ANISOTROPIES OF SOLAR COSMIC RAYS*

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Abstract. The behavior of the anisotropy during solar cosmic-ray events is discussed in terms of a simple model in which cosmic-ray particles propagate along the mean interplanetary magnetic field, undergoing pitch-angle scattering. It is shown that a generalized form of the telegraph equation should be used when the anisotropy is large (i.e. greater than 30%), but that the usual diffusion equation is adequate otherwise. The behavior of the anisotropy during the decay phase of solar cosmic-ray events is then considered, and several effects which can give rise to a small persisting anisotropy are described. Finally, observations of solar cosmic-ray anisotropies following flares are reviewed and it is concluded that the simplified mathematical treatment presented here adequately describes some of the general features of the behavior of the anisotropy, but does not provide a detailed quantitative description of the particle behavior, especially during the highly anisotropic phase of an event.

1. Introduction

Pronounced anisotropies of the directional intensity are a characteristic feature of solar cosmic-ray events. The observed directions of the anisotropies have been used as evidence supporting models for cosmic-ray propagation involving diffusion which is primarily along, and not across, the magnetic-field lines. However, there has been very little discussion of the expected temporal variation of the anisotropy other than to note that it should decay inversely with time for certain simple models of solar cosmic-ray events (DORMAN, 1962; AXFORD, 1965). Indeed, there is evidently some misapprehension concerning the status of anisotropies (i.e. streaming) in diffusion theory, since a number of authors are apparently under the impression that the presence of anisotropies invalidates the diffusion concept entirely.

In this paper we discuss the behavior of the anisotropy during model solar cosmic-ray events in which the cosmic-ray particles propagate along the mean interplanetary magnetic field, undergoing pitch-angle scattering. Enlarging slightly upon the treatment of AXFORD (1965) and SHISHOV (1966), we show that a generalized form of the telegraph equation should be used when the anisotropy is large (i.e. greater than 30%), but that the usual diffusion equation is adequate otherwise. The behavior of the anisotropy during the decay phase of solar cosmic-ray events is then considered; in particular, the influence of convection by the solar wind, non-impulsive source functions, and non-uniform diffusion coefficients is discussed. Finally, we review observations of solar cosmic-ray anisotropies following flares, and conclude that the simplified mathematical treatment presented here is an adequate description of some of the general features of the behavior of the anisotropy but fails to describe the

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particle behavior in quantitative detail, particularly during the highly anisotropic phase of an event.

2. Equations of Motion and Some Representative Solutions

Consider the model for the propagation of solar cosmic rays described by AXFORD (1965) in which the guiding centers of the particles are assumed to move along the quasi-radial, average interplanetary magnetic field, with pitch-angle scattering resulting from the irregular component of the field. It is assumed that the energy of the particles remains constant, and (as a first approximation) that the average field diverges spherically.

The behavior of the particles can be described by a distribution function $f(r, \theta, t)$, which represents the number of particles at time t in $(r, r+\mathrm{d}r)$, where r is the distance from the sun, measured along the average magnetic-field line, with pitch angle in $(\theta, \theta+\mathrm{d}\theta)$ and with velocity in $(v, v+\mathrm{d}v)$. We suppose that the distribution function satisfies the Boltzmann equation:

$$\frac{\partial f}{\partial t} + v\cos\theta\,\frac{\partial f}{\partial r} - \frac{v\sin\theta}{r}\,\frac{\partial f}{\partial \theta} = 2\pi N v \int_0^{\pi} \sin\theta'\sigma(\theta, \theta')\{f(r, \theta', t) - f(r, \theta, t)\}\,\mathrm{d}\theta', \tag{1}$$

where v is the constant particle velocity, $N = N(r)$ the number density of 'scattering centers', and $\sigma(\theta, \theta')$ the differential cross-section for scattering from pitch angle θ to θ'.

The anisotropy of the particle flux is defined to be:

$$\xi(r, t) = \{f(r, 0, t) - f(r, \pi, t)\}/\{f(r, 0, t) + f(r, \pi, t)\}. \tag{2}$$

Clearly, for the case of solar cosmic rays, $f(r, 0, t)$ and $f(r, \pi, t)$ are proportional to the instantaneous directional intensities measured by detectors looking into and away from the direction of maximum flux, respectively.

A. THE CASE OF BI-DIRECTIONAL SCATTERING

It is instructive to consider first the simple case in which the pitch-angle distribution is constrained to be bi-directional; that is, $\theta = 0, \pi$ only, with equal probabilities that a particle has either of these directions following a collision. Equation (1) can then be replaced by the two equations:

$$\frac{\partial f_+}{\partial t} + v\,\frac{\partial f_+}{\partial r} = \frac{f_- - f_+}{2\tau}, \tag{3}$$

$$\frac{\partial f_-}{\partial t} - v\,\frac{\partial f_-}{\partial r} = \frac{f_+ - f_-}{2\tau}, \tag{4}$$

where $f_+ = f(r, 0, t)$ and $f_- = f(r, \pi, t)$; $4\pi(f_+ + f_-)\,\mathrm{d}r$ is the number of particles with velocity v in $(r, r+\mathrm{d}r)$, and $\tau = 1/(N\sigma v)$ is the mean collision time. Adding and subtracting (3) and (4), we obtain:

$$\frac{\partial n}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J) = 0, \tag{5}$$

$$\frac{\partial J}{\partial t} + \frac{J}{\tau} = -\frac{v^2}{r^2}\frac{\partial}{\partial r}(r^2 n), \tag{6}$$

where $n=(f_+ + f_-)/r^2$ is the number density, and $J=v(f_+ - f_-)/r^2$ is the radial current density. On eliminating J between (5) and (6) we obtain a telegraph equation for $r^2 n$:

$$\left\{\frac{\partial^2}{\partial t^2} + \frac{1}{\tau}\frac{\partial}{\partial t} - v^2\frac{\partial^2}{\partial r^2}\right\}(r^2 n) = 0. \tag{7}$$

The solution of Equation (7) for the case in which $4\pi\eta$ particles are released at $r=0$ at time $t=0$ and τ is a constant is (Axford, 1965):

$$\left.\begin{aligned} n(r,t) &= \frac{\eta}{vr^2}\exp(-r/2\tau v)\,\delta(t-r/v) \\ &\quad + \frac{\eta}{2\tau v r^2}\exp(-t/2\tau)\left\{I_0(\varphi) + \frac{tI_1(\varphi)}{2\tau\varphi}\right\}H(t-r/v), \end{aligned}\right\} \tag{8}$$

where $\varphi=(t^2-r^2/v^2)^{1/2}/2\tau$, $I_\mu(\varphi)$ is the modified Bessel function of the first kind and order μ, and $H(t-r/v)$ is the Heaviside unit function. The anisotropy is:

$$\left.\begin{aligned} \xi(r,t) = J/nv &= 1,\ t=r/v \\ &= rI_1(\varphi)/\{2v\tau\varphi I_0(\varphi) + vtI_1(\varphi)\},\ t>r/v. \end{aligned}\right\} \tag{9}$$

Note that the number density contains a pulse (i.e. the delta-function component), propagating at the particle speed v, and decaying with distance. The pulse comprises those particles which at time t have survived scattering following their initial release at the origin; the scattered particles make up the remainder of the number density (i.e. the unit-function component). The anisotropy reaches a maximum value of unity in the pulse, but immediately drops to a smaller value behind the pulse and thereafter decays asymptotically to zero.

For large t (i.e. $t \gg \tau$ and $t \gg r/v$), these solutions have the asymptotic forms:

$$n(r,t) \sim \frac{\eta}{r^2\sqrt{(\pi\kappa t)}}\exp(-r^2/4\kappa t), \tag{10}$$

$$\xi(r,t) \sim r/2vt, \tag{11}$$

where we define $\kappa=v^2\tau$ as the diffusion coefficient. This asymptotic form for n is also a solution of the diffusion equation:

$$\frac{\partial(r^2 n)}{\partial t} = \kappa\frac{\partial^2(r^2 n)}{\partial r^2}, \tag{12}$$

which can be obtained from (7) by suppressing the term $\partial^2(r^2 n)/\partial t^2$. The most obvious difference between the solution of the diffusion equation and the solution of the telegraph equation is that the former implies prompt arrival of the particles at all r with infinite anisotropy initially, whereas the latter indicates that the particles do not appear before $t=r/v$ and that the anisotropy is initially unity. Clearly the diffusion solution is invalid for small t, but as is shown in Figure 1, it merges rapidly with the

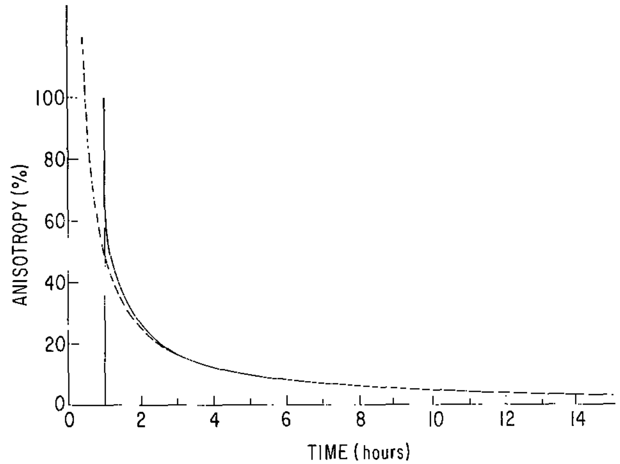

Fig. 1a. A comparison of the anisotropies predicted by the telegraph equation and the diffusion equation in the bi-directional scattering model. The solid line represents the telegraph solution (Equation (9)) and the dashed line, the diffusion solution (Equation (11)). It is assumed that the energy of the particles is 10 MeV, $\tau v=0.1$ AU, and $r=1$ AU.

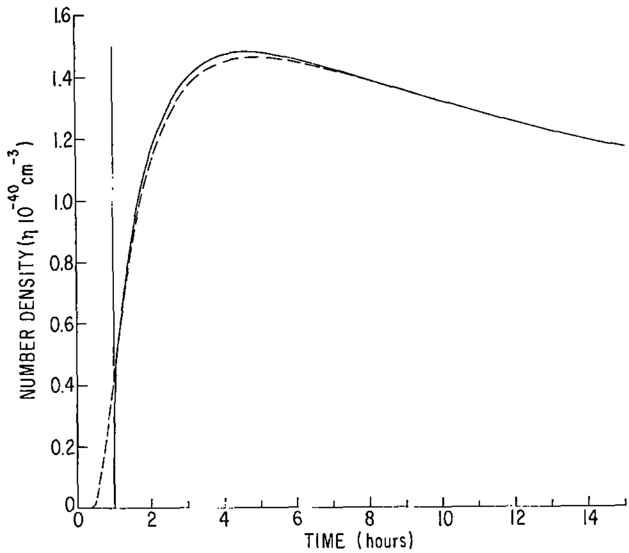

Fig. 1b. A comparison of the number densities predicted by the telegraph equation and the diffusion equation in the bi-directional scattering model. The solid line represents the telegraph solution (Equation (8)), and the dashed line, the diffusion solution (Equation (10)). It is assumed that the energy of the particles is 10 MeV, $\tau v=0.1$ AU, and $r=1$ AU.

telegraph solution and the two are scarcely distinguishable when the anisotropy is less than about 30%.

B. THE CASE OF ISOTROPIC SCATTERING

Consider the more realistic case in which the particles can have any pitch angle and the scattering is isotropic, so that Equation (1) becomes:

$$\frac{\partial f}{\partial t} + v\cos\theta\frac{\partial f}{\partial r} - \frac{v\sin\theta}{r}\frac{\partial f}{\partial\theta} = \frac{1}{\tau}\left[\frac{1}{2}\int_0^\pi f(r,\theta',t)\sin\theta'\,d\theta' - f\right], \tag{13}$$

where $\tau = 1/(4\pi vN\sigma)$. On taking the zeroth and first moments of this equation with respect to $v\cos\theta$, we obtain:

$$\frac{\partial n}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J) = 0 \tag{14}$$

$$\frac{\partial J}{\partial t} + \frac{J}{\tau} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_\parallel^2 n) + \frac{q_\perp^2 n}{r} = -\frac{\partial}{\partial r}(q_\parallel^2 n) + (q_\perp^2 - 2q_\parallel^2)\frac{n}{r}, \tag{15}$$

where $q_\parallel^2$ and $q_\perp^2$ are the mean square particle-velocity components, parallel and perpendicular to the radial vector, respectively. In the absence of any further relationship between $q_\parallel$ or $q_\perp$ and n and J, it is not possible to solve Equations (14) and (15). However, appropriate solutions can be obtained in the two limiting cases, namely $q_\perp^2 \ll q_\parallel^2 \simeq v^2$, and $(q_\perp^2 - 2q_\parallel^2) \ll v^2$, which correspond to extreme anisotropy and near isotropy of the flux, respectively.

Following an impulsive release of particles at $r = r_0$, $t = 0$, the first particles to arrive at any point $r \gg r_0$ are those which have undergone no scattering and thus have essentially zero pitch angle. The group of unscattered particles takes a finite time to pass a given point, since their initial pitch-angle distribution might for example be isotropic. For $r \gg r_0$, however, the spherical divergence of the magnetic field causes all pitch angles to become small irrespective of their initial values; hence, the unscattered particles pass within a time interval which is short compared with r/v, and effectively appear as a pulse. This pulse corresponds to the delta-function component of the solution (8), which is appropriate in these circumstances since with $q_\parallel^2 \simeq v^2$ and $q_\perp^2 \ll q_\parallel^2$ Equation (15) approximates to Equation (6).

When the initial pulse of unscattered particles has passed the point of observation, the particles arriving are those which have undergone some scattering. While $q_\perp^2$ remains small in comparison with $q_\parallel^2$, one can expect the solutions (8), (9) to provide a reasonably accurate description of the temporal behavior of n, J, and ξ. Eventually, however, the terms of Equation (15) involving $q_\perp^2$ become important and these approximate solutions are no longer valid.

For a nearly isotropic distribution we can neglect the last term on the right of Equation (15) (since $(q_\perp^2 - 2q_\parallel^2) \ll v^2$), and put $q_\parallel^2 = v^2/3$ in the first term. In this case the solution of Equations (14) and (15) for an impulsive release of particles at $r = 0$,

$t=0$ (τ is a constant) is, for $t>r/v$:

$$n_2(r,t) = \frac{\eta}{8\tau^3 v'^3} \frac{\exp(-t/2\tau)}{\varphi}\left[I_1(\varphi) + \frac{t}{2\tau\varphi} I_0(\varphi) - \frac{t}{\tau\varphi^2} I_1(\varphi)\right] H(t - r/v'), \tag{16}$$

where $v' = v/\sqrt{3}$ and $\varphi = (t^2 - r^2/v'^2)^{1/2}/2\tau$ (SHISHOV, 1966). The delta-function component of this expression for n has been ignored since it is associated with a large anisotropy. To determine the anisotropy we note that:

$$\xi = \alpha J/nv, \tag{17}$$

where $1 \leqslant \alpha \leqslant 3$. When the anisotropy is large (as in the first approximation) we put $\alpha=1$ and thus obtain ξ in the form given in Equation (9). When the anisotropy is small (as in the second approximation) we put $\alpha=3$ (cf. GLEESON and AXFORD, 1967). Accordingly, for $t>r/v'$ the anisotropy corresponding to the solution (16) is:

$$\xi(r, t) = \frac{3r}{2\tau v}\left[\frac{I_0(\varphi) - (2/\varphi) I_1(\varphi)}{\varphi I_1(\varphi) + (t/2\tau) I_0(\varphi) - (t/\tau\varphi) I_1(\varphi)}\right]. \tag{18}$$

We would expect to obtain a reasonably accurate description of the actual behavior of n, J, and ξ by combining these solutions, as in Figure 2. As in the bi-directional case the solutions merge with those of appropriate diffusion equations when $t \gg \tau$, r/v and the anisotropy is sufficiently small (i.e. $\xi \lesssim 30\%$).

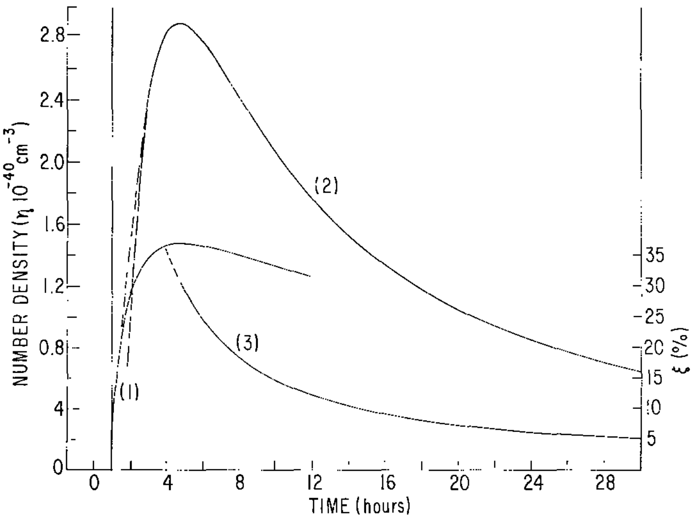

Fig. 2. A plot of the two approximations to the number density predicted in the isotropic scattering model. The curve marked (1) represents the highly anisotropic solution for the number density (Equation (8)) and the curve marked (2), the nearly isotropic solution (Equation (16)). It is assumed that the energy of the particles is 10 MeV, $\tau v = 0.1$ AU, and $r = 1$ AU. The dashed line is a suitable interpolation between the two solutions. The curve marked (3) represents the anisotropy corresponding to the nearly isotropic solution. Note that for these low-energy particles the anisotropy decays as t^{-1} but even after 30 hours is 5%.

C. THE CASE OF SMALL-ANGLE SCATTERING

For small-angle rather than isotropic scattering, Equations (14) and (15) remain valid provided we define τ as (SALPETER and TREIMAN, 1964):

$$\tau = 2\left\{\pi N \int_0^\pi \psi^2 \sigma(\psi) \sin\psi \, d\psi\right\}^{-1}, \tag{19}$$

where the differential scattering cross-section $\sigma(\psi)$ is assumed to be strongly peaked around $\psi=0$. This form for τ requires that the distribution function $(f(r, \theta, t))$ is not strongly peaked, and hence we should not expect the equations to describe the initial particle behavior $(t \leqslant r/v + \tau)$ accurately in this case.

3. The Decay of the Anisotropy

It is evident from the results obtained above that the diffusion equations describe the particle behavior reasonably well, provided $t \ll \tau, r/v$ and ξ is sufficiently small (e.g. $\xi \lesssim 30\%$). Thus, we can expect to be able to examine the influence of (a) convection, (b) a source with an extended time structure, and (c) a diffusion coefficient which depends on r, using the appropriate diffusion equations, with the simple proviso that the anisotropy should be small. In particular, we should be able to determine the relative importance of these effects in producing a small persisting anisotropy.

A. THE INFLUENCE OF CONVECTION

In the diffusion approximation, the effect of moving scatterers is to add an extra 'convection' term to the Equations for n and J (or ξ). Thus, in the case of bi-directional scattering:

$$\frac{\partial}{\partial t}(r^2 n) + V \frac{\partial}{\partial r}(r^2 n) = \frac{\partial^2}{\partial r^2}(\kappa r^2 n), \tag{20}$$

and

$$\xi = -\frac{v\tau}{r^2 n} \frac{\partial}{\partial r}(r^2 n) + \frac{V}{v}, \tag{21}$$

where V is the radial speed of the scatterers (i.e. the solar-wind speed), and $\kappa = v^2 \tau$ (in the bi-directional case $\alpha \equiv 1$). If κ is a constant, then following the impulsive release of particles at $r=0$, $t=0$,

$$\xi \sim (V + r/t)/2v \quad \text{for} \quad t \gg \tau, r/v. \tag{22}$$

Thus, the effect of convection is to produce a persisting anisotropy equal to $V/2v$ when $t \to \infty$. Note that the ultimate streaming speed of the particles is less than the speed of the scatterers (V). This result holds generally; indeed, in one case FISK and AXFORD (1968) have shown that $\xi \to 0$ as $t \to \infty$, when the effects of energy changes are taken into account.

B. THE EFFECT OF A SOURCE WITH AN EXTENDED TIME STRUCTURE

For the case of isotropic scattering, without convection, the diffusion equation is:

$$\frac{\partial n}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\kappa r^2 \frac{\partial n}{\partial r}\right), \tag{23}$$

where $\kappa = v^2\tau/3$. The solution of this equation when κ is constant and the source at $r=0$ has an extended time structure $h(t)$, rather than being a delta-function, is:

$$n(r, t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{h(t-u)\exp(-r^2/4\kappa u)\,\mathrm{d}u}{(\kappa u)^{3/2}}. \tag{24}$$

The asymptotic behavior of $n(r, t)$ for large t depends critically on the asymptotic form of $h(t)$. Thus if $h(t) \sim t^{-\zeta}$ with $\zeta > \frac{3}{2}$, the main contribution to the integral (which comes from values of u near t) is virtually independent of r, implying that $\xi = -(v\tau/n)\,\partial n/\partial r \to 0$ as $t \to \infty$. However, if $\zeta < \frac{3}{2}$, the main contribution to the integral comes from small values of u (i.e. $u \sim r^2/6\kappa$), hence:

$$n(r, t) \sim 1/(\kappa r t^{\zeta}),\ \xi(r, t) \sim \tau v/r \quad \text{as} \quad t \to \infty. \tag{25}$$

Consequently, if the source does not decay too rapidly, we can expect the anisotropy to persist at large times.

C. THE EFFECT OF A NON-UNIFORM DIFFUSION COEFFICIENT

If the diffusion coefficient has the form $\kappa = \kappa_0 r^{\beta}$, with $0 \leqslant \beta \leqslant 2$, then the solution of (23) corresponding to an impulsive release of $4\pi\eta$ particles at $r=0$, $t=0$ is easily found to be (PARKER, 1963):

$$n(r, t) = \frac{\eta \exp[-r^{2-\beta}/(2-\beta)^2 \kappa_0 t]}{(2-\beta)^{(4+\beta)/(2-\beta)}\Gamma(3/(2-\beta))\,(\kappa_0 t)^{3/(2-\beta)}} \tag{26}$$

with

$$\xi(r, t) = -\frac{v\tau}{n}\frac{\partial n}{\partial r} = \frac{3r}{(2-\beta)\,vt}. \tag{27}$$

Thus, the anisotropy decreases inversely with time and asymptotically approaches zero. It has been found that this solution for $n(r, t)$ can in many instances provide a good fit to observations of the temporal behavior of the particle intensity during solar cosmic-ray events (e.g. KRIMIGIS, 1965). Apart from a case discussed in Section 4 of this paper, it is not generally known whether this agreement between theory and observation also applies to the anisotropy. However, observations which show that a persistent anisotropy occurs in some events suggest that this very simple model might be inadequate, since there is no persisting anisotropy when the diffusion coefficient is of the form assumed.

A persisting anisotropy is possible if the diffusion coefficient increases sufficiently rapidly with r. This can be seen for the case of bi-directional scattering (Equations

(20), (21)) with $\kappa = \kappa_0 \exp(r/r_0)$ and $V=0$, where following impulsive release of $4\pi\eta$ particles at $r=0$, $t=0$,

$$n(r, t) = \frac{\eta z^{1/2}}{\kappa_0 r_0 t (\ln(z))^2} I_1(\rho) \exp[-(1+z) r_0^2/\kappa_0 t] \tag{28}$$

and

$$\xi(r, t) = r_0 [I_0(\rho) - z^{1/2} I_1(\rho)]/[vtz^{1/2} I_1(\rho)], \tag{29}$$

where $z = \exp(-r/r_0)$ and $\rho = 2r_0^2 z^{1/2}/\kappa_0 t$. For large values of t we find that

$$\xi(r, t) \sim (\kappa_0/r_0 v) \exp(r/r_0) = (\kappa/r_0 v) \tag{30}$$

and hence there is a persisting anisotropy.

On examining the asymptotic solutions (25) and (30) it can be seen that the values of the residual anisotropy are comparable. This might have been expected since the main consequence of having a diffusion coefficient of the form $\kappa = \kappa_0 \exp(r/r_0)$ is to retard the escape of particles from the region $r < r_0$ and to allow them to move relatively freely in $r \gg r_0$, thus producing (as far as distant regions are concerned) the effect of a source which appears extended in time rather than impulsive. In both cases the residual anisotropy is approximately (κ/Vr) times that expected from convection acting alone (V/v).

4. Observations of Anisotropies

Anisotropies have almost always been evident in ground-level observations of solar cosmic-ray events (McCracken, 1963). A clear example is shown in Figure 3, which is taken from a paper by Burlaga (1967) (the observations, which were originally described by McCracken, 1962, refer to the event of 12 November, 1960). In this event the neutron monitor at College was looking approximately along the local interplanetary magnetic-field direction, towards the sun, while the neutron monitor at Mawson looked in the opposite direction. Thus the College observations refer to $f_+(t)$ and the Mawson observations to $f_-(t)$.

The manner in which Burlaga has been able to plot these data suggests that this portion of the event fits the diffusive phase of a model having impulsive release of particles at $r=0$, $t=0$, with $\kappa = \kappa_0 r^\beta$, $\beta \simeq \frac{4}{5}$ (see Equation (26)) and r being measured along the interplanetary magnetic-field lines leading to the earth (see Axford, 1965). The anisotropy at earth is obtained by taking the ratio of the difference to the sum of the College and Mawson observations (cf. Equation (2)). It can be shown from Equations (2) and (27) that the difference in the slopes of the straight lines drawn through the College and Mawson data should be approximately $2\xi t = 6r/(2-\beta)v \simeq 5r/v$. Since r/v is the minimum transit time from the source to the earth along the interplanetary magnetic field ($r/v \simeq 12$–14 min), the predicted slope difference is 60–70 min. According to Figure 3, the observed slope difference is about 63 min, which must be considered very satisfactory agreement, although it may well be fortuitous.

Detailed observations of anisotropies of low-energy (1–100 MeV) solar cosmic

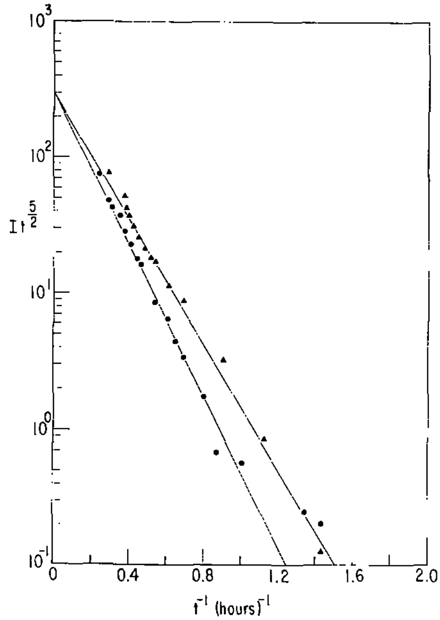

Fig. 3. The neutron monitor data from College and Mawson for the event of 12 November 1960 (taken from BURLAGA, 1967). Here, I is the intensity observed at each of the two stations. The difference in the slopes of the straight lines through the two sets of data is about 63 min.

rays have been made from space probes Pioneer VI and VII (MCCRACKEN and NESS, 1966; BARTLEY *et al.*, 1966; MCCRACKEN *et al.*, 1967; FAN *et al.*, 1966, 1968). It is typically found that the anisotropy is pronounced in the early phases of solar cosmic-ray events and decays to a value of 5–10% rather than to zero. The observed pitch-angle distributions of the particles arriving from the sunward direction (FAN *et al.*, 1968) and the observed backscatter of the particles beyond the point of observation (MCCRACKEN *et al.*, 1967) indicate that the mean collision time for low-energy solar cosmic rays is quite long (a sizeable fraction of an hour). The direction of the anisotropy is strongly controlled by the interplanetary magnetic field, and the particles are observed to arrive from 40–50° West of the sun-satellite line even if the flare is on the east side of the sun; late in the event the anisotropy will frequently lie in a direction closer to the sun-satellite line (MCCRACKEN *et al.*, 1967).

Certain aspects of these space-probe observations are well described by models of the type described in Sections 2 and 3. The tendency of the observed anisotropies to persist can be interpreted as being due to (a) convection by the solar wind parallel to the interplanetary magnetic field lines; (b) continued emission of particles from the source (BARTLEY *et al.*, 1966); and (c) a steep increase of the diffusion coefficient with heliocentric distance. In addition one must take into account anisotropies resulting from particle drifts perpendicular to the magnetic-field lines. These drifts can arise from (d) the presence of an electric field (i.e. an $\mathbf{E} \times \mathbf{B}$ drift) and from (e) density gradients perpendicular to the magnetic field. In the direction parallel to the magnetic field (b) and (c) are likely to be the most important, with (c) being consistent with a long mean collision time for scattering beyond 1 AU, deduced from the observations of McCracken *et al.* (1967). The effect (d) can be quite important, especially if the particle-energy spectrum is soft (Forman, 1968). Late in the event, the direction of the observed anisotropy will presumably lie closer to the sun-satellite line when the anisotropy due to (d) becomes comparable with the anisotropy parallel to the magnetic field (McCracken *et al.*, 1967).

It is unlikely, however, that the simplified mathematical treatment given here will adequately describe the highly anisotropic phase of a low-energy solar cosmic-ray event. The long mean collision times which are observed are consistent with the effective mean collision time for small-angle scattering given in Equation (19). As we have pointed out, our equations will not describe the initial behavior of the particles when there is small-angle scattering. Further, since a likely cause for the observed persisting anisotropies is a diffusion coefficient which increases rapidly with distance from the sun, the solution (16) for constant τ, although indicating some of the features of the particle behavior to be expected, is unlikely to be useful in a quantitative sense.

Finally, we wish to emphasize that it is necessary to determine whether a model explains the behavior of the anisotropy as well as the behavior of the intensity before deciding that it is acceptable. An event is cited by McCracken *et al.* (1967), in which the variation of the intensity with time can adequately be described in terms of a simple diffusion model, which is, however, inconsistent with observations of the anisotropy.

McCracken *et al.* have described a method of estimating the local mean free path from observations of the rate at which a highly anisotropic beam of solar cosmic rays becomes more isotropic due to back-scattering of the particles beyond the point of observation.* From the Pioneer VI and VII observations they deduce that for several events in 1966, the mean free paths were on the order of 1 AU. It should be remembered, however, that this result refers to the mean distance a particle must travel to have its pitch angle turned through 180°, whereas, in general, one would be satisfied to call scattering through 90° or perhaps even one radian an effective 'collision'. There is, in addition, the complication that small increment pitch-angle scattering is not uniform with respect to angle (cf. Jokipii, 1968). Nevertheless, one must conclude that the mean free paths which existed near the orbit of earth during 1966 were quite large and certainly much larger than the values of about 0.1 AU deduced by McCracken *et al.* (1967) on the basis of a simple diffusion model with a diffusion coefficient independent of r. This model provides a very good fit to the observations of the behavior of the intensity in one case, but as McCracken *et al.* suggest, since it does not at the same time fit the observations of the anisotropy, it cannot be regarded as being correct. This demonstrates very clearly the shortcoming of much of the work which has been done in the past on fitting diffusion models to observed intensity-time variations and thereby deducing information concerning the

* This is discussed in terms of the bi-directional scattering model in the appendix to this paper.

diffusion coefficient $\kappa(r)$. Since diffusion theory makes specific predictions about the behavior of the anisotropy, these must also agree with the observations if a particular model is to be considered acceptable.

Appendix

In the bi-directional scattering model, both f_+ and f_- satisfy the telegraph equation. Hence, given $f_+(t)$ at $r=0$, we can determine $f_+(r, t)$ by solving the telegraph equation as a boundary value problem. Using Laplace transform methods (see CARSLAW and JAEGER, 1953), we find that

$$\left.\begin{aligned} f_+(r, t) &= f_0(t - r/v)\, e^{-r/2\tau v} \\ &+ \frac{r}{2\tau v} \int_{r/v}^{t} f_0(t - u)\, e^{-u/2\tau} \frac{I_1\left[1/2\tau(u^2 - r^2/v^2)^{1/2}\right]}{(u^2 - r^2/v^2)^{1/2}}\, \mathrm{d}u, \qquad t > r/v, \end{aligned}\right\} \tag{31}$$

where we have assumed that $f_+ = \partial f_+/\partial t = 0$ at $t=0$, $f_+(r, t) = f_0(t)$ at $r=0$, $t>0$, and $f_+ \to 0$ as $r \to \infty$. Substituting into Equation (4) above, we find:

$$f_-(r, t) = \int_0^t f_+(r, t - u) \frac{e^{-u/2\tau}}{u} I_1\left[u/2\tau\right] \mathrm{d}u\,. \tag{32}$$

MCCRACKEN *et al.* (1967) obtained a similar result by a physical argument. The only essential difference is that we define the probability per unit length that a particle is back-scattered to be $1/2\tau v$, while MCCRACKEN *et al.* considered it to be $1/\tau v$. Our theory, like most diffusion theories, assumes that the probability that a particle should be back-scattered at the end of a mean free path is $\frac{1}{2}$.

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