PAUL EHRLICH

METRIC DEFORMATIONS OF CURVATURE
II: *Compact 3-Manifolds*

1. Introduction

Let (M, g_0) be a smooth compact Riemannian manifold. Given a symmetric 2-tensor $h \in C^\infty(S^2(T^*M))$ we can perturb g_0 by the first order metric deformation $g(t) = g_0 + th$. To linearize the problem and see at first order what this deformation is doing to the curvature quantities of (M, g_0) it is possible to calculate the first derivative K' of the sectional curvature K^t of $g(t)$ at time $t=0$ or the first derivative Ric$'$ of the Ricci curvature Ric^t of $g(t)$ at time $t=0$, using formulas found, for instance, in [7] and/or [10].

Let $G_2(M)$ be the Grassman bundle of 2-planes P in TM. Let (M, g_0) be a Riemannian manifold with sectional curvature $K_{g_0} \geq 0$. We say a deformation $g(t)$ of g_0 is *positive at first order for the sectional curvature* if for all $P \in G_2(M)$, $K_{g_0}(P) = 0$ implies $K'(P) > 0$. It is clear how to give analogous definitions for non-negative, negative, or vanishing at first or higher orders for the sectional and Ricci curvature.

In [4], using differential operators

$$\delta^* : C^\infty(T^*M) \to C^\infty(S^2(T^*M)) \quad \text{and}$$

$$\delta' : C^\infty(S^2(T^*M)) \to C^\infty(T^*M)$$

a splitting $C^\infty(S^2(T^*M)) = \operatorname{Im}\delta^* \oplus \ker \delta'$ was constructed. Explicitly, if η is a 1-form on M, $\delta^*\eta$ is a symmetric 2-tensor given by $\delta^*\eta = \frac{1}{2}\mathscr{L}_{\eta^\#} g_0$ where $\eta^\#$ is the vector field associated to η by g_0. Following [5] we call a first order deformation $g(t) = g_0 + th$ *geometric* if $h \notin \operatorname{Im}\delta^*$. The point of the definition is to take into account the action of the diffeomorphism group on the space of metrics for M. Let X be a global vector field on M with flow φ_t. Set $g(t) = \varphi_t^* g_0$. Assuming $K_{g_0} \geq 0$, $g(t)$ will be a deformation of g_0 that vanishes at first order and is non-negative at second order so that it might appear that $g(t)$ is increasing the positive curvatrue. But since $\varphi_t : (M, g_0) \to (M, g_t)$ is an isometry, we really do nothing at all to the sectional curvature. In this example, the 1-jet of $g(t)$ is just $\mathscr{L}_X g$ so that the definition of geometric deformation rules out this kind of trivial deformation.

In [10], we saw that it is possible to find local geometric deformations of the Ricci curvature positive and negative at first order on the outer annulus of a convex metric disk (compare also [1]). However, we saw that in general there are no local convex deformations of sectional curvature for manifolds M^n if $n \geq 3$.

Geometriae Dedicata **5** (1976) 147–161. *All Rights Reserved*

Thus, the method of proof given in [10] using local convex deformations that a compact manifold M^n admitting a metric of non-negative Ricci curvature and all Ricci curvatures positive at a point admits a metric of everywhere positive Ricci curvature does not generalize to prove the analogous theorem for the sectional curvature if $n \geq 3$.

Two lines of investigation are then suggested. Can the problem of perturbing from non-negative to positive sectional curvature be solved under stronger conditions, such as if the Ricci curvature is positive? Second, are there results which hold in dimension 3 because the Ricci and sectional curvatures are nicely related in 3-dimensions?

In order to answer the first question affirmatively by finding a geometric deformation positive at first order, one might attempt to find a symmetric 2-tensor h such that $K(x, y)=0$ implies

$$K'(x, y) = \operatorname{Ric}(x, x) \operatorname{Ric}(y, y) - (\operatorname{Ric}(x, y))^2 + Q(x, y),$$

where $Q(x, y) \geq 0$. But $S^2 \times S^2$ with the canonical Riemannian structure g_{can} satisfies $K \geq 0$ and $\operatorname{Ric} > 0$ and Berger, [3], has shown that any deformation of g_{can} non-negative at first order vanishes identically at first order. Hence, we cannot in general find such a 2-tensor h.

In [6], Bourguignon showed that $\operatorname{Ric}_{g_0} \notin \operatorname{Im} \delta^*$ unless $\operatorname{Ric}_{g_0} \equiv 0$ so that $g(t) = g_0 + t \operatorname{Ric}_{g_0}$ is a geometric deformation. This suggests we should calculate K' and Ric' for this deformation. In Section 2 we calculate K' in general. In Section 3 we observe that in dimension 3 if $K_{g_0} \geq 0$ and $\operatorname{Ric}_{g_0} > 0$, $g(t) = g_0 - t \operatorname{Ric}_{g_0}$ is a positive deformation at first order for the sectional curvature and surprisingly enough,

$$K'(x, y) = (D^*DR)(x, y, y, x) + \operatorname{Ric}(x, x) \operatorname{Ric}(y, y) - (\operatorname{Ric}(x, y))^2$$

showing that in dimension 3, $-\operatorname{Ric}_{g_0}$ will do as the tensor h mentioned above.

In Section 4 we sketch the calculation of Ric' for the Ricci deformation. Complete details are found in [9].

There is a recent notion of 'rigidity' in connection with the relation between curvature and topology in Riemannian geometry suggested to us by D. Gromoll. If a global geometric or topological result is implied by an 'open' curvature inequality such as $\frac{1}{4} < K \leq 1$, then the result should fail to be true if equality holds in the curvature condition, i.e., $\frac{1}{4} \leq K \leq 1$, only in a 'rigid' way. For example, if a complete simply connected Riemannian manifold M^n satisfies the curvature inequality $\frac{1}{4} < K \leq 1$, then M^n is homeomorphic to S^n. If $\frac{1}{4} \leq K \leq 1$ only and M^n is not homeomorphic to S^n, then M^n is isometric to a symmetric space of rank 1. The 'rigidity' is expressed here in that if M^n

fails to be homeomorphic to S^n, M is not just homeomorphic to a symmetric space of rank 1, but is isometric to a symmetric space of rank 1.

Given the perturbation theorem for $K \geqslant 0$ and $\mathrm{Ric} > 0$ in Section 3 (which succeeds because $K' > 0$ on the zero two-planes), we should thus ask what can happen if $\mathrm{Ric} \geqslant 0$ only so that on the set of zero two-planes $K' \geqslant 0$ only. One such case in which $K' = 0$ on the zero two-planes is the standard Riemannian structure on $S^1 \times S^2$. This raises the question as to whether our deformation theorem fails to be true only if the manifold is locally isometric to a product. In Section 5 we answer a particular case affirmatively; namely, we assume that g_0 is a metric with certain curvature properties of the canonical Riemannian structure on $S^1 \times S^2$ such that all deformations nonnegative at first order vanish at first order. We believe this is a natural hypothesis to consider in view of the observation of Berger mentioned above that product manifolds have this property.

I would like to thank Jean-Pierre Bourguignon for suggesting the use of the second Bianchi identity to study the Ricci deformation.

2. A PRELIMINARY CALCULATION

Fix a manifold M^n, $n \geq 2$, and a metric g_0 for M. Let D be the Levi-Civita connection determined by g_0. We will write $\langle\ ,\ \rangle$ for g_0, R for R_{g_0}, K for K_{g_0}, and Ric for Ric_{g_0}. We will use the sign convention

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

for the curvature tensor. For definitions and/or conventions regarding the sectional curvature K and Ricci tensor Ric, we will follow [11]. For a tensor h and an orthonormal basis $\{e_1, \ldots, e_n\}$ for M_p, define D^*Dh at M_p by

$$(D^*Dh)(v_1, v_2, \ldots) = \sum_i (DDh)(e_i, e_i; v_1, v_2, \ldots).$$

Also, define a (4, 0) tensor from R and $\langle\ ,\ \rangle$ which we will also call R without danger of confusion, by $R(u, v, w, z) := \langle R(u, v)w, z\rangle$. Then for all vector fields X, Y, S, T, U, V

$$(*) \qquad \begin{cases} (D_X D_Y R)(S, T, U, V) - (D_Y D_X R)(S, T, U, V) \\ \quad - (D_{[X,Y]} R)(S, T, U, V) \\ = -R(R(X, Y)S, T, U, V) - R(S, R(X, Y)T, U, V) \\ \quad - R(S, T, R(X, Y)U, V) - R(S, T, U, R(X, Y)V). \end{cases}$$

Given (M, g_0) and $h \in C^\infty(S^2(T^*M))$, let $g(t) = g_0 + th$ and define as in [7]

$$(\Sigma(h))(x, y) = \frac{d}{dt} g(t)(R^t(x, y)\, y, x) \Big|_{t=0}$$
$$= (DDh)(x, y; x, y) - \tfrac{1}{2}(DDh)(x, x; y, y)$$
$$- \tfrac{1}{2}(DDh)(y, y; x, x) + h(R(x, y)\, y, x).$$

Then if $\{x, y\}$ is a g_0-orthonormal basis for a two-plane $P \in G_2(M)$,

$$K'(P) = K'(x, y) = (\Sigma(h))(x, y)$$
$$- K(x, y)(h(x, x)\, h(y, y) - (h(x, y))^2).$$

Thus, if $K(x, y) = 0$, then $K'(x, y) = (\Sigma(h))(x, y)$.

In this section, given (M, g_0) we calculate $\Sigma(\mathrm{Ric}_{g_0})$. Fix g_0-orthonormal vectors $x, y \in M_p$ and extend to a g_0-orthonormal basis $\{e_1, \ldots, e_n\}$ for M_p. Extend to local vector fields $X, Y, E_1, \ldots, E_n$ whose Lie brackets vanish near p and with $DX_{|p} = DY_{|p} = DE_{i|p} = 0$ for all i. We will call this a 'good extension' following [7], Definition 3.5.

Now

$$(\Sigma(\mathrm{Ric}))(x, y)$$
$$= \tfrac{1}{2} \sum_i \{D_X D_y R(E_i, X, Y, E_i)$$
$$+ D_X D_Y R(E_i, X, Y, E_i) - D_X D_X R(E_i, Y, Y, E_i)$$
$$- D_Y D_Y R(E_i, X, X, E_i)\} + \mathrm{Ric}(R(x, y)\, y, x),$$

where all expressions involving the extended vector fields are understood to be evaluated at p.

Note that

$$\mathrm{Ric}(R(x, y)\, y, x) = \sum_i R(e_i, R(x, y)\, y, x, e_i)$$
$$= \sum_i \langle R(x, e_i)\, e_i, R(x, y)\, y \rangle$$

so that

$$(\Sigma(\mathrm{Ric}))(x, y)$$
$$= \tfrac{1}{2} \sum_i \{D_X D_Y R(E_i, X, Y, E_i)$$
$$+ D_X D_Y R(E_i, X, Y, E_i) - D_X D_X R(E_i, Y, Y, E_i)$$
$$- D_Y D_Y R(E_i, X, X, E_i) + 2 \langle R(X, E_i)\, E_i, R(X, Y)\, Y \rangle\}.$$

The idea is to use (*) and the second Bianchi identity to write this as $\sum_i D_{E_i} D_{E_i} R(X, Y, Y, X)$ plus curvature terms. For convenience we will calculate twice the ith term in this sum which we will denote Sum(i). Explicitly with X, Y, and E_i as above, let

$$\begin{aligned}\operatorname{Sum}(i) := {} & D_X D_Y R(E_i, X, Y, E_i) + D_X D_Y R(E_i, X, Y, E_i) \\ & - D_X D_X R(E_i, Y, Y, E_i) - D_Y D_Y R(E_i, X, X, E_i) \\ & + 2\langle R(X, E_i) E_i, R(X, Y) Y\rangle .\end{aligned}$$

First

$$\begin{aligned} & D_X D_Y R(E_i, X, Y, E_i) \\ & \quad = D_Y D_X R(E_i, X, Y, E_i) - R(R(X, Y) E_i, X, Y, E_i) \\ & \qquad - R(E_i, R(X, Y) X, Y, E_i) - R(E_i, X, R(X, Y) Y, E_i) \\ & \qquad - R(E_i, X, Y, R(X, Y) E_i) \\ & \quad = D_Y D_X R(E_i, X, Y, E_i) + \langle R(Y, E_i) X, R(X, Y) E_i\rangle \\ & \qquad - \langle R(Y, E_i) E_i, R(X, Y) X\rangle + \langle R(E_i, X) E_i, R(X, Y) Y\rangle \\ & \qquad - \langle R(E_i, X) Y, R(X, Y) E_i\rangle \end{aligned}$$

so that

$$\begin{aligned} & \operatorname{Sum}(i) \\ & \quad = D_Y D_X R(E_i, X, Y, E_i) + D_X D_Y R(E_i, X, Y, E_i) \\ & \qquad - D_X D_X R(E_i, Y, Y, E_i) - D_Y D_Y R(E_i, X, X, E_i) \\ & \qquad + \langle R(X, E_i) E_i, R(X, Y) Y\rangle + \langle R(Y, E_i) X, R(X, Y) E_i\rangle \\ & \qquad - \langle R(Y, E_i) E_i, R(X, Y) X\rangle - \langle R(E_i, X) Y, R(X, Y) E_i\rangle . \end{aligned}$$

Using the second Bianchi identity and then (*),

$$\begin{aligned} & D_X D_Y R(E_i, X, Y, E_i) \\ & \quad = -D_X D_X R(Y, E_i, Y, E_i) - D_X D_{E_i} R(X, Y, Y, E_i) \\ & \quad = D_X D_X R(E_i, Y, Y, E_i) - D_{E_i} D_X R(X, Y, Y, E_i) \\ & \qquad + R(R(X, E_i) X, Y, Y, E_i) + R(X, R(X, E_i) Y, Y, E_i) \\ & \qquad + R(X, Y, R(X, E_i) Y, E_i) + R(X, Y, Y, R(X, E_i) E_i) \\ & \quad = D_X D_X R(E_i, Y, Y, E_i) - D_{E_i} D_X R(X, Y, Y, E_i) \\ & \qquad - \langle R(Y, E_i) Y, R(X, E_i) X\rangle + \langle R(Y, E_i) X, R(X, E_i) Y\rangle \\ & \qquad - \langle R(X, Y) E_i, R(X, E_i) Y\rangle + \langle R(X, Y) Y, R(X, E_i) E_i\rangle . \end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{Sum}\,(i)\\
&\quad = D_Y D_X R(E_i, X, Y, E_i) - D_{E_i} D_X R(X, Y, Y, E_i)\\
&\qquad - D_Y D_Y R(E_i, X, X, E_i) + 2\langle R(X, E_i) E_i, R(X, Y) Y\rangle\\
&\qquad + \langle R(Y, E_i) X, R(X, Y) E_i\rangle - \langle R(Y, E_i) E_i, R(X, Y) X\rangle\\
&\qquad - \langle R(Y, E_i) Y, R(X, E_i) X\rangle + \langle R(Y, E_i) X, R(X, E_i) Y\rangle.
\end{aligned}$$

Now

$$\begin{aligned}
&D_Y D_X R(E_i, X, Y, E_i)\\
&\quad = -D_Y D_Y R(E_i, X, E_i, X) - D_Y D_{E_i} R(E_i, X, X, Y)\\
&\quad = D_Y D_Y R(E_i, X, X, E_i) - D_Y D_{E_i} R(E_i, X, X, Y)
\end{aligned}$$

and

$$\begin{aligned}
&-D_{E_i} D_X R(X, Y, Y, E_i)\\
&\quad = D_{E_i} D_{E_i} R(X, Y, X, Y) + D_{E_i} D_Y R(X, Y, E_i, X)
\end{aligned}$$

so that

$$\begin{aligned}
&D_Y D_X R(E_i, X, Y, E_i) - D_{E_i} D_X R(X, Y, Y, E_i)\\
&\qquad - D_Y D_Y R(E_i, X, X, E_i)\\
&\quad = -D_{E_i} D_{E_i} R(X, Y, Y, X) - D_Y D_{E_i} R(E_i, X, X, Y)\\
&\qquad + D_{E_i} D_Y R(E_i, X, X, Y)\\
&\quad = -D_{E_i} D_{E_i} R(X, Y, Y, X) - R(R(E_i, Y) E_i, X, X, Y)\\
&\qquad - R(E_i, R(E_i, Y) X, X, Y) - R(E_i, X, R(E_i, Y) X, Y)\\
&\qquad - R(E_i, X, X, R(E_i, Y) Y)\\
&\quad = -D_{E_i} D_{E_i} R(X, Y, Y, X) + \langle R(X, Y) X, R(E_i, Y) E_i\rangle\\
&\qquad - \langle R(X, Y) E_i, R(E_i, Y) X\rangle + \langle R(E_i, X) Y, R(E_i, Y) X\rangle\\
&\qquad - \langle R(X, E_i) X, R(Y, E_i) Y\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathrm{Sum}\,(i)\\
&\quad = -D_{E_i} D_{E_i} R(X, Y, Y, X) + 2\langle R(X, E_i) E_i, R(X, Y) Y\rangle\\
&\qquad + 2\langle R(Y, E_i) X, R(X, Y) E_i\rangle + 2\langle R(Y, E_i) E_i, R(Y, X) X\rangle\\
&\qquad - 2\langle R(X, E_i) X, R(Y, E_i) Y\rangle + 2\langle R(Y, E_i) X, R(X, E_i) Y\rangle
\end{aligned}$$

But

$$R(X, Y) E_i = -R(Y, E_i) X - R(E_i, X) Y$$
$$= -R(Y, E_i) X + R(X, E_i) Y$$

so that

$$\begin{aligned}\operatorname{Sum}(i) &= -D_{E_i} D_{E_i} R(X, Y, Y, X) + 2\langle R(X, E_i) E_i, R(X, Y) Y\rangle \\ &\quad - 2\langle R(Y, E_i) X, R(Y, E_i) X\rangle + 4\langle R(Y, E_i) X, R(X, E_i) Y\rangle \\ &\quad + 2\langle R(Y, E_i) E_i, R(Y, X) X\rangle - 2\langle R(X, E_i) X, R(Y, E_i) Y\rangle.\end{aligned}$$

We have shown

PROPOSITION 1. *For* $g(t)=g_0+t(-\operatorname{Ric}_{g_0})$ *and* $\{x, y\}$ g_0*-orthonormal vectors in* M_p

$$(\Sigma(-\operatorname{Ric}_{g_0}))(x, y) = \tfrac{1}{2}(D^*DR)(x, y, y, x) + \operatorname{Curv}(x, y),$$

where for an orthonormal basis $\{e_1, \ldots, e_n\}$ *for* M_p

$$\begin{aligned}\operatorname{Curv}(x, y) &= \sum_i \{\langle R(x, e_i) x, R(y, e_i) y\rangle + \langle R(y, e_i) x, R(y, e_i) x\rangle \\ &\quad - 2\langle R(x, e_i) y, R(y, e_i) x\rangle + \langle R(x, y) x, R(y, e_i) e_i\rangle \\ &\quad + \langle R(y, x) y, R(x, e_i) e_i\rangle\}.\end{aligned}$$

Remark. Although $\langle R(y, e_i)x, R(y, e_i)x\rangle$ is not symmetric in x and y, $\sum_i \langle R(y, e_i)x, R(y, e_i)x\rangle$ is symmetric in x and y.

Given e_i, x, y as before, if we make good extensions to local vector fields E_i, X, Y, then

$$\begin{aligned}(DDR)(e_i, e_i; x, y, y, x) &= e_i(E_i(R(X, Y, Y, X))) - 2\langle R(x, y) y, R(x, e_i) e_i\rangle \\ &\quad - 2\langle R(y, x) x, R(y, e_i) e_i\rangle.\end{aligned}$$

Recall that if $K \geq 0$, then $K(x, y)=0$ implies $R(x, y)y=R(y, x)x=0$. Hence, if $K \geq 0$ and $K(x, y)=0$, we have $(DDR)(e_i, e_i; x, y, y, x)=e_i(E_i(R(X, Y, Y, X)))$ ≥ 0. This proves

LEMMA 2. *If* $K \geq 0$, *then* $K(x, y)=0$ *implies*

$$(D^*DR)(x, y, y, x) \geq 0.$$

3. Applications to compact 3-manifolds

We prove

THEOREM 3. *Let (M, g_0) be a compact 3-manifold with non-negative sectional curvature and everywhere positive Ricci curvature. Then M admits a metric of everywhere positive sectional curvature.*

Remark. It is *false* that $\mathrm{Ric}>0$ implies $K>0$ in 3-dimensions, see [9], Chapter 7, Part 1, for instance, for a counterexample.

Proof. Since M is compact, for small $t>0$, $g(t)=g_0-t\,\mathrm{Ric}_{g_0}$ will be a metric for M. It is enough to show that under the hypotheses, this is a positive deformation at first order for the sectional curvature.

For a 3-manifold, an elementary calculation shows that if $\{x, y, z\}$ are any triple of g_0-orthonormal vectors, then

$$(*)\qquad \mathrm{Curv}\,(x, y) = \mathrm{Ric}\,(x, x)\,\mathrm{Ric}\,(y, y) - (\mathrm{Ric}\,(x, y))^2 - 2\,(K(x, y))^2 - 2K(x, y)\,\mathrm{Ric}\,(z, z).$$

Let P be a zero two-plane for K_{g_0} with orthonormal basis $\{x, y\}$.

Then by Proposition 1 and formula $(*)$

$$(**)\qquad K'\,(x, y) = \tfrac{1}{2}\,(D^*DR)\,(x, y, y, x) + \mathrm{Ric}\,(x, x)\,\mathrm{Ric}\,(y, y) - (\mathrm{Ric}\,(x, y))^2.$$

By Lemma 2 and the hypothesis that the Ricci curvature is everywhere positive, $g(t)$ is a positive geometric deformation at first order. Q.E.D.

COROLLARY 4. *Let (M, g_0) be a compact 3-manifold that is $\frac{1}{2}$ positively Ricci pinched. Then M admits a metric of everywhere positive sectional curvature.*

Proof. It is well-known that $\frac{1}{2}$ positive Ricci-pinching implies $K_{g_0}\geq 0$ in dimension 3. Q.E.D.

In [10] using the method of local convex deformations, we proved several Ricci curvature deformation theorems which imply in 3-dimensions for compact manifolds M that:

(1) if M admits a metric of non-negative Ricci curvature and all Ricci curvatures positive at some point, then M admits a metric of everywhere positive Ricci curvature, and

(2) if M is d-positively Ricci pinched with $0<d<\frac{1}{2}$ and at some point for all vectors the pinching is not attained, then the Ricci pinching can be improved.

Combining (1) and (2), and Corollary 4, it is not unreasonable to conjecture

CONJECTURE. *Let* (M, g_0) *be a compact 3-manifold with non-negative Ricci curvature and all Ricci curvatures positive at some point. Then M admits a metric of everywhere positive sectional curvature.*

Remarks. (1) The proof of Theorem 3 does not generalize to dimensions greater than 3. Indeed the lemma of Berger (see [3]) implies that for the canonical Riemannian structure on $S^2 \times S^2$, all deformations non-negative at first order vanish at first order. For the 'mixed two-planes' which are the zeroes of the sectional curvature of the canonical Riemannian structure on $S^2 \times S^2$, a simple calculation shows that Curv$=0$ and the Ricci deformation $g(t) = g_0 - t \operatorname{Ric}_{g_0}$ does indeed vanish at first order.

(2) Unlike the local convex deformations of [10], there is no symmetry in the cases $K_{g_0} \leq 0$, $\operatorname{Ric}_{g_0} < 0$ and $K_{g_0} \geq 0$, $\operatorname{Ric}_{g_0} > 0$ from the viewpoint of the Ricci deformation $g(t) = g_0 - t \operatorname{Ric}_{g_0}$. In the case $K_{g_0} \leq 0$, $\operatorname{Ric}_{g_0} < 0$ the terms $(D^*DR)(x, y, y, x)$ and $\operatorname{Ric}(x, x)\operatorname{Ric}(y, y)$ in formula (**) will have opposite signs so the proof of Theorem 3 does not work.

(3) Notice that if $K_{g_0} \geq 0$, $\operatorname{Ric}_{g_0} \geq 0$, formula (**) shows that $g(t) = g_0 - t \times \operatorname{Ric}_{g_0}$ is a deformation that is non-negative at first order. This is studied in some detail in [9], Chapter 7, Part 2.

4. The Ricci curvature tensor and the Ricci deformation

For completeness, we sketch the calculation of Ric′ for $g(t) = g_0 - t \operatorname{Ric}_{g_0}$. For more details, see [9], Chapter 7, Part 2. Given a symmetric two tensor h, define a 1-form $\delta' h$ by $(\delta' h)(w) := \sum_i (D_{e_i} h)(e_i, w)$ for any $w \in M_p$ where $\{e_i\}$ is an orthonormal basis for M_p.

First recall the classical

LEMMA 5. *Let* $\{e_i\}$ *be an orthonormal basis for* M_p. *Then*

$$(\delta' \operatorname{Ric})(w) := \sum_i (D_{e_i} \operatorname{Ric})(e_i, w) = \tfrac{1}{2} w(\tau).$$

In [2], Berger gives (with a sign mistake) the Classical formula which for our sign convention for δ' becomes

$$\operatorname{Ric}' = -\tfrac{1}{2} D^*Dh + \tfrac{1}{2} \operatorname{Ric} \otimes h - R \otimes h - \delta^* \delta' h - \tfrac{1}{2} \operatorname{Hess}(\operatorname{tr} h)$$

where for an orthonormal basis $\{e_i\}$ for M_p,

$$(\operatorname{Ric} \otimes h)(x, y) := \sum_i (\operatorname{Ric}(x, e_i)\, h(y, e_i) + \operatorname{Ric}(y, e_i)\, h(x, e_i))$$

and

$$(R \otimes h)(x, y) := \sum_{i,j} R(x, e_i, e_j, y)\, h(e_i, e_j).$$

Since $h = -\mathrm{Ric}$, by Lemma 1, $\delta' h = \frac{1}{2} d\tau$ so that

$$\delta^* \delta' h + \tfrac{1}{2} \operatorname{Hess}(\operatorname{tr} h) = \tfrac{1}{2}\delta^*(d\tau) - \tfrac{1}{2}\delta^*(d\tau) = 0.$$

Thus,

$$\text{(1)} \qquad \mathrm{Ric}' = \tfrac{1}{2} D^* D\, \mathrm{Ric} - \tfrac{1}{2} \mathrm{Ric} \otimes \mathrm{Ric} + R \otimes \mathrm{Ric}.$$

Suppose (M, g_0) is a compact 3-manifold with $\mathrm{Ric}_{g_0} \geq 0$. It is not difficult to see that if $\mathrm{Ric}(x, x) = 0$, then $(D^* D\mathrm{Ric})(x, x) \geq 0$ (compare Lemma 2 above).

Given an orthonormal basis $\{u, v, w\}$ for M_p, recall that

$$\text{(2)} \qquad K(u, v) = \tfrac{1}{2}(\mathrm{Ric}(u, u) + \mathrm{Ric}(v, v) - \mathrm{Ric}(w, w)).$$

Let $x \in M_p$ be given. Extend to an orthonormal basis $\{x, y, z\}$ for M_p. Writing out formula (2) in 3 dimensions, we obtain

$$\begin{aligned} &\mathrm{Ric}'(x, x) \\ &= \tfrac{1}{2}(D^* D\, \mathrm{Ric})(x, x) - (\mathrm{Ric}(x, x))^2 - (\mathrm{Ric}(x, y))^2 \\ &\quad - (\mathrm{Ric}(x, z))^2 + R(x, y, y, x)\, \mathrm{Ric}(y, y) \\ &\quad + R(x, z, z, x)\, \mathrm{Ric}(z, z) + 2R(x, y, z, x)\, \mathrm{Ric}(y, z) \\ &= \tfrac{1}{2}(D^* D\, \mathrm{Ric})(x, x) - (\mathrm{Ric}(x, x))^2 - (\mathrm{Ric}(x, y))^2 \\ &\quad - (\mathrm{Ric}(x, z))^2 + K(x, y)\, \mathrm{Ric}(y, y) \\ &\quad + K(x, z)\, \mathrm{Ric}(z, z) + 2(\mathrm{Ric}(y, z))^2. \end{aligned}$$

Substituting (2) for $K(x, y)$ and $K(x, z)$ we obtain

$$\text{(3)} \qquad \begin{aligned} &\mathrm{Ric}'(x, x) \\ &= \tfrac{1}{2}(D^* D\, \mathrm{Ric})(x, x) - (\mathrm{Ric}(x, x))^2 - (\mathrm{Ric}(x, y))^2 \\ &\quad - (\mathrm{Ric}(x, z))^2 + \tfrac{1}{2} \mathrm{Ric}(x, x)(\mathrm{Ric}(y, y) + \mathrm{Ric}(z, z)) \\ &\quad + \tfrac{1}{2}(\mathrm{Ric}(y, y) - \mathrm{Ric}(z, z))^2. \end{aligned}$$

Recall that if $\mathrm{Ric} \geq 0$, then $\mathrm{Ric}(x, x) = 0$ implies $\mathrm{Ric}(x, v) = 0$ for all v. Thus given a g_0-unit vector $x \in M_p$ with $\mathrm{Ric}(x, x) = 0$, let $l_1(x)$ and $l_2(x)$ be the non-zero eigenvalues for $\mathrm{Ric}: M_p \times M_p \to R$. We may then choose vectors y, z so that $\{x, y, z\}$ forms an orthonormal basis for M_p and diagonalizes $\mathrm{Ric}: M_p \times M_p \to R$, with $\mathrm{Ric}(y, y) = l_1(x)$ and $\mathrm{Ric}(z, z) = l_2(x)$. With this choice of $\{x, y, z\}$, (3) reduces to

PROPOSITION 6. *Suppose* $\mathrm{Ric}_{g_0} \geq 0$. *If* x *is a* g_0*-unit vector with* $\mathrm{Ric}_{g_0}(x, x) = 0$, *then*

$$\mathrm{Ric}'(x, x) = \tfrac{1}{2}(D^*D\,\mathrm{Ric})(x, x) + \tfrac{1}{2}(l_1(x) - l_2(x))^2.$$

Hence, the Ricci curvature deformation of the Ricci curvature is non-negative at first order.

Remark. Consider $S^1 \times S^2$ with the standard product metric. Let X be the global unit parallel field produced on $S^1 \times S^2$ from a trivialization of TS^1. Then the line field determined by X on $S^1 \times S^2$ is precisely the set of zero vectors of the Ricci tensor. Here $l_1(X) = l_2(X)$ and

$$(D^*D\,\mathrm{Ric})(X, X) = \tau\,\|DX\|^2 = 0 \quad \text{since} \quad DX = 0$$

(where τ is the scalar curvature) so $\mathrm{Ric}'(X, X) = 0$. For more on the study of 'Ricci product-like metrics' whose curvature behavior is modelled on this example, see [8] and [9].

For completeness we note that in n dimensions ($n \geq 2$) for the Ricci deformation $g(t) = g_0 - t\,\mathrm{Ric}_{g_0}$, the classical formula ([4], formula 5.3, p. 385) for the first derivative τ' of the scalar curvature for an arbitrary deformation reduces to

$$\tau' = \tfrac{1}{2}\Delta\tau_{g_0} + \|\mathrm{Ric}_{g_0}\|^2.$$

Here we use the sign convention $\Delta = \mathrm{tr} \circ \mathrm{Hess}$ opposite to the sign convention in [4]. Thus if $\tau \geq 0$ (but is not everywhere positive), the Ricci deformation is non-negative at first order for the scalar curvature. If we define critical metrics for manifolds M in the class of Riemannian manifolds (M, g_0) with $\tau_{g_0} \geq 0$ (but τ_{g_0} is not everywhere positive) analogous to our definition in Section 5 below for the sectional curvature, it is immediate that g_0 is a critical metric iff Ric_{g_0} vanishes identically on the set of vectors lying above the set of points of zero scalar curvature.

As in the case of the sectional curvature in Section 3, the cases of non-negative and non-positive scalar curvature are not symmetric. Indeed for (M, g_0) with non-positive (but not everywhere negative) scalar curvature, we can be sure the first derivative τ' of the Ricci deformation has a definite sign as points p of zero scalar curvature only if (M, g_0) is Ricci flat in which case the first derivative vanishes identically.

5. A CONVERSE TO A LEMMA OF BERGER IN 3-DIMENSIONS

In [3], Berger proved

LEMMA. *Let* (M_1, g_1) *and* (M_2, g_2) *be two compact Riemannian manifolds. Let* $(M, g) = (M_1 \times M_2, g_1 \times g_2)$ *be the Riemannian product manifold. Let* $g(t)$

be a deformation of g_0 that is non-negative at first order. Then $g(t)$ vanishes identically at first order on the zero two-planes of K_g.

In 3-dimensions, we have seen that if (M, g_0) is compact and $K_{g_0} \geq 0$, $\mathrm{Ric}_{g_0} > 0$, then M admits a metric of everywhere positive sectional curvature. However, we noted above that if $\mathrm{Ric} \geq 0$ only, the proof of Theorem 3 does not work. Indeed with a standard product metric g_0 on $S^1 \times S^2$, $K \geq 0$, $\mathrm{Ric} \geq 0$ but since $b_1(M) = 1$, g_0 does not admit a metric of positive sectional or even positive Ricci curvature. Hence, there is no geometric deformation of g_0 that is positive at first order. As we remarked in Section 4, the Ricci deformation vanishes on the zero Ricci directions at first order as it must by Berger's Lemma. In fact, if we let X be the vector field of the remark at the end of Section 4, $\mathrm{Ric}^t(X, X) = 0$ so the Ricci deformation vanishes on the zero Ricci directions at all orders.

It is then natural given a compact 3-manifold (M, g_0) with certain curvature properties of the product metric on $S^1 \times S^2$ and such that for all geometric deformations of g_0 any deformation non-negative at first order vanishes identically at first order (as in Berger's Lemma) to ask if (M, g_0) is locally isometric to a product.

Let $(M, g) := (S^1 \times S^2, g_1 \times g_2)$ where g_1 is the usual metric on S^1 and g_2 is any metric for S^2 with everywhere positive Gaussian curvature. Then $K_g \geq 0$, $\tau_g > 0$, and there is a zero two plane $P \subset M_p$ with $K_g(P) = 0$ for all $p \in M$. By the Bochner theory, if g_0 is any metric on $S^1 \times S^2$ with $K_{g_0} \geq 0$, there does not exist any point $p \in S^1 \times S^2$ with all sectional curvatures positive at p.

We thus consider the set $\mathscr{P} := \{(M^3, g_0);\ M$ is compact, $K_{g_0} \geq 0$, $\tau_{g_0} > 0$ and $\exists$ 2-plane $P \subset M_p$ for all $p \in M$ with $K_{g_0}(P) = 0\}$.

DEFINITION. Let $(M, g_0) \in \mathscr{P}$. We say g_0 is a *critical metric* for M iff for all symmetric 2-tensors h, if the deformation $g(t) = g_0 + th$ of g_0 is non-negative at first order, then it vanishes identically at first order on all the zero two-planes for K_{g_0}.

THEOREM 7. *Suppose $(M, g_0) \in \mathscr{P}$ and g_0 is a critical metric for M. Then M is locally isometrically a product* (in the sense of Proposition 5.1 (2), [12]).

Remark. It is clear that if we replace the condition $\tau_{g_0} > 0$ by the condition $\tau_{g_0} \geq 0$ in the definition of $\mathscr{P}$, if g_0 is not a flat metric then the appropriately modified version of Theorem 7 holds on the open set where $\tau_{g_0} > 0$.

First we state three elementary lemmas, the proofs of which may be safely left to the reader.

LEMMA 8. *Let* $\{x, v, w\} \in M_p$ *be a triple of* g_0*-orthonormal vectors. Then*

$$(D^*D\,\mathrm{Ric})(x, x) = (D^*DR)(v, x, x, v) + (D^*DR)(w, x, x, w).$$

LEMMA 9. *Given* (M^3, g_0) *with* $K_{g_0} \geq 0$. *Let* $p \in M$. *Then either*

(1) $\mathrm{Ric}_{g_0}(v, v) >$ *for all* $v \neq 0$ *in* M_p,
(2) $\exists x \neq 0$ *in* M_p *with* $\mathrm{Ric}_{g_0}(x, x) = 0$ *and* $\mathrm{Ric}_{g_0}(v, v) = 0$ *iff* $v \in Rx$, *or*
(3) $\mathrm{Ric}_{g_0}(v, v) = 0$ *for all* $v \in M_p$.

In case (2), $K_{g_0}(x, v) = 0$ *iff* $g_0(x, v) = 0$.

Suppose the Ricci tensor of (M^3, g_0) satisfies the following two conditions: (1) the scalar curvature $\tau = \tau_{g_0}$ is never zero, and (2) for each $p \in M$, one of the eigenvalues of the Ricci tensor on M_p is zero and the other two eigenvalues are equal and non-zero. We call such a metric *almost Ricci product-like.*

If (M, g_0) is almost Ricci product-like, we can find a local unit vector field X such that $\mathrm{Ric}_{g_0}(X, X) = 0$. Then X satisfies $D_X X = 0$ and $\mathrm{div}\, X = -X(\tau)/\tau$ (see [9]). Also, locally $\mathrm{Ric}_{g_0} = (\tau/2)(g_0 - X^b \otimes X^b)$ where X^b is the 1-form associated to X by g_0, i.e., $(X^b)(Y) := g_0(X, Y)$. A simple calculation gives

LEMMA 10. *Let* $g(t) = g_0 - t\,\mathrm{Ric}_{g_0}$ *where* $\mathrm{Ric} = \mathrm{Ric}_{g_0} = (\tau/2)(g_0 - X^b \otimes X^b)$ *as above. Then*

$$\mathrm{Ric}'(X, X) = \tfrac{1}{2}(D^*D\,\mathrm{Ric})(X, X) = \tfrac{1}{2}\tau\, \|DX\|^2.$$

Proof of Theorem 7

Let $(M, g_0) \in \mathscr{P}$. We write K for K_{g_0} and Ric for Ric_{g_0}. Given any $p \in M$, $\exists$ a two plane $P \subset M_p$ with $K(P) = 0$. Choose a g_0-orthonormal basis $\{x, y\}$ for P with $\mathrm{Ric}(x, y) = 0$. Let $g(t) = g_0 - t\,\mathrm{Ric}$. From Section 3, this a non-negative variation at first order. Hence

$$K'(P) = \tfrac{1}{2}(D^*DR)(x,y,y,x) + \mathrm{Ric}(x,x)\,\mathrm{Ric}(y,y) = 0$$

so $\mathrm{Ric}(x, x)\,\mathrm{Ric}(y, y) = 0$. Since $\tau(p) > 0$ by hypothesis, assume $\mathrm{Ric}(x, x) = 0$, $\mathrm{Ric}(y, y) \neq 0$. By Lemma 9, Rx can be the only zero line for Ric in M_p and it follows that $(M, g_0) \in \mathscr{P}$ implies (M, g_0) is almost Ricci product-like.

Now let X be a local unit vector field with $\mathrm{Ric}(X, X) = 0$ and $\mathrm{Ric} = (\tau/2) \times (g_0 - X^b \otimes X^b)$ as above. Then by Lemma 10, $(D^*D\,\mathrm{Ric})(X, X) = \tau \|DX\|^2$. But now take local fields Y and Z orthonormal to X. By Lemma 8,

$$\tau\, \|DX\|^2 = (D^*D\,\mathrm{Ric})(X, X) = (D^*DR)(Y, X, X, Y) + (D^*DR)(Z, X, X, Z).$$

But $K(X, Y)=0$ from Lemma 9, so that

$$0 = K'(X, Y) = \tfrac{1}{2}(D^*DR)(X, Y, Y, X)$$

since $(M, g_0)\in\mathscr{P}$. Similarly $K(X, Z)=0$ implies $(D^*DR)(X, Z, Z, X)=0$. Hence $\tau\|DX\|^2=0$. But then since $\tau>0$, this forces $DX\equiv 0$ and we are done by the deRham Decomposition Theorem. Q.E.D.

We briefly comment on this type of rigidity for the Ricci curvature in 3 dimensions. Fix a non-flat metric g_0 for M with non-negative Ricci curvature. We say g_0 is *strongly critical at first order* if every geometric deformation of g_0 non-negative at first order vanishes at first order. It follows from the local convex deformations of [10] that if g_0 is a critical metric, there is a zero Ricci curvature at each point of M.

Let us also say g_0 is *Ricci product-like at p M* iff (1) the scalar curvature is non-zero at p, and (2) one eigenvalue of $\text{Ric}: M_p\times M_p\to R$ is zero and the other two eigenvalues are equal and non-zero. Let $Z(g_0)$ be the closed set of points of M at which all Ricci curvatures vanish. Arguments similar to those given above for the sectional curvature show

THEOREM 11. *Given (M^3, g_0) compact with $\text{Ric}_{g_0}\geqslant 0$, suppose g_0 is a metric strongly critical at first order for M. Then for all $p\in M$, either*

(i) *all Ricci curvatures are zero at p, or*

(ii) *there is an open neighborhood U about p such that g_0 is Ricci product-like on U and a unit parallel vector field X on U with* $\text{Ric}(X, X)=0$.

Hence in the non-flat part of M, namely $M-Z(g_0)$, M is locally a product in the sense of (ii).

BIBLIOGRAPHY

1. Aubin, T., 'Metriques riemannienes et courbure', *J. Diff. Geom.* 4 (1970), 383–424.
2. Berger, M., 'Sur les varietes d'Einstein compactes', *Comptes Rendues des Mathematiciens d'Expression Latine* (1965), 35–55.
3. Berger, M., 'Trois remarques sur les variétés riemannienes à courbure sectionelle positive', *Compt. Rend. Acad. Sci. Paris*, 263 Series A (11 juillet, 1966), 76–78.
4. Berger, M. and Ebin, D., 'Some Decompositions of the Space of Symmetric Tensors on a Riemannian Manifold', *J. Diff. Geom.* 3 (1969), 379–392.
5. Bourguignon, J.-P., 'Some Constructions Related H. Hopf's Conjecture on Product-Manifolds', to appear in *Proceedings of the A.M.S. Summer Institute in Differential Geometry* (1973).
6. Bourguignon, J.-P., 'Sur le vecteur de courbure d'une variété riemanniene compacte', *Compt. Rend. Acad. Sci.* (1971).
7. Bourguignon, J.-P., Deschamps, A., and Sentenac, P., 'Conjecture de H. Hopf sur les variétés produits', *Ann. Sci. Ecole Normale Supérieure* (1972), 277–302.
8. Bourguignon, J.-P. and Ehrlich, P., 'Constant Nullity Metrics on Complete 3-Manifolds', to appear.

9. Ehrlich, P., 'Metric Deformations of Ricci and Sectional Curvature on Compact Riemannian Manifolds', thesis, S.U.N.Y., at Stony Brook, 1974.
10. Ehrlich, P., 'Metric Deformations of Curvature. I: Local Convex Deformations', *Geometriae Dedicata* **5** (1976), 1–23.
11. Gromoll, D., Klingenberg, W., and Meyer, W., *Riemannsche Geometrie im Großen*, Springer-Verlag, Lecture Notes in Mathematics, No. 55, 1968.
12. Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, Vol. I, Interscience Tracts in Pure and Applied Mathematics, No. 15, Interscience, New York, 1963.

Author's address:

Paul Ehrlich,
Mathematisches Institut der Universität Bonn,
5300 Bonn,
Wegelerstraße 10,
Federal Republic of Germany

(Received April 17, 1974; in revised form February 1, 1975)