITION. *Let H be a hyperplane of $(P, \mathscr{L})$. Then $H = x^\perp$ for some $x \in P$.*

Proof. Suppose H is a nondegenerate hyperplane of P of rank 2. By Lemma 5.6 there are noncollinear $x, y \in P$ with $\{x, y\}^{\perp} \subseteq H$. Take $u \in \{x, y\}^{\perp}$ and consider $\mathcal{O} = (H \cap u^{\perp})/u$ of $u^{\perp}/u$. The latter space is isomorphic to Q^* and its subspace $\mathcal{O}$ must be a hyperplane of rank 1, so is an ovoid, and contains $\{xu, yu\}^{\perp}$. By Lemmas 5.7 and 5.9(ii) H is $G(H)$-invariant, so $\mathcal{O}$ is $C_{G(H)}(u)$-invariant. In particular, $C_{G(\{x,y\})}(u)$ induces an algebraic k-group of automorphisms of $u^{\perp}/u$ of type $D_4A_1T_1$ fixing the points xu and yu, and stabilizing $\mathcal{O}$. But then Lemma 5.3(ii) asserts that no such $\mathcal{O}$ exists, a contradiction. Hence the proposition. □

5.11. REMARK. Part of the difficulties in establishing the above result arise from ignorance: we do not know whether, apart from the obvious algebraic subgroups, there are other overgroups in G of the algebraic subgroup $G(\{x, y\}^{\perp})$ of type D_6. (Here x and y are as in the above proposition.) If the classification of all Moufang generalized quadrangles were available, another way to circumvent this problem would be to use that then the overgroup $G(H)$ of the hypothetical nondegenerate hyperplane H of $(P, \mathcal{L})$ containing $\{x, y\}^{\perp\perp}$ is known:

LEMMA. *Any nondegenerate hyperplane H of rank* 2 *is a Moufang generalized quadrangle* (cf. p. 274 of Tits [7]) *and the group of all root automorphisms U_α (α a root of H; notation of* [loc. cit.]) *belongs to $G(H)$.*

Proof. Suppose $a \perp u \perp b \perp v \perp a$ are the points of a quadrangle in H. Then the points and lines of this quadrangle form an apartment of H. To verify that H is Moufang, we shall consider the representatives $\pi = \{u, ua, a, av, v\}$ and $\lambda = \{ub, u, ua, a, av\}$ of the two kinds of roots, and verify that $G(\{a, u, v\} \cup ua)$ contains the subgroup U_α for both $\alpha = \pi, \lambda$.

Assume $\alpha = \pi$. Then, by definition, U_α is the kernel of the action of the pointwise stabilizer in Aut(H) of $\{u, a, v\} \cup ua \cup va$ on the set of all lines through $a^{\perp}$. Assume $b' \in \{u, v\}^{\perp} \cap H$. It suffices to find an element in $U_\alpha \cap G(ua)$ moving b to b'. The subgroup $C_G(u, v)$ of G centralizing u and v contains the orthogonal group D of type D_6 over k of Witt index 2; the root groups a, b, and b' belong to it and are 'classical root groups' of the classical group D (centers of the unipotent radical of the stabilizer of a line in the classical embeddable generalized quadrangle Q associated with D). In particular, there is $A \in a \in D$, fixing $\{a\}^{\perp}$ pointwise, such that $b^A = b'$. But then $A \in a \subseteq U_\alpha \cap G(a)$, as required.

$\alpha = \lambda$. Then U_α is the kernel of the action of the pointwise stabilizer of ua on the set of all lines passing through u or a. Assume $b' \in ub \backslash \{u\}$ and $v' \in av \backslash \{a\}$ are collinear. We need to find $B \in U_\alpha$ moving bv to $b'v'$. By taking $x, y \in \{a, b, u, v\}^{\perp}$, and considering $C_G(x, y)$ (again an algebraic k-group of type D_6 corresponding to the 12-dimensional orthogonal group over k of Witt index 2), an element $B \in G(ua) \cong (k^+)^{10}$ can be found that fixes every line on a and on u lying in $\{x, y\}^\perp$, and satisfies $b^B = b'$ and $v^B = v'$. As $ua \subseteq H$ and B also fixes x and y, we have $B \in U_\alpha \cap G(ua)$. Hence the lemma. □

We end the remark by indicating how we could use the above lemma to prove that every hyperplane of $(P, \mathscr{L})$ is of the form $x^\perp$ if the only Moufang generalized quadrangles are the known ones: Suppose H is a hyperplane without a deep point. Then, as in the proof of Proposition 5.9 we can show that H is a nondegenerate generalized quadrangle and that there exist noncollinear $x, y \in P \backslash H$ with $\{x, y\}^\perp \subseteq H$. Now, $D := G(\{x, y\}^\perp)$ is an algebraic k-group of type D_6 (Witt index 2) contained in H. But then, by the (*assumed*) classification of Moufang generalized quadrangles, $G(H)$ is an algebraic k-group which is an overgroup in G. The only maximal proper algebraic k-groups which are overgroups of D are parabolics of type D_6 and $DC_G(D) \cong D$. SL$(2, k)$. If $G(H)$ is a parabolic R of type D_6, we get $H \subseteq R \cap P = z^\perp$ for some root z in the center of the unipotent radical of R, and if $G(H) \subseteq DC_G(D)$, then $H \subseteq G(H) \cap P \subseteq (DC_G(D)) \cap P = \{x, y\}^\perp \cup \{x, y\}^{\perp\perp}$. In both cases, according to Lemma 5.9, the rank of H must be 3, a contradiction. Since also $G(H) = G$ leads to the contradiction $H = P$, this again establishes the nonexistence of rank 2 hyperplanes in $(P, \mathscr{L})$.

Summarizing the three propositions in this section, we obtain

5.12. THEOREM. *Every hyperplane of a polar space of rank at least* 3 *that is not of the form* $x^\perp$ *for some point x, arises from a suitable embedding of the polar space in a projective space by intersecting it with a hyperplane of that projective space.*

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