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CONDORCET'S PARADOX*

1. INTRODUCTION

Voting systems can produce many results that seem to be paradoxical. Some of the more noted paradoxes can be found in Fishburn (1974) and Niemi and Riker (1976). None of these paradoxes has received as much attention as the paradox attributed to Condorcet (1785). To give some background on how Condorcet's paradox can result, consider an example of an election on three alternatives $\{A, B, C\}$ with n voters. We assume that each voter's preferences on the alternatives can be represented by a linear ranking on the alternatives. That is, no voter is indifferent between any pair of alternatives. We also assume throughout that all voters cast votes in agreement with their preferences and that their preferences are independent of the preferences of other voters.

There are six possible preference rankings on the three alternatives. When $A > B$ denotes a voter preference for A over B these six orders are:

$$A > B > C: n_1^3, p_1^3, \qquad A > C > B: n_2^3, p_2^3$$

$$B > A > C: n_3^3, p_3^3, \qquad C > A > B: n_4^3, p_4^3$$

$$B > C > A: n_5^3, p_5^3, \qquad C > B > A: n_6^3, p_6^3$$

where p_i^3 is the proportion of the population with the ith linear ranking representing their preferences on the alternatives. Similarly, n_i^3 is the number of actual voters with the ith linear ranking representing their preferences. Let p^3 be the six dimensional vector of p_i^3's with $\Sigma_{i=1}^6 p_i^3 = 1$. For an election with an n voter turnout $\Sigma_{i=1}^6 n_i^3 = n$. For the linear preference rankings, a given set of n_i's is referred to as a voter profile.

Condorcet's paradox occurs when we obtain cycles in group preference when pairwise comparisons are made on the basis of simple majority voting. For example, with a three voter turnout we could have $n_1^3 = 1$, $n_4^3 = 1$ and

Theory and Decision **15** (1983) 161–197. 0040–5833/83/0152–0161$03.70.

$n_5^3 = 1$. In this example $A >_s B$, $B >_s C$ and $C >_s A$ where $A >_s B$ denotes group simple majority preference for A over B. As a result, we find that there is no alternative capable of defeating all other candidates on the basis of pairwise simple majority elections. If $n_1^3 = 1, n_2^3 = 1, n_6^3 = 1$, we find that A is the Condorcet winner or simple majority winner since $A >_s C$ and $A >_s B$. This form of the paradox has received the most attention in the literature. This paradox has also been referred to as 'cyclical majorities', 'l'effect Condorcet', 'Arrow's Paradox', and simply 'the paradox of voting'. To avoid confusion with other paradoxes of voting, and to give credit to the individual who was reported to be the first to discover the paradox, we use the name 'Condorcet's paradox' throughout the study. In particular, we specify the form of the paradox described above as the no-winner form of Condorcet's paradox.

A second form of Condorcet's paradox is the majority cycle paradox. To observe this form of Condorcet's paradox we must consider more than three alternatives. For four alternatives $\{A, B, C, D\}$ there are 24 linear preference orders given by:

$A > B > C > D : n_1^4, p_1^4$	$C > A > B > D : n_{13}^4, p_{13}^4$
$A > B > D > C : n_2^4, p_2^4$	$C > A > D > B : n_{14}^4, p_{14}^4$
$A > C > B > D : n_3^4, p_3^4$	$C > B > A > D : n_{15}^4, p_{15}^4$
$A > C > D > B : n_4^4, p_4^4$	$C > B > D > A : n_{16}^4, p_{16}^4$
$A > D > B > C : n_5^4, p_5^4$	$C > D > A > B : n_{17}^4, p_{17}^4$
$A > D > C > B : n_6^4, p_6^4$	$C > D > B > A : n_{18}^4, p_{18}^4$
$B > A > C > D : n_7^4, p_7^4$	$D > A > B > C : n_{19}^4, p_{19}^4$
$B > A > D > C : n_8^4, p_8^4$	$D > A > C > B : n_{20}^4, p_{20}^4$
$B > C > A > D : n_9^4, p_9^4$	$D > B > A > C : n_{21}^4, p_{21}^4$
$B > C > D > A : n_{10}^4, p_{10}^4$	$D > B > C > A : n_{22}^4, p_{22}^4$
$B > D > A > C : n_{11}^4, p_{11}^4$	$D > C > A > B : n_{23}^4, p_{23}^4$
$B > D > C > A : n_{12}^4, p_{12}^4$	$D > C > B > A : n_{24}^4, p_{24}^4$.

As in the three alternative cases $\Sigma_{i=1}^{24} p_i^4 = 1$ where p^4 is the 24 dimensional vector of p_i^4's associated with the linear preference orders listed above and $\Sigma_{i=1}^{24} n_i^4 = n$.

Suppose $n_1^4 = 1$, $n_4^4 = 1$ and $n_5^4 = 1$. Then, the no-winner form of Condorcet's paradox does not exist since $A >_s B$, $A >_s C$ and $A >_s D$. However, a cycle still exists since $B >_s C$, $C >_s D$ and $D >_s B$. In this situation we have the majority cycle form of Condorcet's paradox. That is, we have a Condorcet winner but a majority cycle exists on some set of alternatives that does not contain the Condorcet winner. Obviously, if the no-winner paradox results then the majority cycle form of the paradox also exists. If the majority cycle form of Condorcet's paradox does not hold, then the simple majority relation on the pairs of alternatives is transitive.

The situation in which there are only two candidates can be considered as a special case. While there can be no majority cycle on the candidates in the two alternative cases, a simplified form of Condorcet's paradox can occur. One possibility is that due to ties, there may be no majority rule winner when there are an even number of votes. Another possibility is that when abstentions are allowed, the elected majority candidate may be different than the overall majority candidate of the population.

As was noted above, the discovery of this paradox is credited to Condorcet (1785). The paradox has had an interesting history of rediscovery by individuals such as C. L. Dodgson [Lewis Carrol] (discussed in Black (1958), Huntington (1938) and in a series of articles by Black (1948a, 1948b, 1948c, 1948d, 1949a, 1949b, 1949c). Thorough reviews of the history of Condorcet's paradox can be found in Granger (1956), Black (1958) and Riker (1961). In the time since Riker's (1961) study, a great deal of work has been published that relates to Condorcet's paradox. Studies have been concerned with the relationship between Condorcet's paradox and factors related to societies such as social homogeneity [Jamison and Luce (1972), Fishburn (1973c), Abrams (1976), and Fishburn and Gehrlein (1980a)] and voter antagonism [Kuga and Nagatani (1974)]. Other studies have examined the relationship between Condorcet's paradox and logrolling [Bernholz (1973, 1974), Koehler (1975), Sullivan (1976)]. Another area of interest has been the relationship between Arrow's Possibility Theorem [Arrow (1963)] and Condorcet's paradox [Campbell and Tullock (1965), Tullock (1967) and Tullock and Campbell (1970)]. Argument both for and against the use of majority rule decisions to determine winners can be found in Tullock (1959), Downs (1961) and Sen (1970). A number of studies have been concerned with the propensity of simple voting systems to elect the Condorcet winner when there is one

[Paris (1975), Ludwin (1976), Chamberlin and Cohen (1968), Gehrlein (1981d) and Fishburn and Gehrlein (1982)]. Craven (1971) and Rosenthal (1975) consider variations of majority rule which attempt to reduce the probability of Condorcet's paradox. The current study concerns itself with the simplest of all questions about Condorcet's paradox, specifically, "How likely is the paradox to occur?"

The discussion of the likelihood of Condorcet's paradox takes place on two levels. The first level is to consider how often the paradox has occurred in practice. The second level addresses analytical representations for the probability of the paradox. When addressing this second level, Williamson and Sargent (1967) point out that a distinction must be made between developing representations for the probability of the paradox and finding conditions which require that the paradox cannot occur. The latter has been studied extensively and a general discussion of the topic can be found in Sen (1970) and Fishburn (1973a). The current study considers actual occurrences of Condorcet's paradox and analytical representations for the likelihood of the paradox.

The remainder of the paper is organized as follows. In Section 2, a review is presented of studies which analyze cases where the paradox of voting have occurred or may have occurred. All remaining sections are generally related to analytical representations for the probability of Condorcet's paradox. Section 3 is devoted to studies which have been concerned with specific forms of analytical representations for the probability of the paradox. In this section attention is restricted to a finite number of voters with linear orders for preferences. Section 4 considers approximations that have been developed for the probability of the paradox. Section 5 concerns itself with the probability of the paradox when voters have linear orders for preferences and the number of voters is large (approaching infinity). Section 6 considers work that has been done on recursion relations for the probability of Condorcet's paradox. That is, for a given number of alternatives and voters the paradox probability can sometimes be expressed in terms of the probability of the paradox with fewer alternatives or fewer voters.

Section 7 considers work which concerns itself with the general behavior of representations for the probability of the paradox. The behavior of importance is the determination of how the probability of the paradox changes as we increase the number of alternatives or the number of voters. The final section is devoted to the consideration of representations for the probability of Condorcet's paradox as we remove the strict restriction of linear preference rankings. Specifically, we consider the case where abstentions are allowed and where preferences other than linearly ordered preferences can represent voters' preferences.

2. ACTUAL OCCURRENCES OF CONDORCET'S PARADOX

Actual observations of either form of Condorcet's paradox are difficult to find. This difficulty results from the complexity of the voting system that would be required to observe the paradox. One procedure to observe the paradox would be to require voters to use majority voting on all parts of candidates. Another procedure would be to require all voters to rank all candidates in order of preference. Under the ranking procedure we would have to assume that rankings on pairs of candidates would remain the same as their relative ranking on the overall set of candidates. Actual election results for elections meeting either of these conditions are scarce. Marz *et al.* (1973) consider the number of pairwise comparisons that must be made to be assured that the Condorcet winner is found if there is one.

Some studies have been conducted to determine if the preferences of individuals in a group lead to either form of Condorcet's paradox. Most studies of this type have proceeded by obtaining the linear preference rankings on the items under consideration. Jamison (1975) considered both forms of Condorcet's paradox while examining the preferences of two groups of individuals on three sets of objects. The two groups consisted of a set of 67 graduate students and a set of 42 undergraduate students. Each subject was required to rank order (no ties permitted) preferences on 9 possible presidential candidates, 12 types of soup, and 11 European cities which could be toured.

For each of the six special cases in Jamison's (1975) experiment, the preference rankings that were obtained became the basis of a computer simulation experiment. The simulation analysis was performed in the same fashion for each of the six cases. For each number of individuals, n, with $n = \{3, 5, \ldots, 15\}$, select n individuals at random, without replacement, from the respondents. For a predetermined number of alternatives, m, with $m = \{3, 4, 5, 6\}$, randomly select a set of m alternatives. Reduce the preference rankings for the n individuals by moving all alternatives not in the chosen subset. For the n individuals and m alternatives, determine if the no-winner form of Condorcet's paradox exists and determine if the majority cycle form of Condorcet's paradox exists. Repeat the process 4,500 times and determine the proportion of time that each form of Condorcet's paradox results.

Over all six cases that were considered, the probability of the no-winner form of the paradox generally ranges between 0.02 and 0.15 while the majority cycles form of the paradox generally ranges between 0.02 and 0.40. The general trends in the simulation results suggested that the probability of observing either form of Condorcet's paradox is minimized when there are five voters. For all numbers of alternatives considered, the probability of observing either form of the paradox tended to decrease monotonically as the number of voters was increased or decreased from five.

Niemi (1970) examined voting results from university elections in which individuals were elected to committees. In these results, voters were not required to vote for all candidates. Voters were required to rank only as many candidates as they wished. All unranked candidates were assumed to be tied with each other at the bottom of each individual's preference order. That is, if an individual responded with $A > B > C$ in a five-candidate election, then the remaining two candidates would be assumed to be tied with each other and each would be ranked below C.

Using these preference assumptions, Niemi (1970) considered actual election results for 22 elections. These elections were held on three to 36 candidates with the number of voters ranging from 81 to 463. With this modification in the assumption of linear preferences, ties could now exist in pairwise comparisons. There were 18 elections with six or fewer candidates. Of the 18 elections, there was one strict occurrence of majority cycles form of the paradox and three occurrences of the majority cycles form of the paradox which involved ties. In all four of these elections, there would have been no strict Condorcet winner. That is, there was no candidate that could defeat all remaining candidates in majority voting when ties are not considered a victory. The results suggest that the majority cycles form of the paradox tends to become more likely as the number of candidates involved in the election increases.

Gehrlein and Bonwit (1981) obtained the pairwise preference comparisons of juveniles for time spent in various activities: with their family, reading, watching television, in church, and with friends. The 154 subjects were given the option of not responding on any of the 15 pairs presented to them. During the test, 19 subjects gave responses that contained cyclic preferences in their individual preference ranking. These 19 subjects were removed from consideration. The pairwise responses of the remaining 135 subjects gave a clearly transitive result on the basis of majority comparison.

Dobra and Tullock (1981) considered the results of a departmental election for a new chairman at a university. Pairwise comparisons were deduced from 1 to 10 scale measurement that voters gave to each of 37 candidates. Since an individual voter could give the same score to two candidates, ties were assumed in the preference rankings. On one criterion, six voters found a clear Condorcet winner. On a second criterion, four voters found a candidate that could not be defeated by another candidate on the basis of pairwise comparison, but this candidate did tie with three other candidates when compared to them.

Riker (1958) examined congressional voting results for the Committee of the Whole on the Agricultural Appropriation Act of 1953. In the original bill, \$250 million was to be provided to the Soil Conservation Service. Four amendments were put forth to modify the appropriation to \$142,410,000; \$100 million; \$200 million; and \$225 million. Because of the nature of the process used in voting on the amendments, it cannot be definitely determined that the majority preferences of the group contained a majority cycle. However, Riker (1958) argues convincingly that some of the preferences on budgeted amounts that are given in the amendments could have been involved in a majority cycle. That cycle may well have included the winning option, which was the original bill. Riker (1958) estimated that the House of Representatives and Senate may have voting results which contain majority voting cycles in more than 10 percent of the cases when two or more amendments are considered with the original bill. By the rules of the House of Representatives, no more than four amendments may be considered at one time, along with the original bill (five alternatives for monetary bills as described above).

Blydenburgh (1971) examined congressional voting results on two major revenue bills when roll-call voting was used to vote on amendments to the proposed bills. By requesting the imposition of the closed rule, the Ways and Means Committee can prevent the addition of amendments to bills from the floor. Since this request is usually made, few observations of multiple alternative elections are observed in major revenue bills. The Revenue Acts of 1932 and 1938 were both voted on without the imposition of the closed rule. Three amendments were attached to the Revenue Act of 1932 and four amendments were attached to the Revenue Act of 1938. Blydenbugh (1971) uses the roll-call results recorded on the votes and attempted to reconstruct the preferences of the voters on various forms of taxation. The results of voting on the 1932 Revenue Act suggest that majority preference probably cycles on three available forms of taxation that were considered (Excise Tax, Sales Tax and Income Tax).

The results of voting on the 1938 Revenue Act resulted in a comparison of the original act, the act with the deletion of a corporate tax, and the act with the addition of an excise tax on pork. By reconstructing likely preferences on the basis of the roll-call votes on the amendments, the majority preference relation did not cycle.

Bowen (1972) examined roll-call votes from bills considered by the United States Senate. Only those bills were considered which had one or more amendments attached to them. For the one amendment case, the bills considered allowed for three alternatives: the original bill, the amendment bill, and the status quo (neither). Two amendment bills have five alternatives: the original bill, the bill with the first amendment, the bill with the second amendment, the bill with both amendments, and the status quo.

Based upon the roll-call vote, Bowen (1972) made some assumptions to find estimates of the p_i^m probabilities for the m alternative cases. For bills which contained amendments between 1958 and 1966, 98 had a probability of no Condorcet winner that was not significantly different than zero. There were 13 which had a probability of no Condorcet winner that was significantly different than zero. Significance was determined with the probability of a type-one error equal to 0.05.

The notable cases showing a high probability of exhibiting the no-winner form of Condorcet's paradox are the Wheat Act of 1960 (3 amendments with probability 0.940), Housing Act of 1960 (3 amendments with probability 0.460), Food and Agriculture Act of 1962 (9 amendments with probability 0.965), the Economic Opportunity Amendments of 1965 (16 amendments with probability 0.630), and the Alternate Crops Act of 1966 (5 amendments with probability 0.390). The results generally suggest that it is very unlikely that the no-winner form of Condorcet's paradox occurs with two or fewer amendments, or with five or fewer alternatives in Senate voting. The probability is more likely, but still small when more alternatives are considered.

Weisberg and Niemi (1972) consider elections held in the House of Representatives and the Senate. Their analysis generally follows the format of Bowen's (1972) study. The major difference in the two studies concerns the assumptions made in estimating the p_i^m values. Weisberg and Niemi (1972) use a more sophisticated model to deduce their version of the p_i^m's from the roll-call vote results. Their results generally produce substantially lower estimates for the probability of the no-winner form of Condorcet's paradox. For example, Weisberg and Niemi estimate the probability of no-Condorcet winner to be 0.198 for the Wheat Act of 1960.

3. ANALYTICAL REPRESENTATIONS FOR FINITE VOTERS WITH LINEAR ORDERED PREFERENCE

To describe analytical representations for the probability of Condorcet's paradox we can just as well consider the probability that the paradox does not exist. For the no-winner form of Condorcet's paradox, let $P(m, n, p^m)$ be the probability that there is a Condorcet winner for m alternatives with n voters and the $m!$ dimensional vector of probabilities, p^m, attached to the linearly ordered preference relations. Similarly, let $P^t(m, n, p^m)$ be the probability of a transitive simple majority relation for m alternatives with n voters and probability vector p^m associated with the $m!$ linear preference orders. It is generally assumed that voters act independently and that they vote according to their true preferences. Unless it is stated to the contrary, n is assumed to be odd and the election proceeds by taking voters with replacement after their preferences are noted.

It is apparent that $P(m, n, p^m)$ and $P^t(m, n, p^m)$ will depend on m, n and p^m. Calculated values for these probabilities frequently make some simplifying assumptions about p^m. The most common assumptions are:

> *Impartial Culture Condition* (*IC*) – Under this condition, each of the $m!$ linear preference orders are assumed to be equally likely so $p_i^m = 1/m!$ for $i = 1, 2, \ldots, m!$.

Impartial Anonymous Culture Condition (IAC) – Under this condition, m is fixed and all combinations, with repetition of orders allowed, of n linear orders are assumed to be equally likely. This condition cannot be described for any fixed p^m.

Dual Culture Condition (DC) – Under this condition, we require $p_i^m = p_j^m$ when the ith and jth linear preference orders are the duals of one another. The dual of an order is obtained by reversing all preferences on pairs in that order.

Let $P(m, n, \text{IC})$, $P(m, n, \text{IAC})$ and $P(m, n, \text{DC})$ define $P(m, n, p^m)$ where p^m meets the conditions of impartial culture, impartial anonymous culture, and dual culture respectively. Define $P^t(m, n, \text{IC})$, $P^t(m, n, \text{IAC})$ and $P^t(m, n, \text{DC})$ in a similar fashion. Many studies were conducted to find computer simulation estimates for $P(m, n, \text{IC})$ and $P^t(m, n, \text{IC})$. These simulation studies were conducted by Campbell and Tullock (1965), Klahr (1966), Williamson and Sargent (1967), Pomeranz and Weil (1970). Buckley and Westen (1979) present simulation estimates of $P(m, n, \text{IC})$ for even values of n. Some complexities that arise for the case of even n will be developed in detail in Section 7. Computer enumeration was used to find $P(m, n, \text{IC})$ for m and n up to seven by Sevcik (1969).

The results of these simulation studies give a good general idea of the likelihood of Condorcet's paradox under the condition of impartial culture. Naturally, exact calculations for the probability of Condorcet's paradox are more appealing than simulation estimates. A number of approaches were developed to obtain exact analytical representations for $P(m, n, p^m)$. These studies included Campbell and Tullock (1966), Garman and Kamien (1968), Niemi and Weisberg (1968), Hansen and Prince (1973), and DeMeyer and Plott (1970).

The most tractable representation for $P(m, n, p^m)$ for fixed m equal to three or four is found in Gehrlein and Fishburn (1976b) where it is shown that

$$P(3, n, p^3) = \Sigma^3 \frac{n!}{a_1!a_2!a_3!a_4!} \begin{bmatrix} (p_5 + p_6)^{a_1} p_3{}^{a_2} p_4{}^{a_3} (p_1 + p_2)^{a_4} + \\ + (p_2 + p_4)^{a_1} p_1{}^{a_2} p_6{}^{a_3} (p_3 + p_5)^{a_4} + \\ + (p_1 + p_3)^{a_1} p_5{}^{a_2} p_2{}^{a_3} (p_4 + p_6)^{a_4} \end{bmatrix}$$

where the superscript on the p_i^3 values have been removed to simplify notation and where Σ^3 is a triple sum with indicies a_1, a_2 and a_3 and limits

$$0 \leqslant a_1 \leqslant (n-1)/2$$

$$0 \leqslant a_2 \leqslant (n-1)/2 - a_1$$

$$0 \leqslant a_3 \leqslant (n-1)/2 - a_1$$

with $a_4 = n - a_1 - a_2 - a_3$.

The representation for $P(4, n, p^4)$ is given in terms of the probability that a specific alternative is the Condorcet winner. Let $P_A(4, n, p^4)$ be the probability that alternative A is the Condorcet winner. Then

$$P_A(4, n, p^4) = \Sigma^3 \Sigma^4 n! \sum_{\substack{i=1,2,3,4 \\ j=5,6}} \frac{p_{ij}^{a_{ij}}}{a_{ij}!}$$

where

$$p_{15} = p_{10}^4 + p_{12}^4 + p_{16}^4 + p_{18}^4 + p_{22}^4 + p_{24}^4$$

$$p_{16} = p_{15}^4 + p_9^4 \qquad p_{25} = p_{21}^4 + p_{11}^4.$$

$$p_{26} = p_7^4 + p_8^4 \qquad p_{35} = p_{23}^4 + p_{17}^4$$

$$p_{36} = p_{13}^4 + p_{14}^4 \qquad p_{45} = p_{19}^4 + p_{20}^4$$

$$p_{46} = p_1^4 + p_2^4 + p_3^4 + p_4^4 + p_5^4 + p_6^4$$

and Σ^4 is a four summation function with indices a_{15}, a_{25}, a_{35} and a_{45} and limits

$$0 \leqslant a_{15} \leqslant a_1 \qquad 0 \leqslant a_{25} \leqslant a_2$$

$$0 \leqslant a_{35} \leqslant \text{Min}\,\{a_3; (n-1)/2 - a_{15} - a_{25}\}$$

$$0 \leqslant a_{45} \leqslant \text{Min}\,\{a_4; (n-1)/2 - a_{15} - a_{25} - a_{35}\}.$$

Here $a_{i6} = a_i - a_{i5}$ and Min $\{x; z\}$ is the minimum of the two values x and z.

Representations for $P_B(4, n, p^4), P_C(4, n, p^4)$ and $P_D(4, n, p^4)$ are not really needed since they can be calculated by the representation for $P_A(4, n, p^4)$ with a suitable change in subscript for the p_i^4's. That is, to find $P_B(4, n, p^4)$ use the representation given above but interchange p_i^4 values for pairs of linear orders that are identical except that A and B are interchanged. Specifically, we would interchange values for p_1^4 and p_7^4, for p_2^4 and p_8^4, for p_3^4 and p_9^4, and so on. Then we simply use the fact that

$$P(4, n, p^4) = P_A(4, n, p^4) + P_B(4, n, p^4) + P_C(4, n, p^4) + \\ + P_D(4, n, p^4).$$

Studies have also been conducted to consider representations for $P(m, n, p^4)$ for fixed n. These studies have been restricted to the condition of impartial culture. Gehrlein and Fishburn (1979a) use a variation of a representation developed by DeMeyer and Plott (1970) and May (1971) to show that

$$P(m, 3, \mathrm{IC}) = \Sigma^2 \frac{(m-1-m_1)!(m-1-m_2)!}{m!(m-1-m_1-m_2)!(m_1+m_2+1)}$$

where Σ^2 is a double sum with indices m_1 and m_2 and limits

$$0 < m_1 < m-1 \qquad 0 < m_2 < m-1-m_1.$$

Similarly, we find a relationship for $P(m, 5, \mathrm{IC})$ is given by

$$P(m, 5, \mathrm{IC}) = \Sigma^6 \frac{b_4!(m-1-b_4)!b_5!(m-1-b_5)!b_6!(m-1-b_6)!}{b_1!b_2!b_3!b_{12}!b_{13}!b_{23}!m^*!(m!)^2} \times$$
$$\times\, Q(m^*, a_1, a_2)$$

where Σ^6 is a six summation function with indices $b_1, b_2, b_3, b_{12}, b_{13}$ and b_{23} and limits

$$0 \leqslant b_1 \leqslant m-1$$
$$0 \leqslant b_2 \leqslant m-1-b_1$$
$$0 \leqslant b_3 \leqslant m-1-b_1-b_2$$
$$0 \leqslant b_{12} \leqslant m-1-b_1-b_2-b_3$$
$$0 \leqslant b_{13} \leqslant m-1-b_1-b_2-b_3-b_{12}$$
$$0 \leqslant b_{23} \leqslant m-1-b_1-b_2-b_3-b_{12}-b_{13}$$

with

$$b_4 = b_1 + b_{12} + b_{13} \qquad b_5 = b_2 + b_{12} + b_{23}$$
$$b_6 = b_3 + b_{13} + b_{23}$$
$$m^* = m - b_1 - b_2 - b_3 - b_{12} - b_{13} - b_{23}$$
$$Q(m^*, a_1, a_2) = \sum_{\alpha=0}^{a_1} \frac{a_1!(m-1-m^*-\alpha)!}{(a_1-\alpha)!(m-m^*)!(\alpha+a_2+1)}.$$

Gehrlein and Fishburn (1979a) also developed similar representations for $P(m, 7, \mathrm{IC})$ and $P(m, 9, \mathrm{IC})$. Computationally efficient forms for $P(3, n, \mathrm{IC})$ and $P(5, n, \mathrm{IC})$ that do not have an elegant form were also developed. These relations were then used to generate $P(m, n, \text{IC})$ values for odd n up to 35 for m equal to 3 and 5, for odd m up to 449 for n equal to 3 and odd m up to 39 for n equal to 5. Other calculated values for odd n up to 49 and odd m up to 25 are shown as five digit entries in Table I. Entries for even m are not listed since they are directly obtainable as a linear combination of the $P(m^i, n, \text{IC})$ for all odd m^i less than m. This linear combination relation will be developed in detail in Section 6. Gillett (1977) calculated the probability of a unique Condorcet winner for three alternatives under impartial culture with an even number of voters for n up to 20. Gillett (1979) calculates $P(3, n, p^3)$ for some interesting p^3 other than impartial culture. Gillett (1980b) used simulation in an attempt to find the p^3 which minimize $P(3, n, p^3)$ for various n and conjectured that $P(3, n, p^3)$ is minimized by $p_1^3 = p_4^3 = p_5^3 = \frac{1}{3}$ or $p_2^3 = p_3^3 = p_6^3 = \frac{1}{3}$. Buckley (1975) proved Gillett's conjecture true for the special case of n equal to three.

A study by Gehrlein and Fishburn (1976a) developed an analytical representation for the probability of a Condorcet winner under the impartial anonymous culture condition for three and four alternatives with odd n

$$P(3, n, \text{IAC}) = \frac{15}{16} \frac{(n+3)^2}{(n+2)(n+4)}$$

and

$$P(4, n, \text{IAC}) = \frac{7n^2 + 42n + 71}{8(n+2)(n+4)}.$$

The development of representations for $P^t(m, n, p^m)$ remains open. Other than a solution for $P^t(4, \infty, \text{IC})$ that will be given in Section 5, the only representation for $P^t(m, n, p^m)$ is given by DeMeyer and Plott (1980). This representation involves a 24 summation function which makes it generally intractable for use, even with a high-speed computer. DeMeyer and Plott used their representation to calculate some values of $P^t(4, n, p^4)$ under the assumpton of impartial culture. Due to the complexity of their representation, the only results obtained were $P^t(4, 3, \text{IC}) = 0.8298$ and $P^t(4, 5, \text{IC}) = 0.7896$.

4. APPROXIMATIONS FOR $P(m, n, p^m)$

All of the analytical representations for $P(m, n, p^m)$ described in the last chapter become very difficult computationally for all but relatively small values of m and n. This leaves open the question of how we might obtain values

TABLE I

Values of $P(m, n, \mathrm{IC})^a$

	m											
n	3	5	7	9	11	13	15	17	19	21	23	25
3	0.94444	0.84000	0.76120	0.70108	0.65356	0.61484	0.58249	0.55495	0.53111	0.51021	0.49168	0.47511
5	0.93056	0.80048	0.70424	0.63243	0.57682	0.53235	0.45983	0.46521	0.43908	0.41647	0.39667	0.37915
7	0.92498	0.78467	0.68168	0.60551	0.54703	0.50063	0.46280	0.43128	0.406	0.382	0.362	0.345
9	0.92202	0.77628	0.66976	0.59135	0.534	0.486	0.447	0.415	0.388	0.365	0.345	0.327
11	0.92019	0.77108	0.664	0.585	0.524	0.475	0.436	0.404	0.377	0.354	0.334	0.316
13	0.91893	0.76753	0.659	0.578	0.517	0.468	0.429	0.397	0.369	0.346	0.326	0.309
15	0.91802	0.76496	0.655	0.574	0.512	0.463	0.424	0.391	0.364	0.341	0.321	0.304
17	0.91733	0.76300	0.642	0.570	0.508	0.459	0.420	0.387	0.360	0.337	0.317	0.300
19	0.91678	0.76146	0.650	0.568	0.505	0.456	0.417	0.384	0.357	0.334	0.314	0.297
21	0.91635	0.76023	0.648	0.566	0.503	0.454	0.414	0.382	0.354	0.331	0.311	0.294
23	0.91599	0.75920	0.646	0.564	0.501	0.451	0.412	0.379	0.352	0.329	0.309	0.292
25	0.91568	0.75835	0.645	0.562	0.499	9.450	0.410	0.378	0.351	0.328	0.308	0.290
27	0.91543	0.75763	0.644	0.561	0.498	0.448	0.409	0.376	0.349	0.326	0.306	0.289
29	0.91521	0.75700	0.643	0.560	0.497	0.447	0.407	0.375	0.348	0.325	0.305	0.288
31	0.91501	0.75646	0.642	0.559	0.496	0.446	0.406	0.374	0.347	0.324	0.304	0.287
33	0.91484	0.75598	0.642	0.448	0.495	0.445	0.405	0.373	0.346	0.323	0.303	0.286
35	0.91470	0.75556	0.641	0.557	0.494	0.444	0.405	0.372	0.345	0.322	0.302	0.285
37	0.91456	0.754	0.640	0.557	0.493	0.444	0.404	0.371	0.344	0.321	0.301	0.284
39	0.91444	0.754	0.640	0.556	0.492	·0.443	0.403	0.371	0.344	0.321	0.301	0.283
41	0.91434	0.753	0.639	0.556	0.492	0.442	0.403	0.370	0.343	0.320	0.300	0.283
43	0.91424	0.753	0.639	0.555	0.491	0.442	0.402	0.369	0.342	0.319	0.299	0.282
45	0.91415	0.753	0.639	0.555	0.491	0.441	0.401	0.369	0.342	0.319	0.299	0.282
47	0.91407	0.753	0.638	0.554	0.490	0.441	0.401	0.369	0.341	0.318	0.299	0.281
49	0.91399	0.752	0.638	0.554	0.490	0.440	0.401	0.368	0.341	0.318	0.298	0.281
Limit	0.91226	0.74869	0.63082	0.54547	0.48129	0.43131	0.39127	0.35844	0.33100	0.30771	0.28768	0.27025

[a] Five decimal place entries are exact and three decimal place entries are approximations.

for $P(m, n, p^m)$ if m and n are not relatively small. One approach would be to resort to computer simulation. A second approach, the development of approximations, is the topic of the current section.

If we limit attention to the restriction of the impartial culture condition, Gehrlein and Fishburn (1979a) give an approximation for $P(m, n, \text{IC})$ as

$$P(m, n, \text{IC}) \doteq \frac{9.33}{m + 9.53} + (0.63)^{(m-3)/2} + \frac{\dfrac{83.23}{m + 154.6} - \ln[1 + (0.55)^{(m-1)/2}]}{n - 1 + 0.9^{(m-1)/2}}.$$

This approximation was developed to predict all known values of $P(m, n, \text{IC})$ to within 0.5 percent accuracy for all known exact values of $P(m, n, \text{IC})$ with m less than 50. This approximation was used to generate the three decimal place entries in Table I.

The development of an approximation for $P(m, n, p^m)$ for general p^m becomes difficult simply due to the fact that there are $m!$ dimensions to p^m. As a result, attention to an approximation for $P(m, n, p^m)$ has been limited to m equal to three. All studies considering the general problem have suggested that we appeal to the limiting behavior of $P(m, n, p^m)$ to avoid the combinatorial aspects of calculation. A number of studies have addressed this issue, including Garman and Kamien (1968), Niemi and Weisberg (1968), DeMeyer and Plott (1970), Weisberg and Niemi (1973) and Gillett (1980a). However, it was May (1971) who spelled out the procedure for obtaining the limiting behavior of $P(m, n, p^m)$ to use as an approximation. To obtain May's approximation, define the following variables:

$$x_1 = p_1^3 + p_2^3 - p_3^3 + p_4^3 - p_5^3 - p_6^3 \qquad a_2 = nx_2$$

$$x_2 = p_1^3 - p_2^3 + p_3^3 - p_4^3 + p_5^3 - p_6^3 \qquad a_1 = nx_1$$

$$x_3 = -p_1^3 - p_2^3 - p_3^3 + p_4^3 + p_5^3 + p_6^3 \qquad a_3 = nx_3$$

$$Q_i^2 = n[1 - (a_i/n)^2], \quad i = 1, 2, 3$$

$$b_{11} = b_{22} = b_{33} = 1$$

$$b_{12} = b_{21} = p_1^3 - p_2^3 - p_3^3 - p_4^3 - p_5^3 + p_6^3$$

$$b_{23} = b_{32} = -p_1^3 + p_2^3 - p_3^3 - p_4^3 + p_5^3 - p_6^3$$

$$b_{13} = b_{31} = -p_1^3 - p_2^3 + p_3^3 + p_4^3 - p_5^3 - p_6^3$$

$$\mathrm{Cov}\,(a_i a_j) = n\left[b_{ij} - \left(\frac{a_i}{n}\right)\left(\frac{a_j}{n}\right)\right], \quad i,j = 1,2,3, \text{ and } i \neq j.$$

$$r_{ij} = [\mathrm{Cov}\,(a_i a_j)]/(Q_i Q_j), \quad i,j = 1,2,3, \text{ and } i \neq j.$$

$$\pi_i = x_i(1 - x_i^2)^{-1/2}.$$

Using these definitions, we obtain estimates for the probability that each alternative (A, B, C) is the Condorcet winner as

$$P_A(3, n, p^3) \doteq L(-n^{1/2}\pi_1, n^{1/2}\pi_3, -r_{13})$$

$$P_B(3, n, p^3) \doteq L(-n^{1/2}\pi_2, n^{1/2}\pi_1, -r_{12})$$

$$P_C(3, n, p^3) \doteq L(-n^{1/2}\pi_3, n^{1/2}\pi_2, -r_{23})$$

where $L(h, k, r)$ is the bivariate normal probability with parameters h, k and r. By definition, the probability $L(h, k, r)$ is given by

$$\int_h^\infty \int_{-\infty}^{-k} \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[-\frac{1}{2}\,\frac{x^2 - 2rxy + y^2}{1-r^2}\right] dx\, dy.$$

This function is not generally integrable, but extensive tables of values of $L(h, k, r)$ have been compiled. For example, see National Bureau of Standards (1959). May (1971) showed that the limit of accuracy of this approximation was of order $1/n$ so that it becomes exact in the limit of n. The fact will be used extensively in the next section.

Another approximation was developed by Gehrlein and Fishburn (1979a). Using a result of Bacon (1963), an approximation for $P(m, n, \mathrm{IC})$ is found in the limit of voters $(n \to \infty)$ with

$$P(m, \infty, \mathrm{IC}) \doteq m2^{-m+1}\left[1 + \sum_{k=1}^{(m-1)/2} \frac{(m-1)!\theta^k}{(m-1-2k)!\,\Pi_{i=0}^{k-1}(1-4i\theta)}\right]$$

where $\theta = [\sin^{-1}(1/3)]/\pi$. The problem of finding representations for the probability of Condorcet's paradox in the limit of n has received much attention in the literature and is the topic of the next section.

Finally, approximation methods have been used to find the probability of a tied majority election for two candidates with an even number of votes

[Beck (1975), Margolis (1977) and Chamberlain and Rothschild (1981)]. To describe these results, let $P(2,n,p)$ be the probability of a majority winner on two alternatives $\{A,B\}$ when there is a probability p of voter preference $A > B$ and a probability $1-p$ of voter preference $B > A$.

Chamberlain and Rothschild (1981) show that for even n,

$$P(2,n,\tfrac{1}{2}) \doteq 1-(\pi n)^{-1/2}$$

and if $p \neq \frac{1}{2}$,

$$P(2,n,p) \doteq 1-P(2,n,\tfrac{1}{2})\, e^{nc}$$

where

$$c = 2\log(2)+\log(p)+\log(1-p).$$

More sophisticated forms for obtaining voter profiles are also considered in Chamberlain and Rothschild (1981), but they are not presented in the current study.

5. ANALYTICAL REPRESENTATIONS FOR LIMITING CASE IN VOTERS WITH LINEAR PREFERENCE ORDERS

All of the work which deals with precise analytical representations for Condorcet's paradox has centered around the use of limiting distributions. Specifically, some sets of variables are defined and the specific probability we are seeking is found as a probability from a multivariate normal distribution on the defined variables as n goes to infinity. A good background in limiting distributions and multivariate normal distributions can be found in Johnson and Kotz (1972). The approximation developed for $P(3,n,p^3)$ in Section 4 follows this procedure. That is, the variables x_1, x_2 and x_3 were defined and $P_A(3,n,p^3)$, $P_B(3,n,p^3)$ and $P_C(3,n,p^3)$ were described as a bivariate normal probability with parameters given in terms of the x_i terms.

The first representation of Condorcet's paradox for the limiting case in voters was given by Guilbaud (1952). Guilbaud was concerned with the probability that there was a Condorcet winner on three alternatives in the limit of voters under impartial culture. He stated, without proof, that

$$P(3,\infty,\mathrm{IC}) = \tfrac{3}{4}+\tfrac{3}{2}\pi\,\sin^{-1}(\tfrac{1}{3})$$

which is evaluated as 0.91226.

We reiterate that May's approximation procedure becomes exact in the limit of n and can easily obtain Guilbaud's result. Using May's method with $p_i^3 = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$, we find

$$x_1 = x_2 = x_3 = a_1 = a_2 = a_3 = 0$$

$$Q_1^2 = Q_2^2 = Q_3^2 = n$$

$$b_{11} = b_{22} = b_{33} = 1$$

$$b_{12} = b_{21} = b_{23} = b_{32} = b_{13} = b_{31} = -\tfrac{1}{3}$$

$$\mathrm{Cov}\,(a_1 a_2) = \mathrm{Cov}\,(a_1 a_3) = \mathrm{Cov}\,(a_2 a_3) = -n/3$$

$$r_{12} = r_{13} = r_{23} = -\tfrac{1}{3} \qquad \pi_1 = \pi_2 = \pi_3 = 0.$$

Thus,

$$P(3, \infty, \mathrm{IC}) = 3L(0, 0, \tfrac{1}{3}).$$

In the case that $L(h, k, r)$ has the special form of h and k equal to zero, we can appeal to Sheppard's Theorem of Median Dichotomy [Kendall and Stuart (1963)] which states

$$L(0, 0, r) = \tfrac{1}{4} + \tfrac{1}{2}\pi \sin^{-1}(r).$$

Guilbaud's result now follows in a straightforward manner.

May (1971) and Weisberg and Niemi (1973) both pointed out that the sign of the π_i's will determine the value of $P(3, \infty, p^3)$. Consider the value of $P_A(3, \infty, p^3)$. Similar observation will hold by symmetry for $P_B(3, \infty, p^3)$ and $P_C(3, \infty, p^3)$. Since we require $n \to \infty$, May's results require that:

(1) If $\pi_1 > 0$ and $\pi_3 > 0$, then $P_A(3, n, p^3) = L(-\infty, \infty, -r_{13}) = 0$.
(2) If $\pi_1 > 0$ and $\pi_3 < 0$, then $P_A(3, n, p^3) = L(-\infty, -\infty, -r_{13}) = 1$.
(3) If $\pi_1 < 0$ and $\pi_3 > 0$, then $P_A(3, n, p^3) = L(\infty, \infty, -r_{13}) = 0$.
(4) If $\pi_1 < 0$ and $\pi_3 < 0$, then $P_A(3, n, p^3) = L(\infty, -\infty, -r_{13}) = 0$.

These results require that if none of the π_i's are zero, then the probability of observing the no-winner form of Condorcet's paradix is either 0 or 1 for any p^3 vector. Forms like that shown in Guilbaud's result only exist if both of the π_i's under consideration are zero.

(5) If $\pi_1 = 0$ and $\pi_3 = 0$, then $P_A(3, \infty, p^3) = L(0, 0, -r_{13}) = \frac{1}{4} + \frac{1}{2}\pi \sin^{-1}(-r_{13})$.

If only one of the π_i's under consideration is zero, we appeal to additional information about $L(h, k, r)$ [Srivastava and Kharti (1979)]. Specifically, $L(0, -\infty, r) = L(-\infty, 0, r) = \frac{1}{2}$. Thus

(6) If $\pi_1 = 0$ and $\pi_3 > 0$, then $P_A(3, \infty, p^3) = L(0, \infty, -r_{13}) = 0$.
(7) If $\pi_1 = 0$ and $\pi_3 < 0$, then $P_A(3, \infty, p^3) = L(0, -\infty, -r_{13}) = \frac{1}{2}$.
(8) If $\pi_1 > 0$ and $\pi_3 = 0$, then $P_A(3, \infty, p^3) = L(-\infty, 0, -r_{13}) = \frac{1}{2}$.
(9) If $\pi_1 < 0$ and $\pi_3 = 0$, then $P_A(3, \infty, p^3) = L(\infty, 0, -r_{13}) = 0$.

Since the nine specific situations for π_1 and π_3 listed above cover all possible events, the determination of $P(3, \infty, p^3)$ reduces to determining the signs of the π_i values and finding r_{ij} if the appropriate π_i's are equal to zero.

Gehrlein (1978) showed that $\pi_1 = \pi_2 = \pi_3 = 0$ if and only if p^3 meets the dual culture condition. Fishburn and Gehrlein (1980a) showed that with the dual culture condition $P(3, \infty, \mathrm{DC})$ is given by

$$P(3, \infty, \mathrm{DC}) = \tfrac{3}{4} + \tfrac{1}{2}\pi\{\sin^{-1}[2(p_1 + p_2 - p_3)] + \sin^{-1}[2(p_1 + p_3 - p_2)] + \sin^{-1}[2(p_2 + p_3 - p_1)]\}.$$

A further study by Gehrlein (1981) found the analytical representation for four alternatives in the limit of voters under dual culture as

$$P(4, \infty, \mathrm{DC}) = \tfrac{1}{2} + \tfrac{1}{4}\pi \sum_{i=1}^{12} \sin^{-1}(f_i)$$

where

$$f_1 = 2[p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - p_7 - p_8 + p_9 - p_{11} - p_{13} + p_{15}]$$

$$f_2 = 2[p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - p_7 - p_8 - p_9 + p_{11} + p_{13} - p_{15}]$$

$$f_3 = 2[p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 - p_9 - p_{11} - p_{13} - p_{15}]$$

$$f_4 = 2[p_4 + p_6 + p_7 + p_8 + p_9 + p_{11} - p_1 - p_2 + p_3 - p_5 + p_{13} - p_{15}]$$

$$f_5 = 2[p_4 + p_6 + p_7 + p_8 + p_9 + p_{11} - p_1 - p_2 - p_3 + p_5 - p_{13} + p_{15}]$$

$$f_6 = 2[p_4 + p_6 + p_7 + p_8 + p_9 + p_{11} + p_1 + p_2 - p_3 - p_5 - p_{13} - p_{15}]$$

$$f_7 = 2[p_2 + p_5 + p_8 + p_{11} + p_{13} + p_{15} + p_1 - p_3 - p_4 - p_6 + p_7 - p_9]$$

$$f_8 = 2[p_2 + p_5 + p_8 + p_{11} + p_{13} + p_{15} - p_1 - p_3 - p_4 + p_6 - p_7 + p_9]$$

$$f_9 = 2[p_2 + p_5 + p_8 + p_{11} + p_{13} + p_{15} - p_1 + p_3 + p_4 - p_6 - p_7 - p_9].$$

$$f_9 = 2[p_2 + p_5 + p_8 + p_{11} + p_{13} + p_{15} - p_1 + p_3 + p_4 - p_6 - p_7 - p_9]$$

$$f_{10} = 2[p_1 + p_3 + p_7 + p_9 + p_{13} + p_{15} + p_2 - p_4 - p_5 - p_6 + p_8 - p_{11}]$$

$$f_{11} = 2[p_1 + p_3 + p_7 + p_9 + p_{13} + p_{15} - p_2 + p_4 - p_5 - p_6 - p_8 + p_{11}]$$

$$f_{12} = 2[p_1 + p_3 + p_7 + p_9 + p_{13} + p_{15} - p_2 - p_4 + p_5 + p_6 - p_8 - p_{11}].$$

For further discussion, let $\Phi_g(r)$ describe a probability on a set of g variables from a multivariate normal distribution. The specific probability is that the g standardized variables are all positive. In addition, the terms in the correlation matrix for the multivariate normal distribution are all assumed to be equal to r. Niemi and Weisberg (1968) showed that the probability of a Condorcet winner in the limit of voters under impartial culture is directly related to the $\Phi_g(r)$ probabilities. The specific relationship that they found was that

$$P(m, \infty, \text{IC}) = m\Phi_{m-1}(\tfrac{1}{3}).$$

Guilbaud's result follows directly from this general result. In an unrelated study, Ruben (1954) calculated values of $\Phi_g(\frac{1}{3})$ for g up to 49. These values were used to calculate the $P(m, \infty, \text{IC})$ values for odd m up to 49 in Table I.

Using Niemi and Weisberg's representation for $P(m, \infty, \text{IC})$, Gehrlein and Fishburn (1978a) extend Guilbaud's result to show

$$P(4, \infty, \text{IC}) = \tfrac{1}{2} + 3/\pi \sin^{-1}(\tfrac{1}{3})$$

$$P(5, \infty, \text{IC}) = \tfrac{5}{16} + \frac{15}{4\pi}\sin^{-1}(\tfrac{1}{3}) + \frac{15}{2\pi^2}\int_0^{1/3} \frac{\sin^{-1}[\alpha/(1+2\alpha)]}{(1-\alpha^2)^{1/2}}\, d\alpha.$$

$$P(6, \infty, \text{IC}) = \tfrac{3}{16} + \frac{15}{4\pi}\sin^{-1}(\tfrac{1}{3}) + \frac{45}{2\pi^2}\int_0^{1/3} \frac{\sin^{-1}[\alpha/(1+2\alpha)]}{(1-\alpha^2)^{1/2}}\, d\alpha.$$

All of the probabilities considered to this point have been conditional probabilities. That is, we have representations for the probability of Condorcet's paradox given a specific p^m vector. Buckley (1975) introduced the concept of an unconditional probability of Condorcet's paradox. For this unconditional probability we require that there be a probability distribution over the p^m vectors. The unconditional probability of the paradox is then the expected probability of Condorcet's paradox where the expectation is over all possible p^m vectors. Specifically, let $\overline{P(m, n, p^m)}$ be the unconditional probability that there is a Condorcet winner where we assume that all p^m vectors are equally likely. Gehrlein (1981a) used May's (1981) results for a direct proof that

$$\overline{P(3,\infty,p^3)} = \tfrac{15}{16}.$$

In a follow-up study, Gehrlein (1981b) proved that

$$\overline{P(3,n,p^3)} = P(3,n,\text{IAC})$$

if all vectors are assumed to be equally likely to occur.

Other results dealing with the limiting case of voters for $P(m,n,\text{IC})$ have been found. May (1971) proved that $P(\infty,3,\text{IC}) = 0$ and that $P(\infty,\infty,\text{IC}) = 0$. Bell (1978, 1981) extended May's result that $P(\infty,\infty,\text{IC}) = 0$ and considered the probability that k elements were in the top cycle. That is, we know that there is no Condorcet winner. However, let k be the smallest integer such that we can find a set of k candidates that are in a cycle of length k and such that all of the candidates in the set defeat all remaining candidates on the basis of majority rule. Bell (1978, 1981) proved that the probability that k is less than m goes to zero as $m \to \infty$ and $n \to \infty$. Therefore, as $m \to \infty$ and $n \to \infty$ under impartial culture, the smallest k must equal m.

An additional result of Gehrlein and Fishburn (1978a) deals with the probability of a transitive simple majority relation on four alternatives in the limit of voters under the condition of impartial culture. It is shown that

$$P^t(4,\infty,\text{IC}) = \tfrac{3}{8} + 6/\pi^2 \int_0^{1/3} \frac{\cos^{-1}[-\alpha/(1-2\alpha^2)]}{(1-\alpha^2)^{1/2}}\, d\alpha.$$

Numerical integration shows that $P^t(4,\infty,\text{IC})$ is approximately equal to 0.73946. It follows that, for four alternatives in the limit of voters under impartial culture, a Condorcet winner exists while we observe the cycles form of Condorcet's paradox with probability $P(4,\infty,\text{IC}) - P^t(4,\infty,\text{IC}) = 0.08506$.

Blin (1973) proved that $P^t(m,\infty,\text{IC}) \to 0$ for large m. Williamson and Sargent (1967) showed that $P^t(m,\infty,\text{IC})$ was generally quite small for m at all large but showed that $P^t(m,\infty,p^m)$ increased dramatically as p^m changed even slightly from the impartial culture vector. The specific change they considered was to have the p_i^m for one linear preference ranking increase while each of the other $(m! - 1)$ p_i^m's decreased equally.

6. RECURSION RELATIONS FOR $P(m, n, p^m)$

All work on recursion relations has been concerned with the probability of a Condorcet winner. The recursion relations attempt to find representations for $P(m, n, p^m)$ in terms of smaller m or n. The first published recursion of this type was given by May (1971) without proof. May's Theorem restricted attention to the impartial culture condition and states that

$$P(4, n, \text{IC}) = 2P(3, n, \text{IC}) - 1$$

for odd n. Fishburn (1973b) published a simple proof of May's Theorem.

Gehrlein and Fishburn (1976b) presented a result which generalizes this recursion relation result for all even m. May's theorem is a special case of this general result. For even m and odd n, there are coefficients C_i^m such that

$$P(m, n, \text{IC}) = C_0^m + \sum_{i=1}^{(m-2)/2} C_i^m P(2i+1, n, \text{IC}).$$

Therefore, for even m, $P(m, n, \text{IC})$ is expressible as a linear combination of $P(m^i, n, \text{IC})$ for all odd m^i less than m. Table II shows the C_i^m terms of the recursion relation for $P(m, n, \text{IC})$ for m up to 12. The coefficients for m up to 24 can be found in Gehrlein and Fishburn (1976b). It was also shown that recursion relations of the linear combination type do not exist for even n. These coefficients remain the same for recursion relations for $P(m, n, \text{IAC})$ with even m and odd n.

TABLE II
C_i^m coefficients of recursion relations for $P(m, n, \text{IC})$

m	C_0^m	C_1^m	C_2^m	C_3^m	C_4^m	C_5^m
4	−1	2				
6	3	−5	3			
8	−17	28	−14	4		
10	155	−255	126	−30	5	
12	−2073	3410	−1683	396	−55	6

The notion of recursion relations for $P(m, n, p^m)$ based on smaller n was developed by Gillett (1978) for m equal to three and four. To develop these recursion relations we reintroduce the definition of $P_A(m, n, p^m)$ as the probability that alternative A is the Condorcet winner. For three alternatives we drop the superscripts from the p_i^3's for convenience and find for odd n that

$$P_A(3,n,p^3) = P_A(3,n-1,p^3) +$$

$$+ \sum_{C_1=0}^{k-1} \sum_{C_2=0}^{k-1-C_1} \frac{(n-1)!(p_5+p_6)^{C_1}(p_1+p_2)^{k-C_2}}{C_1!C_2!(k-C_1)!(k-C_2)!} \times$$

$$\times [(p_1+p_2+p_3)p_3^{C_2}p_4^{k-C_1} + (p_1+p_2+p_4)p_3^{k-C_1}p_4^{C_2}] +$$

$$+ \sum_{C_1=0}^{k} \frac{(n-1)!(p_5+p_6)^{C_1}p_3^{k-C_1}p_4^{k-C_1}(p_1+p_2)^{C_1+1}}{C_1!(k-C_1)!(k-C_1)!C_1!}$$

where $k = (n-1)/2$.

Similarly, when n is even $P_A(3,n,p^3)$ is the probability that A is the unique Condorcet winner and

$$P_A(3,n,p^3) = P_A(3,n-1,p^3) -$$

$$- \sum_{C_1=0}^{k-1} \sum_{C_2=0}^{k-1-C_1} \frac{(n-1)!(p_5+p_6)^{C_1}(p_1+p_2)^{k-C_2+1}}{C_1!C_2!(k-C_1)!(k-C_2+1)!} \times$$

$$\times [(p_4+p_5+p_6)p_3^{C_2}p_4^{k-C_1} + (p_3+p_5+p_6)p_3^{k-C_1}p_4^{C_2}] -$$

$$-(p_3+p_4+p_5+p_6) \times$$

$$\times \sum_{C_1=0}^{k} \frac{(n-1)!(p_5+p_6)^{C_1}p_3^{k-C_1}p_4^{k-C_1}(p_1+p_2)^{C_1+1}}{C_1!(k-C_1)!(k-C_1)!(C_1+1)!} .$$

By interchanging subscripts on the p^3 vector, as described in Section 3, we can obtain recursion relations for $P_B(3,n,p^3)$ and $P_C(3,n,p^3)$. The overall recursion relation for $P(3,n,p^3)$ in terms of probabilities on $n-1$ voters is then the sum of the representations for $P_A(3,n,p^3)$, $P_B(3,n,p^3)$ and $P_C(3,n,p^3)$.

Gillett (1978) also develops representations for recursion relations for $P(4,n,p^4)$ in terms of $P(4,n-1,p^4)$. The recursion relation for four alternatives becomes quite cumbersome and is not included in the current study.

7. GENERAL BEHAVIOR OF $P(m,n,p^m)$ AND $P^t(m,n,p^m)$

In Section 3, it became apparent that direct calculation of $P(m,n,\text{IC})$ and particularly $P^t(m,n,\text{IC})$ can become very difficult. As a result, effort has

has made to attempt to at least describe the general behavior of these probabilities. Of particular interest is the way that these probabilities change as m and n increase. Major progress has only been made in this area by restricting attention to the impartial culture condition.

In this section, we will concern ourselves with the difficulties that can arise when n is even. For even n, Buckley and Westen (1979) define three types of Condorcet winners for a set $X = \{x_2, x_1, x_2, \ldots, x_m\}$.

> *Strong Winner* – A candidate x_1 is a strong winner if $x_1 >_s x_j$ for all x_j in X with $i \neq j$.
>
> *Semi-Strong Winner* – A candidate x_i is a semi-strong winner if $x_i \geqslant_s x_j$ for all x_j in X with $i \neq j$ and $x_i >_s x_j$ for some j.
>
> *Weak Winner* – A candidate x_i is a weak winner if $x_i \geqslant_s x_j$ for all x_j in X with $i \neq j$.

Here $A \geqslant_s B$ denotes that the number of preference orders with A ranked over B is greater than or equal to the number of orders with B ranked over A (allows a tie).

In all discussion to this point, $P(m, n, p^m)$ has referred to the probability of a strong winner. Let $P^s(m, n, p^m)$ and $P^w(m, n, p^m)$ refer respectively to the probability of a semi-strong winner and weak winner for m candidates with n voters. For odd n, all Condorcet winners must be strong winners since ties cannot exist.

Kelly (1974) began the investigation of how $P^w(m, n, \text{IC})$ changed in m and n. Two of the main results were

THEOREM 1. $P^w(m, n+1, \text{IC}) > P^w(m, n, \text{IC})$ *for odd n and* $m \geqslant 3$.

THEOREM 2. $P^w(m, n, \text{IC}) > P^w(m, n+1, \text{IC})$ *for even n and* $m \geqslant 3$.

Kelley (1974) then formalized two conjectures based on the work of Black (1948a, 1958).

CONJECTURE 1. $P^w(m, n, \text{IC}) > P^w(m+1, n, \text{IC})$ for $m \geqslant 2$ and $n = 3$ or $n \geqslant 5$.

CONJECTURE 2. $P^w(m, n, \text{IC}) > P^w(m, n+2, \text{IC})$ for $m \geqslant 3$ and $n = 1$ or $n \geqslant 3$.

Fishburn, Gehrlein and Maskin (1979a, b) made some progress on these two Conjectures for odd n.

THEOREM 3. $P(m, 3, \text{IC}) > P(m+1, 3, \text{IC})$ *for all* $m \geqslant 2$.

THEOREM 4. $P(3, n, \text{IC}) > P(3, n+2, \text{IC})$ *for all odd* n.

THEOREM 5. $P(3, n, \text{IC}) > P(4, n, \text{IC})$ *for odd* $n \geqslant 3$.

THEOREM 6. $P(3, n, \text{IC}) > P(5, n, \text{IC})$ *for odd* $n \geqslant 3$.

THEOREM 7. $P(4, n, \text{IC}) > P(4, n+2, \text{IC})$ *for all odd* n.

Also, for even n

THEOREM 8. $P^w(3, n, \text{IC}) > P^w(3, n+2, \text{IC})$ *for large even* n.

Buckley and Westen (1979) obtained some results about strong winners.

THEOREM 9. $P(m, n, \text{IC}) > P(m, n+1, \text{IC})$ *for odd* n *and* $m \geqslant 3$.

THEOREM 10. $P(m, n+1, \text{IC}) > P(m, n, \text{IC})$ *for even* n *and* $m \geqslant 3$.

The proofs of Theorems 9 and 10 can be generalized from the impartial culture result to any p^m vector for m equal to 3 or 4 by appealing to Gillett's (1978) recursion relations in Section 6. Buckley and Westen (1979) then extended the conjectures given by Kelly (1974) to the other forms of winners:

CONJECTURE 3. $P^s(m, n+1, \text{IC}) > P^s(m, n, \text{IC})$ for odd n.

CONJECTURE 4. $P^s(m, n, \text{IC}) > P^s(m, n+1, \text{IC})$ for even n.

CONJECTURE 5. $P^s(m, n, \text{IC}) > P^s(m, n+2, \text{IC})$.

CONJECTURE 6. $P^s(m, n, \text{IC}) > P^s(m+1, n, \text{IC})$.

CONJECTURE 7. $P(m, n, \text{IC}) > P(m, n+2, \text{IC})$.

CONJECTURE 8. $P(m, n, \text{IC}) > P(m+1, n, \text{IC})$.

A final conjecture that is related to this area is due to Fishburn (1976). To describe this conjecture, we need a definition for the specific case of n voters on n alternatives. Let $P^*(n, n, \text{IC})$ be a conditional probability of a strong winner under the impartial culture condition. The conditional statement is that each of the n voters has a different candidate ranked first in their linear preference ranking. The conjecture can then be stated as

CONJECTURE 9. $P(n, n, \text{IC}) > P^*(n, n, \text{IC})$.

Fishburn, Gehrlein and Maskin (1979a, b) also developed some general relationships that hold up between Conjectures 1 and 2. These relationships were

THEOREM 11. *If* $P(5, n, \text{IC}) > P(6, n, \text{IC})$ *then* $P(4, n, \text{IC}) > P(5, n, \text{IC})$ *for odd* $n \geqslant 3$.

THEOREM 12. $P(3, n, \text{IC}) > P(6, n, \text{IC})$ *if and only if* $P(4, n, \text{IC}) > P(5, n, \text{IC})$ *for odd* $n \geqslant 3$.

THEOREM 13. *If* $P(6, n, \text{IC}) > P(6, n+2, \text{IC})$ *then* $P(5, n, \text{IC}) > P(5, n+2, \text{IC})$ *for odd* $n \geqslant 1$.

THEOREM 14. *If* $P(6, n, \text{IC}) > P(6, n+2, \text{IC})$ *then* $P(5, n, \text{IC}) > P(6, n, \text{IC})$ *for odd* $n \geqslant 3$.

Other relationships of this type were developed by Gehrlein (1981c). For example:

THEOREM 15. $P(3, n, \text{IC})^2 > P(4, n, \text{IC})$ *for odd* $n \geqslant 1$.

THEOREM 16. $P(4, n, \text{IC}) > P(3, n, \text{IC})^3$ *for odd* $n \geqslant 1$.

THEOREM 17. *If* $P(5, n, \text{IC}) \leqslant 0.780625$ *then* $P(5, n, \text{IC}) > P(6, n, \text{IC})$ *for odd* $n \geqslant 1$.

THEOREM 18. *If* $P(5, n, \text{IC}) \leqslant 0.799$ *then* $P(3, n, \text{IC}) > P(7, n, \text{IC})$ *for odd* $n \geqslant 1$.

THEOREM 19. *If* $P(7, n, \mathrm{IC}) \leqslant 0.75$ *then* $P(4, n, \mathrm{IC}) > P(5, n, \mathrm{IC})$ *for odd* $n \geqslant 1$.

THEOREM 20. *If* $P(6, n, \mathrm{IC}) > P(6, n+2, \mathrm{IC})$ *for all odd n then* $P(3, n, \mathrm{IC}) > P(7, n, \mathrm{IC})$.

THEOREM 21. *If* $P(6, n, \mathrm{IC}) > P(6, n+2, \mathrm{IC})$ *for all odd n then* $P(7, n, \mathrm{IC}) > P(8, n, \mathrm{IC})$ *implies that* $P(5, n, \mathrm{IC}) > P(7, n, \mathrm{IC})$.

THEOREM 22. $P(m, \infty, \mathrm{IC}) > m/(2(m-1))\, P(m-1, \infty, \mathrm{IC})$ *for all* $m > 1$.

Kelley's (1974) original study in the area of the general behavior of the probability of Condorcet's paradox made the most progress in considering the probability of a transitive majority relation. It was shown that

THEOREM 23. $P^t(m, n, \mathrm{IC}) > P^t(m+1, n, \mathrm{IC})$ *for* $n \geqslant 3$ *and* $m \geqslant 2$.

THEOREM 24. $P^t(m, n, \mathrm{IC}) > P^t(m, n+1, \mathrm{IC})$ *for odd n and* $m \geqslant 3$.

THEOREM 25. $P^t(m, n+1, \mathrm{IC}) > P^t(m, n, \mathrm{IC})$ *for even n and* $m \geqslant 3$.

THEOREM 26. $P^t(m, n, \mathrm{IC}) > P^t(m, n+2, \mathrm{IC})$ *for all n and* $m \geqslant 3$.

In fact, Theorem 4 follows directly from Theorem 26 since $P(3, n, \mathrm{IC}) = P^t(3, n, \mathrm{IC})$.

Let $P^q(m, n, \mathrm{IC})$ be the probability that the majority rule relation is a weak order for n voters on m alternatives under the impartial culture condition. Kelly (1974) proved that

THEOREM 27. $P^q(m, n, \mathrm{IC}) > P^q(m+1, n, \mathrm{IC})$ *for* $n \geqslant 3$ *and* $m \geqslant 2$.

THEOREM 28. $P^q(m, n, \mathrm{IC}) > P^q(m, n+1, \mathrm{IC})$ *for odd n and* $m \geqslant 3$.

THEOREM 29. $P^q(m, n+1, \mathrm{IC}) > P^q(m, n, \mathrm{IC})$ *for even n and* $m \geqslant 3$.

THEOREM 30. $P^q(m, n, \mathrm{IC}) > P^q(m, n+2, \mathrm{IC})$ *for all n and* $m \geqslant 3$.

8. CONDORCET'S PARADOX WITH OTHER PREFERENCE FORMATS

There are two basic lines of research that have considered variations from the standard format of linear ordered preferences that have been considered to this point. One approach has been to consider the impact of allowing abstentions. The other approach has been to consider voter preferences that are not essentially represented by linear rankings. We shall consider these two approaches in order.

Gehrlein and Fishburn (1978b) consider the effect of abstentions on the existence of majority winners for two candidate elections for several different situations. In general, assume that for candidates A and B we have A as the majority winner for a population of n possible voters. If voters do not necessarily vote, it may well be true that B turns out to be the majority winner for the sample subset of potential voters who actually vote. Let A be defined as the global Condorcet winner since it represents the preferences of the population. Then let B be the local Condorcet winner since it represents the preferences of the sample.

To discuss the possibility that the global and local Condorcet winners are the same, we must be concerned with

e_1 – the number of voters with $A > B$ who vote,
e_2 – the number of voters with $A > B$ who abstain,
e_3 – the number of voters with $B > A$ who vote,
e_4 – the number of voters with $B > A$ who abstain,

where $\Sigma_{i=1}^4 e_i = n$.

Let $P^r(2, n, e^*)$ be the probability that the global and local Condorcet winners coincide with two candidates with n potential voters where e^* denotes the likelihood that various e_i's will obtain. We assume n is odd and that ties are broken randomly when there are an even number of actual voters.

When e^* takes on a variation of impartial anonymous culture which assumes that all combinations of e_i's to be equally likely, we get $P^r(2, n, \text{IAC}^*)$. For all n, Gehrlein and Fishburn (1978b) show that $P^r(2, n, \text{IAC}^*)$ is independent of n with

$$P^r(2, n, \text{IAC}^*) = \tfrac{3}{4}.$$

To consider a different situation, let $P^r(2, n, p, \lambda_A, \lambda_B)$ denote the probability that the global and local Condorcet winners coincide for two candidates with n potential voters where p is the probability of a voter having a preference $A > B$, λ_A is the probability that a voter who prefers A will vote and λ_B is the probability that a voter who prefers B will vote. It is shown that for odd n

$$P^r(2, n, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{3}{4}$$

and that

$$P^r(2, \infty, \tfrac{1}{2}, \lambda, \lambda) = 1/\pi \cos^{-1}(-\lambda^{1/2}).$$

Gehrlein and Fishburn (1978b) presents some other relations for $P(2, n, p, \lambda_A, \lambda_B)$ and gives some values for $P(2, n, \tfrac{1}{2}, \lambda, \lambda)$ for various n and λ.

Gehrlein and Fishburn (1978a, 1979b) consider the probability of the coincidence of the local and global Condorcet winners in three candidate elections. Let $P^r(3, n, p, \text{IC})$ denote the coincidence probability for n voters with preferences meeting the impartial culture condition. Let p be the probability that any potential voter actually votes. The value of p is constant over preference rankings. It is shown that

$$P^r(3, \infty, \text{IC}) = \tfrac{3}{16} + \frac{3}{4\pi}\{\sin^{-1}(p^{1/2}) + \sin^{-1}(\tfrac{1}{3}) + \sin^{-1}(p^{1/2}/3\} + \frac{3}{4\pi^2}\{[\sin^{-1}(\tfrac{1}{3})]^2 + [\sin^{-1}(p^{1/2})]^2 - [\sin^{-1}(p^{1/2}/3)]^2\}.$$

Table III gives $P^r(3, \infty, p, \text{IC})$ values for $p = 0.1(0.2)0.9$.

TABLE III

Values of $P^r(3, \infty, p, \text{IC})$

p	$P^r(3, \infty, p, \text{IC})$
0.1	0.386 449
0.3	0.482 587
0.5	0.564 283
0.7	0.650 084
0.9	0.763 094

May (1970) started investigating the probability of a Condorcet winner when preferences could vary from linear preference rankings. Let $P(m, n, \text{RC})$ be the probability of a Condorcet winner in a random culture. Random culture refers to a situation in which the direction of preference on each pair is determined at random for each voter. Then, $P(m, n, \text{RC})$ is independent of n with

$$P(m, n, \text{RC}) = m2^{-(m-1)}.$$

The probability of Condorcet's paradox for three candidates with all possible voter preference structures allowed has been considered by Fishburn and Gehrlein (1980b) and Gehrlein and Fishburn (1981). To describe the more general situation, we must define voter indifference between candidates A and B, $A \sim B$, as neither $A > B$ nor $B > A$. For three candidates, there are five relevant classes of preference rankings on the candidates. These are defined in Table IV.

TABLE IV
Types of preference relations on three alternatives

Class	Number of Orders of this type	Type of Relation	Probability in Class
1	6	$A \sim B, A > C, B > C$	q_1
2	6	$A > B, B > C, A > C$	q_2
3	6	$A \sim B, B \sim C, A > C$	q_3
4	6	$A > B, B > C, A \sim C$	q_4
5	2	$A > B, B > C, C > A$	q_5

From Table IV we see that there are two orders in Class 5 which represent cyclic preference for an individual. Class 1 represents all the individual preference structures of individuals which can be represented by weak orders with two equivalent classes. The case of total individual indifference between candidates ($A \sim B$, $B \sim C$, $A \sim C$) is ignored since individuals of this type will have no impact on the existence of a Condorcet winner. It follows that $\Sigma_{i=1}^{5} q_i \leqslant 1$.

Fishburn and Gehrlein (1980b) consider the probability that there is a Condorcet winner in the limit of voters under the condition of permutation invariance, $P(3, \infty, \text{PI})$. The condition of permutation invariance assumes that all preference orders within the same class are equally likely. Before continuing, we recall the definition of $\Phi_m(r)$ from Section 5. That is, $\Phi_m(r)$ is the multivariate normal orthant probability of the m-variate normal distribution when all correlation terms equal r. It is shown that

$$P(3, \infty, \mathrm{PI}) = 3\Phi_2(p_1)$$

where

$$p_1 = \frac{q_1 + q_2 - q_4 - 3q_5}{2q_1 + 3q_2 + q_3 + 2q_4 + 3q_5}.$$

Gehrlein and Fishburn (1980a) then show that

$$P(3, \infty, \mathrm{PI}) = \tfrac{3}{4} + \frac{3}{2\pi} \sin^{-1}(p_1).$$

Guilbaud's (1952) result is obtained from this representation with $q_2 = 1$. Also, when $q_1 = 1$, we have $p_1 = \frac{1}{2}$ and $P(3, \infty, \mathrm{PI}) = 1$. This result fits with the finding of Inada (1964) that when voters have dichotmous preferences, there must be a majority winner.

Fishburn and Gehrlein (1980b) extended these results to the four and five candidate case for the limiting case in voters under a variation of the permutation invariance condition. Assume that all voters preferences are represented by a weak order. Let $q(m, i)$ be the probability that a voter has preferences which represent a weak order with $m + 1 - i$ equivalence classes. Then we define $P(m, n, \mathrm{PI}^*)$ as the probability that there is a Condorcet winner when all voters' preferences are weak orders and where all weak orders with the same number of equivalence classes are equally likely. Then

$$P(4, \infty, \mathrm{PI}^*) = 4\Phi_3(p_2)$$

$$P(5, \infty, \mathrm{PI}^*) = 5\Phi_4(p_3)$$

where

$$p_2 = \frac{14q(4,1) + 14q(4,2) + 12q(4,3)}{42q(4,1) + 35q(4,2) + 24q(4,3)}$$

$$p_3 = \frac{50q(5,1) + 50q(5,2) + 48q(5,3) + 40q(5,4)}{150q(5,1) + 135q(5,2) + 114q(5,3) + 80q(5,4)}.$$

Using the trivariate extension of Sheppard's Theorem [Kendall and Stuart (1963)], it can be shown that

$$P(4, \infty, \mathrm{PI}^*) = \tfrac{1}{2} + 3/\pi \sin^{-1}(p_2).$$

A closed form representation of this type does not exist for $P(5, \infty, \mathrm{PI}^*)$. Numerical values of $\Phi_4(r)$ can be obtained for various r from Steck (1962) and others. Using a general representation for $\Phi_4(r)$ developed by Gehrlein (1979), it can be shown that

$$P(5, \infty, \mathrm{PI}^*) = \tfrac{5}{16} + \left(\frac{15}{4\pi}\right)\sin^{-1}(p_3) +$$

$$+\left(\frac{3}{4\pi^2}\right)\int_0^{p_3}[1/1-x^2]^{1/2}\sin^{-1}\left[\frac{p_3-z^2}{1+p_3-2z^2}\right]\mathrm{d}z.$$

Tullock and Campbell (1970) used computer simulation to consider the probability of the no-winner form of Condorcet's paradox for a different set of assumptions on voters' preferences. In that study, alternatives were defined by a vector of attribute values in g-dimensional space. That is, each alternative had value which was determined by the level of each of g attributes that defined it. Let A_i^g denote the g-dimensional vector defining alternative i. Similarly, voters were given ideal points in the same space. Let V_j^g denote the g-dimensional vector defining voter j's ideal point.

For a set of mA_i^g and nV_j^g vectors, the preference rankings on the alternatives are determined for each individual by using Euclidean distance to measure how close alternative points are to the ideal points. For voter j, the most preferred candidate, i, minimizes the sum of the squared differences on attributes between V_j^g and A_i^g. The next most preferred alternative is the second closest A_k^g to V_j^g, and so on.

For a set of random A_i^g and V_j^g vectors, the preference rankings on the alternatives were obtained for each voter. It was then determined as to whether there was a Condorcet winner among the candidates. This process was repeated 1,000 times for each g of 2, 3, 5; m of 3, 4, 5, 6; and odd n ranging up to 25.

The results suggest that the probability of the no-winner form of Condorcet's paradox increases as the number of dimensions in the attribute space increases. The probability also tends to increase as the number of candidates increases. In addition, the probability tends to be smaller for 19 or more voters than for three or five voters. The probability of the no-winner form of Condorcet's paradox ranges from 0.01 to 0.12 with these assumptions.

NOTE

* This research was supported by a grant from the National Science Foundation to the University of Delaware.

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