EXACT STATIC EQUILIBRIUM OF VERTICALLY ORIENTED MAGNETIC FLUX TUBES

I. *The Schlüter-Temesváry Sunspot*

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Abstract. A method is prescribed for generating exact solutions of magnetostatic equilibrium describing a cylindrically symmetric magnetic flux tube oriented vertically in a stratified medium. Given the geometric shape of the field lines, compact formulae are presented for the direct calculation of all the possible distributions of pressure, density, temperature and magnetic field strength compatible with these field lines under the condition of static equlibrium. The plasma satisfies the ideal gas law and gravity is uniform in space. A particular solution is obtained by this method for a medium sized sunspot whose magnetic field obeys the similarity law of Schlüter and Temesváry (1958). With this solution, it is possible for the first time to illustrate explicitly the confinement of the magnetic field of the cool sunspot by the hotter external plasma in an exact relationship involving both magnetic pressure and field tension as well as the support of the weight of the plasma by pressure gradients. It is found that the cool region of the sunspot is not likely to extend much more than a few density scale heights below the photosphere. The sunspot field approaches being potential in the neighbourhood of the photosphere so that the Lorentz force exerting on the photosphere is less than what the magnetic pressure would suggest. This accounts for how the sunspot field can be confined in the photosphere where its magnetic pressure is often observed to even exceed the normal photospheric pressure. The energy mechanism operating in the sunspot and the question of mechanical stability are not treated in this paper.

1. Introduction

The physical problem posed by the sunspot and other related structures is a classical one (Tandberg-Hanssen, 1967; Zwaan, 1968). Interest in this problem was renewed recently by the efforts to understand the remarkable observation that the general photospheric magnetic field resides in isolated narrow flux tubes of 1500 to 2000 G (see e. g. Harvey, 1977). The reader is referred to a recent article of Parker (1977a) for a detailed review of the theoretical questions and some of their possible answers. There is still much physics at the fundamental level that is not understood. The basic effects are nonlinear posing no mean obstacle to quantitative analysis and theoretical development must proceed in small steps. In this paper, we go back to look at the old problem of constructing magnetostatic models of the sunspot. A survey of previous work (Tandberg-Hanssen, 1967) shows that available models either are phenomenological in nature or are based on ingenious theoretical constructions to circumscribe the direct solution of the nonlinear equations of magnetostatics. Although these models have provided valuable insights into the difficult problem, the need remains to obtain exact magnetostatic solutions to illustrate basic physical

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properties with confidence. This paper is aimed at meeting this need. We first adapt a method developed elsewhere (Low, 1980a) to construct exact solutions for a cylindrically symmetric flux tube penetrating vertically through a stratified atmosphere in static equilibrium. Using this method, we present a series of solutions in this paper and another to follow (Low, 1980b). At this stage of development, we will not attempt at building complete models. The complex phenomenon of sunspot-like structures is the result of force balance and energy transport. Our interest here is to look at the simplest analytic particular solutions to learn something of the largely unexplored static interaction between gravity, magnetic field and plasma. Apart from selecting for presentation only those solutions having qualitatively reasonable temperature distributions, we will not treat the question of energy transport.

Let us review briefly the method of constructing exact solutions presented recently (Low, 1980a). We showed that given a magnetic field depending on only two Cartesian coordinates, a simple transformation of functions can be exploited to calculate in closed forms the distributions of pressure, density and temperature required for static equilibrium. The embedding plasma was assumed to satisfy the ideal gas law and gravity was taken uniform in space. Solutions can thus be generated from variously prescribed magnetic fields. For the general magnetic field depending on all three coordinates, essentially the same procedure for generating solutions is possible except that not every magnetic field can be a candidate for equilibrium. Here, we encountered a theorem due to Parker (1972, 1977a, 1979) that a magnetic field in static equilibrium must possess suitable symmetries. We derive the equation stipulating these symmetries in terms of a pair of Euler potentials defining the magnetic field. Starting with a magnetic field that satisfies Parker's theorem, equilibrium is possible and the endowed symmetry of the field makes the construction of the magnetostatic equilibrium a two dimensional problem parallel to the special case of dependence on two Cartesian coordinates where the field symmetry is the invariance in the direction of the third coordinate. Another special case amendable to an analytic treatment is the cylindrically symmetric field which is the subject of the present paper.

The equations of equilibrium are:

$$\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}-\nabla p-\rho g\hat{z}=0\,, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0\,, \tag{2}$$

$$p=\rho kT/m\,, \tag{3}$$

where $\mathbf{B}$ is the magnetic field, p the pressure, ρ the density, and T the temperature. The unit vector $\hat{z}$ points vertically upwards and g is a uniform gravitational acceleration. The ideal gas law (3) is for a fully ionised hydrogen plasma with m as the proton mass and k the Boltzman constant. Equations (1)–(3) make up five equations for the six unknowns p, ρ, T and the three components of $\mathbf{B}$. Taking one of the six unknowns as given, the five equations express the other five unknowns in terms of it.

If the unknown to be given is chosen correctly, the other five unknowns can be integrated for in closed forms for the cylindrically symmetric field. This is demonstrated in Section 2. We point out that we do not consider the energy equation directly. Otherwise, the energy equation together with Equations (1)–(3) form a complete set and all six unknowns have to be solved for together consistently. This formidable task should be postponed until we have explored magnetostatic equilibrium on the limited basis pursued here. Section 3 is devoted to presenting an exact solution for a solar magnetic field concentrated into an isolated flux rope. The starting point is the sunspot model of Schlüter and Temesváry (1958). This model is based on a prescribed similarity law for the distribution of the sunspot field such that the field is uniquely determined by the difference in pressures at the sunspot axis and at infinity at the same height. At the time when it was not feasible to solve the equilibrium equations completely, this model provided a valuable basis for theoretical discussion. Using the method of Section 2, we can now obtain the full distributions of pressure, density and temperature in space. A number of interesting results are obtained based on the particular solution of Section 3. Section 4 presents a discussion.

2. Cylindrically Symmetric Fields

Express the cylindrically symmetric field $\mathbf{B}$ in terms of two scalar functions H and K:

$$\mathbf{B} = B_0\left(-\frac{1}{hr}\frac{\partial H}{\partial hz}, \frac{1}{hr}K, \frac{1}{hr}\frac{\partial H}{\partial hr}\right), \tag{4}$$

where B_0 is a constant field strength and h^{-1} is a suitable scale length. We use cylindrical coordinates r, θ, and z and under the assumed symmetry, H and K are independent of θ. In this representation, Equation (2) is automatically satisfied. From the equation for the field lines,

$$\mathrm{d}r/B_r = r\,\mathrm{d}\theta/B_\theta = \mathrm{d}z/B_z, \tag{5}$$

it is simple to show that H is constant along individual field lines. In particular, the projection of the field lines on the θ = constant plane are contours of constant H. For a given H, an arbitrary function $\Phi\,(r, z)$ may be defined in terms of its values obtained on the contours of constant H at height z,

$$\Phi(r, z) = \Phi\mathbf{[}H, z\mathbf{]}, \tag{6}$$

where we denote with bold face brackets this type of transformation. The reader should bear in mind that there is no loss of generality through this transformation except that $\Phi\mathbf{[}H, z\mathbf{]}$ may be multi-valued. Using this transformation, we showed previously (Low, 1975) that Equations (1)–(3) reduce to:

$$p = p_0 P_0\mathbf{[}H\mathbf{]} \exp\left(-\int_0^z \frac{\mathrm{d}Z'}{\Theta\mathbf{[}H, Z'\mathbf{]}}\right), \tag{7}$$

$$\rho = \rho_0 P_0[H] \exp\left(-\int_0^Z \frac{dZ'}{\Theta[H, Z']}\right) \Big/ \Theta[H, Z] , \tag{8}$$

$$T = T_0 \Theta[H, Z] , \tag{9}$$

$$\mathbf{B} = B_0\left(-\frac{1}{R}\frac{\partial H}{\partial Z}, \frac{1}{R}K[H], \frac{1}{R}\frac{\partial H}{\partial R}\right) , \tag{10}$$

where the functions p_0, Θ, H, and K are related by the equation

$$\frac{\partial^2 H}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial H}{\partial R}\right) + K\frac{\partial}{\partial H}K[H] + + \beta R^2 \frac{\partial}{\partial H}\left[P_0[H] \exp\left(-\int_0^Z \frac{dZ'}{\Theta[H, Z']}\right)\right] = 0 . \tag{11}$$

The partial differentiation with respect to H in Equation (11) is to be performed with Z held constant. We have inserted the constants p_0, ρ_0, T_0 and defined the following to put Equation (11) in the dimensionless form:

$$h = mg/kT_0 , \tag{12}$$

$$R = hr , \qquad Z = hz , \tag{13}$$

$$\beta = 4\pi p_0 / B_0^2 , \tag{14}$$

$$p_0 = \rho_0 k T_0 / m . \tag{15}$$

The above reduction of equations separates force equilibrium into components parallel and perpendicular to the local magnetic field. Equation (7) shows that pressure is scale heighted by graivity along individual field lines as though the latter are rigid tubes. This is force balance along the field lines. Equation (11) gives force balance in the direction perpendicular to the field and lying in the θ = constant plane. Force balance in the second independent perpendicular direction, namely, the one pointing out of the θ = constant plane is automatic because of cylindrical symmetry. The problem is similar to the one where the system depends on only two Cartesian coordinates treated in Low (1980a). The reader is assumed to be familiar with the mathematical steps of this paper and we will be brief in the following description of a similar procedure to construct solution for the cylindrically symmetric field.

Of the functions H, K, P_0, and Θ, we may freely specify one of them. To generate solutions, H is specified and in terms of it, we evaluate,

$$S(R, Z) = \frac{\partial^2 H}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial H}{\partial R}\right) . \tag{16}$$

Transform S into its dependence on H and Z by Equation (6). Invert the function $H(R, Z)$ to give

$$R^2 = R^2[H, Z] . \tag{17}$$

Equation (11) becomes,

$$S[H,Z]+K[H]\frac{\partial}{\partial H}K[H]+$$

$$+\beta R^2[H,Z]\frac{\partial}{\partial H}\left[P_0[H]\exp\left(-\int_0^Z\frac{\mathrm{d}Z'}{\Theta[H,Z']}\right)\right]=0\,. \tag{18}$$

It is evident that for any given pair of $K[H]$ and $P_0[H]$, Equation (18) gives Θ by direct integration with respect to H. From the explicit forms of H, K, P_0, and Θ obtained this way, Equations (7)–(10) give the pressure, density, temperature and the magnetic field in terms of their variation in space.

To be more sophisticated, let

$$H(R,Z)=\tilde{H}(\phi)\,, \tag{19}$$

where $\tilde{H}$ is an arbitrary function of one variable and ϕ is a given function of R and Z. Equation (19) contains the set of all functions H having the same contours ϕ = constant on each of which H has a constant value. The magnetic field Equation (10) becomes

$$\mathbf{B}=B_0\left[-\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\frac{1}{R}\frac{\partial\phi}{\partial Z},\frac{1}{R}K[\tilde{H}(\phi)],\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\frac{1}{R}\frac{\partial\phi}{\partial R}\right]. \tag{20}$$

Note that

$$B_r^2+B_z^2=B_0^2\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\frac{1}{R^2}\left[\left(\frac{\partial\phi}{\partial Z}\right)^2+\left(\frac{\partial\phi}{\partial R}\right)^2\right]. \tag{21}$$

We are therefore looking at the set of all fields having the same field line projection on the θ = constant plane but having different distributions of transverse field strength $(B_r^2+B_z^2)^{1/2}$ on the projected field lines. For a given ϕ, evaluate,

$$M_1=\frac{\partial^2\phi}{\partial Z^2}+R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\phi}{\partial R}\right), \tag{22}$$

$$M_2=\left(\frac{\partial\phi}{\partial Z}\right)^2+\left(\frac{\partial\phi}{\partial R}\right)^2, \tag{23}$$

and use an obvious modification of Equation (6) to transform these quantities into dependence on ϕ and Z. Invert $\phi(R,Z)$ to obtain R^2 in terms of ϕ and Z. For H given by Equation (19), Equation (11) now becomes

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}M_1[\phi,Z]+\frac{\mathrm{d}^2\tilde{H}}{\mathrm{d}\phi^2}M_2[\phi,Z]+K[\tilde{H}]\frac{\mathrm{d}}{\mathrm{d}\tilde{H}}K[\tilde{H}]+$$

$$+\beta R^2[\phi,Z]\frac{\partial}{\partial\tilde{H}}\left[P_0[\tilde{H}]\exp\left(-\int_0^Z\frac{\mathrm{d}Z'}{\Theta[\tilde{H},Z']}\right)\right]=0\,. \tag{24}$$

Multiply across by $\mathrm{d}\tilde{H}/\mathrm{d}\phi$ converts Equation (24) into a partial differential equation with ϕ and Z as the independent variables,

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2 M_1[\phi, Z] + \frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\frac{\mathrm{d}^2\tilde{H}}{\mathrm{d}\phi^2} M_2[\phi, Z] + K[\tilde{H}(\phi)]\frac{\partial}{\partial\phi}K[\tilde{H}(\phi)] + \\ + \beta R^2[\phi, Z]\frac{\partial}{\partial\phi}\left[P_0[\tilde{H}(\phi)]\exp\left(-\int_0^Z \frac{\mathrm{d}Z'}{\Theta[\tilde{H}(\phi), Z']}\right)\right] = 0\,. \tag{25}$$

Compare Equation (25) to the more restricted case of Equation (18). In addition to the free functions K and P_0, we also have the free function $\tilde{H}(\phi)$ corresponding to a freedom to change the loading of the transverse field strength on the field lines. For each choice of these arbitrary functions, it is straight forward to integrate Equation (25) with respect to ϕ to obtain Θ. Substituting ϕ, $\tilde{H}$, K, P_0, and Θ into Equations (7)–(9) and (20) then gives the pressure, density, temperature and magnetic field in terms of their spatial variations. It is convenient to define the dimensionless pressure,

$$P = p/p_0\,. \tag{26}$$

Inspection of Equation (7) shows that Equation (25) can be rewritten,

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2 M_1[\phi, Z] + \frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\frac{\mathrm{d}^2\tilde{H}}{\mathrm{d}\phi^2} M_2[\phi, Z] + K[\tilde{H}(\phi)]\frac{\partial}{\partial\phi}K[\tilde{H}(\phi)] + \\ + \beta R^2[\phi, Z]\frac{\partial}{\partial\phi}P[\tilde{H}(\phi), Z] = 0\,. \tag{27}$$

It becomes apparent that integration with respect to ϕ gives the relation between pressure and magnetic field across field lines.

3. The Schlüter-Temesváry Sunspot

In the Schlüter and Temesváry (1958) model of the sunspot, the ansatz is made that the vertical component of the cylindrically symmetric field obeys the similarity law,

$$B_z(r, z) = B_z(0, z) D[hr\zeta(hz)]/D(0)\,, \tag{28}$$

where D and ζ are functions of one variable. We have inserted the constant h for later convenience. In other words, the ratio of the vertical component $B_z(r, z)$ to its central intensity $B_z(0, z)$ at the same depth depends on z only by a scale factor ζ. The field is untwisted and the azimuthal component B_θ is zero. The field lines in the $\theta = \text{constant}$ plane are given by

$$hr\zeta(hz) = \text{constant} \tag{29}$$

while $1/\zeta$ describes the dependence of the diameter of the flux tube on z. The particular case of

$$B_z(0, z) = B_0\zeta^2(hz) D(0) \tag{30}$$

was considered. Integrate the r component of Equation (1),

$$B_z\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)-4\pi\frac{\partial p}{\partial r}=0\,, \tag{31}$$

with respect to r over the range from 0 to ∞. Impose the boundary condition that $\mathbf{B}$ vanishes at infinity to obtain

$$\zeta\frac{\mathrm{d}^2\zeta}{\mathrm{d}(hz)^2}\int_0^\infty D^2(\alpha)\alpha\;\mathrm{d}\alpha-\tfrac{1}{2}D^2(0)\zeta^4=-\frac{4\pi}{B_0^2}[p(\infty,z)-p(0,z)]\,. \tag{32}$$

The run of pressure with height at the sunspot axis and at infinity therefore determines ζ which in turn completely determine B_z through Equations (28) and (30). In this model, then, the sunspot field may be studied theoretically without having to obtain the full distributions of pressure, density and temperature.

Let us relate the above model to the analysis of Section 2. Identify the contours of constant ϕ with the contours defined by Equation (29), that is, set ϕ to be a function of $hr\zeta(hz)$. For the sake of neatness of mathematical expressions in the equations we will derive, let

$$\phi=R^2F(Z)\,, \tag{33}$$

where

$$F(Z)=\zeta^2(hz)\,. \tag{34}$$

In terms of the function H defined in Equation (4), it follows from Equations (28) and (30) that

$$\frac{1}{R}\frac{\partial H}{\partial R}=\zeta^2(Z)D[R\zeta(Z)]\,, \tag{35}$$

which upon integration, becomes Equation (19) with

$$\tilde{H}=\tfrac{1}{2}\int_0^\phi D(\phi^{1/2})\;\mathrm{d}\phi\,. \tag{36}$$

Direct computation of $M[\phi, Z]$, $M_2[\phi, Z]$ and $R^2[\phi, Z]$ puts Equation (27) in the form

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}+\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\frac{\mathrm{d}^2\tilde{H}}{\mathrm{d}\phi^2}\left[\frac{1}{F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\phi+4F^2\right]+$$

$$+\frac{1}{\phi}K\frac{\partial K}{\partial\phi}F+\beta\frac{\partial P}{\partial\phi}=0\,. \tag{37}$$

Integration with respect to ϕ gives

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\left[\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\phi+2F^2\right]+\int_0^\phi\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\mathrm{d}\phi\left[\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}-\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\right]+$$

$$+\int_0^\phi\frac{1}{\phi}K\frac{\partial K}{\partial\phi}\,\mathrm{d}\phi\,F+\beta P=Q\,, \tag{38}$$

where $Q(Z)$ is an arbitrary function. This integral defines P in terms of the free functions F, K, Q, and $\tilde{H}$ (or D in Equation (35)). Going back to Equations (7) and (26), the temperature Θ is determined in terms of another free function P_0. In the final step, Equations (7)–(9) and (20) give the explicit variation of the plasma and magnetic field in space.

Since $B_\theta=0$ in the Schlüter-Temesváry model, set $K=0$ in Equations (20) and (38),

$$\mathbf{B}=B_0\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\left(-R\frac{\mathrm{d}F}{\mathrm{d}Z},0,2F\right), \tag{39}$$

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\left[\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\phi+2F^2\right]+$$

$$+\int_0^\phi\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\mathrm{d}\phi\left[\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}-\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\right]+\beta P=Q\,. \tag{40}$$

A little algebra transforms Equation (40) into the form

$$\beta P+\tfrac{1}{2}(\mathbf{B}/B_0)^2+\int_0^\phi\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\mathrm{d}\phi\left[\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}-\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\right]=Q \tag{40'}$$

which exhibits the balance of pressure against both magnetic pressure and field tension. The last term on the left side is the field tension. The Schlüter-Temesváry Equation (32) is contained in Equation (40). For an isolated flux tube, **B** vanishes at $R=\infty$. Setting $R=0$ and ∞ successively in Equation (40) and taking the difference,

$$\int_0^\infty\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2\mathrm{d}\phi\left[\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}-\frac{1}{2F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\right]-\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)_{\phi=0}2F^2=$$

$$=-\beta[P(\infty,Z)-P(0,Z)]\,. \tag{41}$$

Then use Equations (34) and (36) to express $\mathrm{d}\tilde{H}/\mathrm{d}\phi$ and F in terms of D and ζ and Equation (32) is reproduced. In fact, Equation (40) contains the complete information on all possible equilibria having the field geometry (33).

We look at the full solution of the following field geometry first considered by Schlüter and Temesváry (1958):

$$F = \frac{1}{a^2 + Z^2}, \tag{42}$$

where a is a constant parameter. The field lines $R^2F = \text{constant}$ are sketched in Figure 1. We have a bundle of lines extending to infinity in both directions and

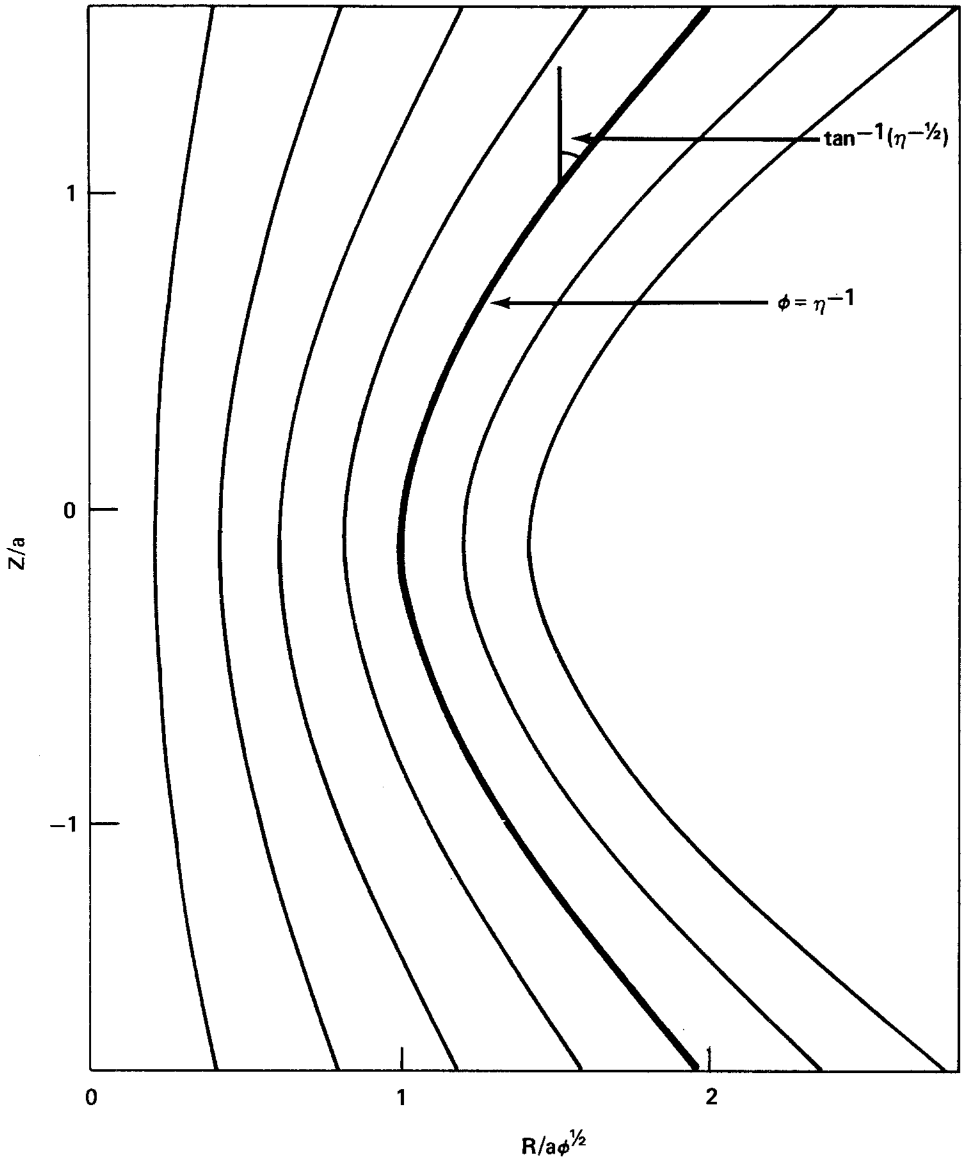

Fig. 1. Field geometry of the Schlüter–Temesváry sunspot.

constricted into a narrow waist at $Z=0$. At infinity, the lines are straight as though radiating from a monopole at the origin. The field lines are perfectly symmetrical about the plane $Z=0$ as evident from Equation (42).

Put Equation (42) into Equation (40),

$$\beta P = Q - \left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2 \left[\frac{2Z^2\phi}{(a^2+Z^2)^3} + \frac{2}{(a^2+Z^2)^2}\right] + \\ + \int_0^{\phi} \left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2 \mathrm{d}\phi \left[\frac{6Z^2}{(a^2+Z^2)^3} - \frac{2}{(a^2+Z^2)^2}\right]. \tag{43}$$

Impose the normalisations,

$$\int_0^{\infty} \left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)^2 \mathrm{d}\phi = \frac{1}{2\eta}, \tag{44}$$

$$\left(\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi}\right)_{\phi=0} = 1, \tag{45}$$

where η is a constant. Schlüter and Temesváry set $\eta = 1$ to obtain the simple expression for the pressure difference,

$$\beta[P(\infty, Z) - P(0, Z)] = \frac{3}{(a^2+Z^2)^3}. \tag{46}$$

For the present we leave η unspecified. The function

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}\phi} = \exp(-\eta\phi) \tag{47}$$

satisfies the above normalisations. Substitute it into Equation (39),

$$\mathbf{B} = 2B_0 \exp\left(-\eta \frac{R^2}{a^2+Z^2}\right) \left[\frac{RZ}{(a^2+Z^2)^2}, 0, \frac{1}{a^2+Z^2}\right], \tag{48}$$

which is a magnetic field whose vertical component at a fixed height has a gaussian decline over a radial distance proportional to $\eta^{-1/2}$. This is an attractive feature consistent with observation (Bumba, 1960). Using Equation (47) and redefining the function $Q(Z)$, Equation (43) becomes

$$\beta P(R, Z) = \beta P(\infty, Z) - \exp(-2\eta\phi) \left[\frac{2\phi + 2 - (2/\eta)}{(a^2+Z^2)^2} + \frac{a^2[(3/\eta) - 2\phi]}{(a^2+Z^2)^3}\right]. \tag{49}$$

The pressure is composed of a component equal to the external pressure and another arising from its interaction with the magnetic field.

Magnetic fields in the photosphere clump into concentrated fluxes of kilogauss fields. The flux diameters range from a few hundred kilometers for the narrow tubes making up the general photospheric field to two orders of magnitude larger for the sunspots. It had been suggested that the field concentration is localised in a thin layer below the photosphere (e.g. Schlüter and Temesváry, 1958; Parker, 1977a). The situation is as shown in Figure 1 where the narrow waist is located near the photospheric level and the constriction is the result of local cooling and compression by the normal photospheric pressure outside. We now generate a solution for a moderate sized sunspot. With P given by Equation (49), Equations (7)–(9) and (26) give

$$p = p_0\left\{P(\infty, Z) - \beta^{-1}\exp(-2\eta\phi)\left[\frac{2[\phi+1-(1/\eta)]}{(a^2+Z^2)^2} + \frac{a^2[(3/\eta)-2\phi]}{(a^2+Z^2)^3}\right]\right\}, \quad (50)$$

$$\rho = \rho_0\left\{-\frac{\partial P(\infty, Z)}{\partial Z} - \beta^{-1}\exp(-2\eta\phi)\times \right.$$

$$\left.\times\left[\frac{8Z[\phi+1-(1/\eta)]}{(a^2+Z^2)^3} + \frac{6a^2Z[(3/\eta)-2\phi]}{(a^2+Z^2)^4}\right]\right\}, \quad (51)$$

$$T = T_0(P/p_0)/(\rho/\rho_0)\,. \quad (52)$$

The external pressure $P(\infty, Z)$ is in pure hydrostatic equilibrium with gravity and decreases with height. Its derivative is negative and the first term on the right side of Equation (51) is therefore positive and in fact is the external density. The choice of $P(\infty, Z)$ is limited to functions such that p and ρ, and hence T, do not take negative values in the domain $\phi \geq 0$ and $-\infty < Z < \infty$. This requirement sets lower bounds on both the external pressure and its gradient at each height Z. The bound on the pressure is the minimum required for field confinement while the bound on its gradient ensures that it is sufficient to support the weight of plasma along field lines. Notice that all terms in Equations (50)–(52) which depend on the magnetic field through ϕ are of bounded variation in the domain $\phi \geq 0$ and $-\infty < Z < \infty$. Many choices of $P(\infty, Z)$ are available such that p and ρ are positive in the domain of interest. Moreover, these magnetic field dependent terms vanish for large Z showing that the magnetic field exerts negligible force on the plasma outside the region of the narrow waist. The field is largely potential in the far away region.

Before proceeding further, let us give meaning to the parameters η and a. Equation (48) shows that the magnetic field strength is proportional to $\exp(-\eta\phi)$ and the field line $\eta\phi = 1$ may be taken to be the flux tube boundary, which is the curve,

$$R = \eta^{-1/2}(a^2+Z^2)^{1/2}\,, \quad (53)$$

shown in heavy line in Figure 1. At large Z, this curve makes an angle of $\tan^{-1}\eta^{-1/2}$ with the vertical. The width of the flux tube at its waist is $\eta^{-1/2}a$ and it follows that the smaller a is the narrower is the waist for the same inclination angle. Observation suggests that the inclination angle at the edge of the sunspot umbra is about 30° (Treanor, 1960; Nishi, 1962; Adam, 1963; see also Henoux, 1963). We shall follow Schlüter and Temesváry (1958) in setting $\eta = 1$ corresponding to an inclination angle of 45°. This choice of parametric value simplifies the expressions in Equations (50)–(52).

Let us take the external pressure in the neighbourhood of the waist of the field to be due to a plasma of 5000 K and identify the reference pressure p_0 to be the external pressure at $Z=0$. We recall from Equations (12) and (13) that the physical length scale of the system is h^{-1} which is determined by the reference temperature T_0. Set $T_0 = 5000$ K and the external pressure is given by

$$P(\infty, Z) = \exp(-Z)\,. \tag{55}$$

Equations (50)–(52) with $\eta = 1$ become

$$p = p_0\left\{\exp(-Z) - \beta^{-1}\exp(-2\phi)\left[\frac{2\phi}{(a^2+Z^2)^2} + \frac{a^2(3-2\phi)}{(a^2+Z^2)^3}\right]\right\}, \tag{56}$$

$$\rho = \rho_0\left\{\exp(-Z) - \beta^{-1}\exp(-2\phi)\left[\frac{8Z\phi}{(a^2+Z^2)^3} + \frac{6a^2Z(3-2\phi)}{(a^2+Z^2)^4}\right]\right\}, \tag{57}$$

$$T = T_0\left\{\frac{\exp(-Z) - \beta^{-1}\exp(-2\phi)\left[\frac{2\phi}{(a^2+Z^2)^2} + \frac{a^2(3-2\phi)}{(a^2+Z^2)^3}\right]}{\exp(-Z) - \beta^{-1}\exp(-2\phi)\left[\frac{8Z\phi}{(a^2+Z^2)^3} + \frac{6a^2Z(3-2\phi)}{(a^2+Z^2)^4}\right]}\right\}. \tag{58}$$

Let us discuss the qualitative properties of this solution. Take the density along the flux axis,

$$\rho_{\phi=0} = \rho_0\left[\exp(-Z) - \beta^{-1}\frac{18a^2Z}{(a^2+Z^2)^4}\right]. \tag{59}$$

Compared to the normal photosphere at the same height, density at the flux axis is depleted for $Z>0$. The surface of constant density therefore has a depression in the flux and this in the Wilson effect (Tandberg-Hanssen, 1967). Surfaces of constant density are defined by $\rho[\phi(R,Z), Z] = \text{constant}$. If we set $Z=a$ in Equation (57), it is easy to show that as ϕ decreases from infinity to zero, density decreases monotonically from its external value. The surfaces of constant density in the neighbourhood of $Z=a$ each has a simple depression at the flux center. Of the two ϕ dependent terms in Equation (57), the term in $(a^2+Z^2)^{-3}$ dominates for sufficiently large Z and the minimum density at a given height is no longer located at the flux center but has shifted to a ring around the flux center. The central depression of the surfaces of constant density at these heights contains a local maximum which by virtue of Equation (59) does not rise above the surface at infinity.

The Wilson effect is caused by the temperature being lower in the flux tube where the density scale height is accordingly smaller. Temperature varies as follows on the level $Z=0$,

$$T_{Z=0} = T_0[1 - \beta^{-1}\,3\exp(-2\phi)/a^4]\,. \tag{60}$$

As the flux axis is approached from infinity, temperature drops from $T_0 = 5000$ K to

$$T_{\substack{\phi=0\\ Z=0}} = T_0(1-\beta^{-1}\,3/a^4)\,. \tag{61}$$

The magnetic field is concentrated by the larger pressure of the hotter exterior. To illustrate this basic effect, we note from Equation (48) that the total flux through the level $Z=0$ is just B_0 apart from numerical factors since the radius of the flux at this level is a. Fix the total flux as well at the external pressure p_0. The parameter β defined in Equation (14) is also fixed. Then reducing the central temperature $T_{\substack{\phi=0\\ Z=0}}$ corresponds to reducing the radius of the flux tube, and the mean field $\mathbf{B}_{\substack{\phi=0\\ Z=0}}$ of Equation (48) is enhanced according to direct compression. This compression effect is reflected by the pressure distribution at the level $Z=0$,

$$p_{Z=0} = p_0[1-\beta^{-1}\,3\exp(-2\phi)/a^4]\,. \tag{62}$$

It has the same dependence on ϕ as $T_{Z=0}$ since $\rho_{Z=0}$ on this level is independent of ϕ.

The photosphere is to be taken at $Z = Z_0$ where Z_0 is a positive constant to be constructed as follows. As noted earlier the ϕ dependent terms in Equations (50)–(52) are bounded in the domain $\phi \geq 0$ and $-\infty < Z < \infty$. For a large enough β with p_0 fixed, p and ρ can be made positive everywhere below a given height because of the rapid rise of e^{-Z} for negative Z. In the lower depths, the constituents of the flux tube do not differ from the external plasma, a point noted by Schlüter and Temesváry. The cool region of the flux therefore does not extend more that a depth equal to the product of a with the density scale height h^{-1}. Above the given height, p and ρ rapidly become negative because of the same mathematical property of e^{-Z}. This only means that the external pressure must be continued smoothly in this region into another function of Z that decreases suitably less rapidly with height than e^{-Z}. This corresponds to requiring the external temperature to rise from 5000 K, in the upper part of the atmosphere, which is a well known feature of the solar atmosphere. Assume that this construction has been done for the level $Z = Z_0$, the pressure at the flux axis is

$$p_{\phi=0} = p_0\left[\exp(-Z) - \beta^{-1}\frac{3a^2}{(a^2+Z^2)^3}\right]. \tag{63}$$

Using Equations (14), (15), and (48), it can be written as

$$p_0P(\infty, Z) = p_0P(0, Z) + \frac{3}{2}\,\frac{a^2}{a^2+Z^2}\left(\frac{\mathbf{B}^2}{8\pi}\right)_{\phi=0}. \tag{64}$$

Look at the pressure balance at $Z=0$ first,

$$p_0P(\infty, 0) = p_0P(0, 0) + \frac{3}{2}\left(\frac{\mathbf{B}^2}{8\pi}\right)_{\phi=0}. \tag{65}$$

Notice the factor $\frac{3}{2}$ indicating the pressure difference for confinement must exceed the magnetic pressure $\mathbf{B}^2/8\pi$. This is due to the fact that external pressure has to counteract field tension. Observation of the sunspot shows that at the photosphere, the field strength is about 1000 to 3000 G with a magnetic pressure comparable to or exceeding the normal photospheric pressure. It is clear from Equation (64) that the pressure difference decreases with height from the $Z = 0$ level and can fall below the magnetic pressure at the same height. It is therefore natural to set the photosphere $Z = Z_0$ to be the level where the external pressure is comparable to the magnetic pressure at the flux axis. What was baffling previously how the sunspot field strength is greater than could be accounted for by $p + \mathbf{B}^2/8\pi = \text{constant}$ has a simple explanation. At the photosphere, the field approaches being potential and the moderate fanning out of the field lines gives a Lorentz force less than the magnetic pressure.

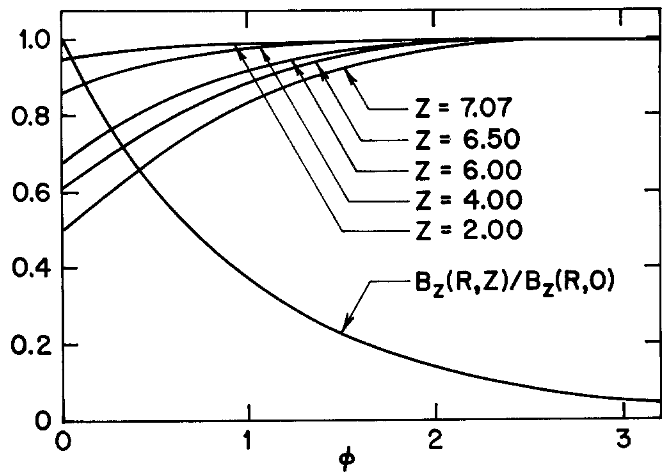

Fig. 2. Profiles of pressure and the vertical magnetic field component.

We carried out the above construction for $a = 5$ and the results are displayed in Figures 2, 3, and 4. We take the photospheric density to be 1.5×10^{17} particles cm^{-3} at the temperature of 5000 K. The pressure external to the flux tube at the photosphere is therefore 10^5 dynes cm^{-2}. The magnetic field strength at the centre of the flux tube is set at 1500 G. With these numerical values, the photospheric level is located at $Z_0 = 7.07$ in the above solution. The flux tube then has a diameter of 2600 km at the photosphere, about the size of a medium sunspot.

A few remarks about the manner in which the figures have been plotted. In Equation (33) where F is defined by Equation (42), $\phi = \text{constant}$ traces individual field lines. At any given level, F is fixed and ϕ measures the square of the distance from the flux axis modulated by the fixed value of F. In Figures 2, 3, and 4, the profiles of pressure, density and temperature are plotted against ϕ for different levels. When any one of these thermodynamic variables is read against a fixed value of ϕ, we will be following the particular variable along a field line through different levels. At a fixed

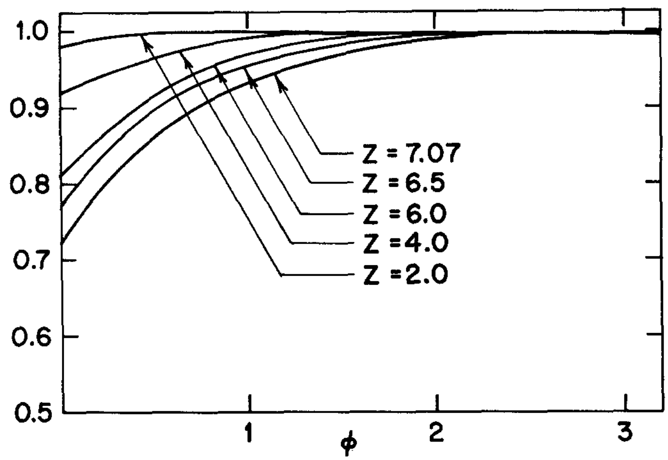

Fig. 3. Profiles of density.

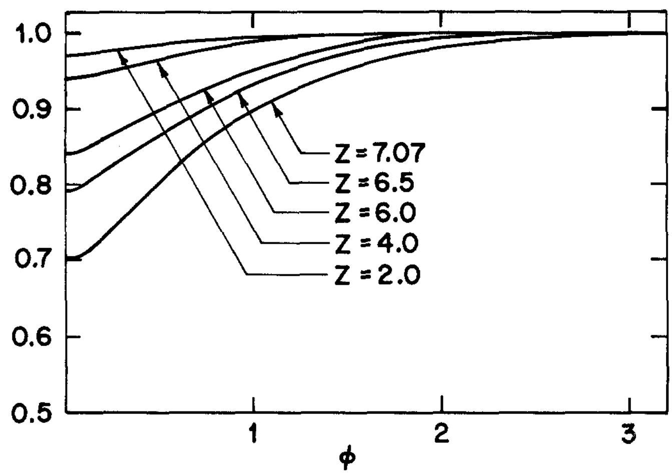

Fig. 4. Profiles of temperature.

level, the profiles in Figures 2, 3, and 4 give the variation against the square of the distance from the flux axis. Since F decreases with positive Z, the same value of ϕ corresponds to larger radii as we move to higher levels. For conversion to physical lengths, the unit length is the scale height $h^{-1} = 150$ km for $T_0 = 5000$ K.

The exponential curve in Figure 2 gives the vertical magnetic field component normalised to its value at the flux axis at the same level. The same profile obtains at all levels. Note that $\phi \sim R^2$ so that the profile is gaussian in R. The set of curves tending to the value 1 for large ϕ give the pressure $p(R, Z)$ normalised to the external pressure $p(\infty, Z)$ at the same level. The pressure ratio at the flux axis is least at the photospheric level with the pressure in the sunspot centre being half the value of the external pressure. As we go deep below the photosphere, the pressure ratio increases monotonically to unity. Five scale heights down, some 750 km to the level $Z = 2$, the pressure at the sunspot centre is about 98% of the external pressure. The compression of the magnetic field increases with depth as we approach $Z = 0$ from above as can be seen in Figure 1. The field strength increases with depth like F as follows from Equations (33) and (42). It is easy to see that the magnetic pressure increases with depth less rapidly than the exponential external pressure. Because of this, the differential between the pressure at the sunspot centre and the external pressure decreases as we go below the photosphere inspite of the enhancement of the magnetic field compression. Extrapolating from the external photospheric pressure of 10^5 dynes cm^{-2} downward, we obtain 10^8 dynes cm^{-2} for the pressure at $Z = 0$ whereas the magnetic field increases from its photospheric value of 1500 G (equivalent to about 10^5 dynes cm^{-2}) to 4400 G (equivalent to about 10^6 dynes cm^{-2}) at $Z = 0$. Figure 3 gives the density $\rho(R, Z)$ normalised to the external density $\rho(\infty, Z)$ at the same level. On any level $Z > 0$, the density decreases as we approach the sunspot centre from the ouside. It is the depletion of the density in the sunspot that gives rise to the Wilson effect. The Wilson effect is small in this solution; a depression of less than a scale height in the photosphere in the sunspot is obtained if we were to look at the contour surfaces of constant density. Figure 3 also shows that the density ratio increases to unity as we go deep below the photosphere along the sunspot axis. The density in the sunspot is barely different from the external density some five scale heights down. In Figure 4, the temperature profile at different levels are presented normalised to the external temperature. Whereas both the external pressure and density increase exponentially with depth, the external temperature is constant at 5000 K. Figure 4 shows that the temperature differential is maximal at the photosphere $Z = Z_0 = 7.07$ with 3500 K at the sunspot centre. As we go down the sunspot axis, the temperature rises to 4500 K two scale heights (300 km) below and is practically 5000 K five scale heights (750 km) further down to $Z = 0$ where the flux tube is most compressed. In these depths, the density is so large that only a slight local cooling is sufficient to generate the necessary pressure differential to confine the magnetic field.

We have also analysed the case of $a = 1$ to find that the waist at $Z = 0$ is narrower and the photospheric level is located at $Z_0 = 10$. Since the waist is situated deeper below the photosphere, the fanning out of the field lines gives a moderately larger diameter for the flux tube at the photosphere, about 3000 km. The other features of the solution are qualitatively similar to the $a = 5$ solution.

For completeness, we consider the solution for $Z > Z_0$ where the isothermal external temperature (55) has been continued smoothly into another function which declines suitably less rapidly than e^{-Z} such as

$$P(\infty, Z) = \frac{P_1}{(a^2 + Z^2)^2} + \frac{P_2}{(a^2 + Z^2)^3}, \tag{66}$$

where P_1 and P_2 are positive constants. Equations (50)–(52) become

$$p = p_0\left[\frac{P_1 - 2\beta^{-1}\phi \exp(-2\phi)}{(a^2 + Z^2)^2} + \frac{P_2 - a^2\beta^{-1}(3 - 2\phi)\exp(-2\phi)}{(a^2 + Z^2)^3}\right], \tag{67}$$

$$\rho = \rho_0 \left\{ \frac{4Z[P_1 - 2\beta^{-1}\phi \exp(-2\phi)]}{(a^2+Z^2)^3} + \frac{6Z[P_2 - a^2\beta^{-1}(3 - 2\phi \exp(-2\phi)]}{(a^2+Z^2)^4} \right\}, \tag{68}$$

$$T = T_0 \left\{ \frac{[P_1 - 2\beta^{-1}\phi \exp(-2\phi)](a^2+Z^2) + P_2 - a^2\beta^{-1}(3-2\phi)\exp(-2\phi)}{4[P_1 - 2\beta^{-1}\phi \exp(-2\phi)](a^2+Z^2) + 6[P_2 - a^2\beta^{-1}(3-2\phi)\exp(-2\phi)]} \right\} \times \times \left(\frac{a^2+Z^2}{Z} \right). \tag{69}$$

The temperature along any field line ϕ = constant for $Z \geq 0$ is infinite at $Z = 0$, decreases to a minimum and then increases without bound as Z goes to infinity. The singularity at $Z = 0$ is irrelevant since the solution does not apply there and is taken over by the solution for the isothermal external pressure (55). Moreover, setting ϕ equal to zero and infinity successively, we see that the flux centre is hotter than the exterior which is consistent with observation (e.g. Vaiana, 1976). The temperature tends to the same value everywhere as Z increases. If P_1 and P_2 satisfy

$$P_1 \geq e^{-1}, \tag{70}$$

$$P_2 \geq 3, \tag{71}$$

p and ρ are positive everywhere in the region $Z \geq Z_0$. The following point should be noted. In Equation (49), the pressure is made up of a magnetic field dependent component superposed on the external pressure $P(\infty, Z)$. The external pressure (66) declines with height at the same rate as the magnetic field dependent component of the total pressure. At each height, then, pressure and magnetic forces are comparable in magnitude. If an even less rapidly decreasing function of Z is employed in place of (66), the magnetic field dependent component becomes negligible compared to the total pressure at some height beyond which the pressure is practically in pure hydrostatic equilibrium with gravity.

4. Discussion

We formulated the technique of generating exact magnetostatic solution (Low, 1980a) for the situation of a cylindrically symmetric flux tube oriented vertically in a stratified atmosphere. The method consists of prescribing the field geometry for the flux followed by analytic construction of the distributions of field strength, pressure, density and temperature required for static equilibrium. The simplest type of field geometry to prescribe is one which satisfies the similarity law put forth by Schlüter and Temesváry (1958) for their sunspot model. Wilson (1977) recently obtained an approximate solution by calculating the pressure, density and temperature from a prescribed cylindrically symmetric field using an expansion procedure. Wilson's solution can be obtained in closed form exactly as we show in the Appendix.

As an application of the formulation, a particular Schlüter–Temesváry field was treated in detail. From the distributions in space of the magnetic field and the plasma, many interesting properties of the sunspot can be illustrated explicitly as we did in

Section 3. The sunspot field is concentrated below the photosphere. The pressure is constrained not only by the requirement to confine the magnetic field but also by the requirement on its vertical gradient to support the weight of matter. Pressure grows exponentially with depth and in a few scale heights below the photosphere, it exceeds the magnetic pressure by orders of magnitude so that the pressure difference needed to confine the magnetic field can be created with an almost negligible temperature variation. It is therefore unlikely that the cool region of the sunspot extends much more than a few density scale heights below the photosphere. The concentrated field fans out as it rises to the photosphere in consequence of the scale height effect of gravity on the pressure difference confining it. At the photosphere, the field approaches being potential. Taking into account the full relation between plasma pressure, magnetic pressure and field tension, the actual Lorentz force bearing on the photosphere is less than the force due to the magnetic pressure alone. It is well known that the sunspot fields have magnetic pressures comparable or even exceeding the normal photospheric pressure. The particular solution presented serves its purpose of illustrating basic properties. More realistic models of the sunspots can be developed through experimenting with other field line geometries than the one prescribed by Equation (33).

The exact solution confirms Parker's conclusion from his analysis of slender flux ropes that moderate cooling over a few scale heights can effectively concentrate the field (Parker, 1976). As to what causes cooling our solution provides no definite answer since the energy equation is not treated. Parker (1974, 1975, 1976) and Roberts (1976) suggest that the cooling is the result of refrigeration by Alfvén waves propagating out of the flux tube rather than the suppression of convective transport of heat (Biermann, 1941). Until it was questioned recently, the latter has been a popular idea for the cooling of the sunspot. The question of what cooling mechanism applies is being debated (e.g. Beckers, 1976; Beckers and Schneeberger, 1977; Boruta, 1977; Cowling, 1976; Giovanelli *et al.*, 1978; Parker, 1977b, 1978; Roberts and Webb, 1978; Spruit, 1977; Webb and Roberts, 1978). It is worthwhile to familiarise ourselves with what possibilities there are for magnetostatic configurations to make way for a clearer discussion of both the questions of energy transport and mechanical stability. The fact that no exact magnetostatic solution of the sunspot has been available until now reflects the difficulty of the quantitative problem. For those static solutions that can be written in explicit forms such as the Schlüter-Temesváry sunspot solution presented here, it will be interesting to go on to consider the question of stability using perturbation methods to expand the dynamical equations about a given static equilibrium configuration.

We emphasise that Equation (41) contains all the possible magnetostatic equilibria having the field geometry (33). Through this equation and Equations (7)–(9), the temperature has a fixed dependence on the free functions $F(Z)$, $\tilde{H}(\phi)$, $Q(Z)$ and their derivatives. We may think of p, ρ, and $\mathbf{B}$ being related by the force equilibrium while p and ρ determine T through the ideal gas law. Only within the freedom of arbitrarily specifying the above free functions can the temperature be varied. In general, it may not be possible to adjust the temperature to satisfy a further equation for energy transport. This point raises the question of consistency when an energy equation is imposed on this type of system (e.g. Deinzer, 1965). This limitation arises from the similarity ansatz (28). The freedom of adjusting the temperature is less limited if this ansatz is given up and Equation (33) is replaced by a broader class of field geometries. See the Appendix for an example. In the most general case, ϕ is completely free and any energy equation can be imposed. The advantage of the Schlüter-Temesváry similarity law is of course the simplification of the problem to the point where analytical methods can be used. We shall take further advantage of this in a paper to follow (Low, 1980b) to isolate and illustrate basic properties. In particular we will investigate the complication arising from field lines that are twisted and that do not extend vertically to infinity. It should be pointed out that the similarity law probably is unlikely to obtain in thick flux tubes such as a large sunspot.

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Appendix

We derive the integral for force balance across field lines for a slightly more general field geometry than the one treated in Section 3. The exact solution which Wilson (1977) approximated will be seen as a particular solution of this type of geometry. Define,

$$\psi = R^2 F(Z)\,, \tag{A1}$$

$$H = KR^2 + W(\psi)\,, \tag{A2}$$

where K is a constant and W is a function of one variable. The field lines projected on the θ = constant plane are contours of constant H and depend on the functional form of W. Following the mathematical steps of Section 2, Equation (11) is transformed into a partial differential equation with ψ and Z as the two independent variables,

$$\left(KF+\frac{\mathrm{d}W}{\mathrm{d}\psi}\right)^2\frac{\mathrm{d}^2F}{\mathrm{d}Z^2}+\left(KF+\frac{\mathrm{d}W}{\mathrm{d}\psi}\right)\frac{\mathrm{d}^2W}{\mathrm{d}\psi^2}\left[\frac{1}{F}\left(\frac{\mathrm{d}F}{\mathrm{d}Z}\right)^2\psi+4F^2\right]+$$

$$+\frac{1}{\psi}K\frac{\partial K}{\partial\psi}+\beta\frac{\partial P}{\partial\psi}=0\,, \tag{A3}$$

where P is defined in Equation (26). Notice that setting $K = 0$ reduces the contours of constant H to the contours of constant ψ. In this case, identify $\psi \equiv \phi$ and $W(\psi) \equiv \tilde{H}(\phi)$. We then have the case treated in Section 3 and Equation (A3) is just Equation (38). The case of a non-zero K can also be readily integrated to generate solutions. Take the example where

$$F \stackrel{*}{=} \exp(-Z)\,, \tag{A4}$$

$$W(\psi) = a\psi + b\psi^2\,, \tag{A5}$$

where a and b are constants. Equation (A3) becomes

$$\begin{aligned}&[K \exp(-Z) + a + 2b\psi]^2 \exp(-Z) + \\ &\quad + 2b[K \exp(-Z) + a + 2b\psi]\,[\exp(-Z)\psi + 4 \exp(-2Z)] + \\ &\quad + \frac{1}{\psi} K \frac{\partial K}{\partial \psi} \exp(-Z) + \beta \frac{\partial P}{\partial \psi} = 0\,.\end{aligned} \tag{A6}$$

It is a straightforward integration with respect to ψ to obtain P in terms of K. Going back to Equations (7)–(10), we can obtain the distributions in space for the plasma and magnetic field. This particular solution is the exact version of Wilson's solution (1977).

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